6. C. H. Edwards, Jr., Advanced Calculus of Several Variables, Academic Press, 1973.
7. F. R. Gantmacher, The Theory of Matrices, Vol. I, Chelsea, New York, 1977.
8. H. Hancock, Theory of Maxima and Minima, Ginn, 1917.
9. ______, Lectures on the Theory of Maxima and Minima of Functions of Several Variables, University of Cincinnati Bulletin No. 13, Series II, Vol. II.
10. H. B. Mann, Quadratic forms with linear constraints, this MONTHLY, 50 (1943) 430–433.
11. J. Marsden and A. J. Tromba, Vector Calculus, 2nd ed., Freeman, 1976.
12. Y. Murata, Mathematics for Stability and Optimization of Economic Systems, Academic Press, 1977.
13. A. L. Ostrosky and J. V. Koch, Introduction to Mathematical Economics, Houghton Mifflin, 1979.
14. Ch. J. de la Vallée Poussin, Cours d'Analyse Infinitésimale, Dover Publications, New York, Vol. 1, 8th ed., Chapter III, Section 4 and Chapter IV, Section 3, 1946.

THE SEARCH FOR A ROLLE'S THEOREM IN THE COMPLEX DOMAIN

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1. Introduction. Our purpose here is to give a brief account of the results obtained over many years in the attempts to create a complex domain analogue to a very familiar theorem of real analysis. The theorem is one which is often introduced in our first calculus courses. It states that between any two real zeros of a differential real function f lies at least one critical point of f (zero of its first derivative f').

This theorem first appeared in a book published in 1691 by the French mathematician Michel Rolle. Its publication had predated the adoption of the geometric representation of complex numbers by about 140 years. For, though this representation was devised by the Norwegian cartographer Caspar Wessel in 1797 and again by the Swiss mathematician Jean Argand in 1806, its universal acceptance had to await its invention in 1831 by none other than the great Karl Friedrich Gauss.

With this representation came the concepts of a complex variable $z = x + iy$ and a function of a complex variable. It is not surprising that some early attention was directed towards constructing in the complex domain counterparts to well-known theorems of real analysis such as Rolle's theorem. However, the generalization of Rolle's theorem to the complex plane is not obvious or trivial, as the following two examples show.

First, take the function $f(z) = e^{zi} - 1$ which has zeros at $z = 0$ and at $z = 2\pi$. If Rolle's theorem were valid, at least one critical point would be situated on the interval $0 < x < 2\pi$. But $f'(z) = ie^{zi}$ so that f' has no zeros whatsoever.

Secondly, take the polynomial $f(z) = (z^2 - 1)(z - i\sqrt{3})$ which has zeros at the vertices $z = \pm 1$, $z = i\sqrt{3}$ of an isosceles triangle. If Rolle's theorem were valid, f would have a critical point on each side of this triangle. But $f'(z) = 3[z - (i/\sqrt{3})]^2$ so that f' has a single zero at $z = i\sqrt{3}$, a point interior to the triangle.

As the second example shows, the concept of a critical point lying between two real zeros (i.e.,

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on the line segment joining the two zeros) generally is replaced in the complex plane by the concept of a critical point situated in some region containing the zeros of the given function. In our second example that region is a closed triangle but in our later examples a polygon or a circular disk may be the most convenient choice.

As to the importance of locating the critical points of a given function f, it is to be recalled that, whereas for a function of a real variable the determination of the critical points helps in locating the maxima and minima, for a function of a complex variable f analytic in a region T finding the critical points aids in determining where the map of T by $w = f(z)$ fails to be conformal.

2. Locating all the critical points. An immediate corollary of Rolle's Theorem is the following result.

THEOREM 2.1. *If a real polynomial f of the real variable x has only real zeros all situated on the interval I: $a \leqslant x \leqslant b$ of the real axis, then all the critical points of f also lie on I.*

It is Theorem 2.1 that was first to be generalized to the complex domain. The earliest such result was due to Gauss who in 1836 stated the following [**2**, pp. 21–24].

THEOREM 2.2 (Gauss). *The critical points of a polynomial f which are not multiple zeros of f are located at the equilibrium positions in a certain field of force. This field is one due to a particle placed at each zero of f, having a mass equal to the multiplicity of the zero and attracting according to the inverse distance law.*

From Gauss' Theorem we may deduce immediately the following fact.

THEOREM 2.3 (Lucas). *The critical points of a polynomial f lie in the convex hull H of the zeros of f.*

The convex hull H of the zeros of f is defined as the smallest convex polygon enclosing all the zeros of f.

Though Theorem 2.3 is a corollary of Theorem 2.2, nowhere in Gauss' papers does one find either a statement or proof of Theorem 2.3. The first published statement and proof of Theorem 2.3 appears to be due to the French engineer F. Lucas in 1874.

Another form of Lucas' Theorem is the following result.

THEOREM 2.4. *The critical points of a polynomial f lie in any circle C enclosing all the zeros of f.*

In fact, Theorem 2.4 is equivalent to Theorem 2.3. For, on the one hand, C encloses H and on the other hand, H is the intersection of all circular disks covering H.

3. Separation of the critical points. If, instead of being given that all the zeros of polynomial f lie in a single circle C, one presupposes that the zeros are distributed over a set of given circular disks, one may expect more specific results on the location of the critical points than given in Theorem 2.4. During the period 1918-1922 Joseph L. Walsh discovered a number of such results, two of which are the following [**2**, pp. 89-92].

THEOREM 3.1 (Walsh). *If a polynomial f of degree n has m_1 zeros in a circle C_1, with center at $z = c_1$ and radius r_1 and the remaining $m_2 = n - m_1$ zeros in a circle C_2 with center at $z = c_2$ and radius r_2, then every critical point of f that does not lie in C_1 or C_2 lies in a third circle C with center at $z = (m_2c_1 + m_1c_2)/n$ and radius $r = (m_2r_1 + m_1r_2)/n$.* (See Fig. 1.)

THEOREM 3.2 (Walsh). *If $C_0, C_1, \dots, C_p$ is a set of circles having a common external center 0 of similitude and if f is a polynomial of degree n that has m_k zeros in circle C_k, $k = 0, 1, \dots, p$, then every critical point of f, not in a circle C_k, lies in a circle C_k' where $C_1', C_2', \dots, C_q'$ is a set of circles also having 0 as an external center of similitude.* (See Fig. 2.)

An interesting new application of Theorem 3.1 is to a real polynomial f with only real zeros.

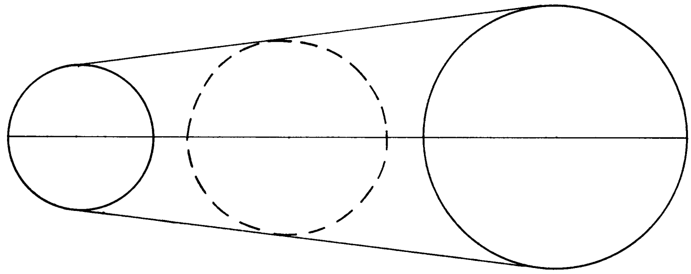

FIG. 1

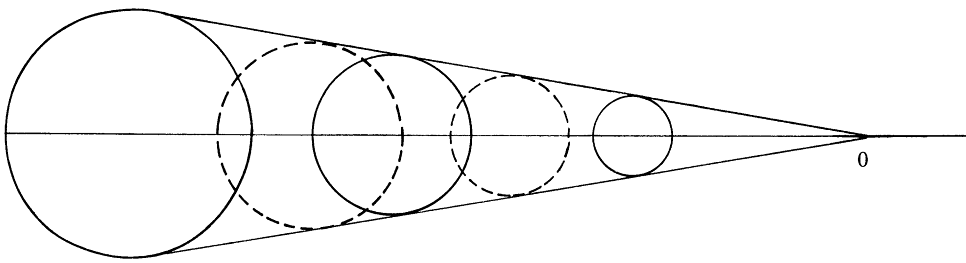

FIG. 2

THEOREM 3.3. *Let f be a real polynomial of degree n with only real zeros of which m_1 are located on the interval I_1: $a_1 \leqslant x \leqslant b_1$ and the remaining $m_2 = n - m_1$ are located on the interval I_2: $a_2 \leqslant x \leqslant b_2$, with $a_2 > b_1$. Then any critical point of f not on I_1 or I_2 is located on the interval I: $a \leqslant x \leqslant b$, where $a = (m_2a_1 + m_1a_2)/n$ and $b = (m_2b_1 + m_1b_2)/n$.*

This theorem supplements Rolle's Theorem. To prove it, let us draw the circle C_1 whose center is at $c_1 + i\gamma$, where $c_1 = (a_1 + b_1)/2$, and which passes through the two points $z = a_1$ and $z = b_1$. Also let us draw the circle C_2 whose center is at $c_2 + i\gamma\lambda$, where

$$c_2 = (a_2 + b_2)/2 \quad \text{and} \quad \lambda = (b_2 - a_2)/(b_1 - a_1),$$

and which passes through the points $z = a_2$ and $z = b_2$. Then according to Theorem 3.1, any critical point of f not in circle C_1 or circle C_2 lies in circle C whose center is at

$$z = (1/n)[m_1c_2 + m_2c_1 + i\gamma(\lambda m_1 + m_2)]$$

and which cuts the real axis in the points $z = a$ and $z = b$. Now, as γ is allowed to vary from $-\infty$ to $+\infty$, the intersection of all disks C_1 is interval I_1, the intersection of all disks C_2 is the interval I_2 and finally the intersection of all the circles C is the interval I. Thus, we have proved Theorem 3.3.

4. Further separation of the critical points. Theorem 3.2 has been generalized to an arbitrary set of circular disks C_k, one that does not necessarily have a common external center of similitude [**2**, pp. 96-106].

THEOREM 4.1 (Marden). *For $j = 0, 1, 2, \ldots, p$ let $f_j(z)$ be a polynomial of degree m_j that has all its zeros on the circular disk $C_j(z) \leqslant 0$, where $C_j(z) \equiv |z - c_j|^2 - r_j^2$. Then, if a critical point of the product*

$$f(z) = f_0(z)f_1(z) \cdots f_p(z) \tag{4.1}$$

is not situated in one of the circular disks $C_0, C_1, \ldots, C_p$, it lies in a simply-connected region bounded *by an oval of the p-circular 2p-ic curve $E(z) = 0$ where*

$$\frac{E(z)}{\prod_0^p C_j(z)} = \sum_{j=0}^{p} \frac{nm_j}{C_j(z)} - \sum_{\substack{j=0\\k=j+1}}^{p} \frac{m_j m_k \tau_{jk}}{C_j(z) C_k(z)} \tag{4.2}$$

and where

$$\tau_{jk} = |c_j - c_k|^2 - (r_j - r_k)^2, \quad n = m_0 + m_1 + \cdots + m_p.$$

It is to be noted that, if $\tau_{jk} > 0$, then τ_{jk} equals the square of the length of the external common tangent of the circles $C_j(z) = 0$ and $C_k(z) = 0$.

To clarify Theorem 4.1, let us consider the special case $p = 2$, that is, of just three given circles $C_j(z) = 0$, $j = 0, 1, 2$. Equation (4.2) then simplifies to

$$E(z) \equiv n^2(x^2 + y^2)^2 + \text{lower degree terms}, \tag{4.3}$$

so that $E(z) = 0$ is the equation of a bicircular quartic. It may happen that (4.3) is factorable, thus:

$$E(z) = (x^2 + y^2 + \alpha_1 x + \beta_1 y + \gamma_1)(x^2 + y^2 + \alpha_2 x + \beta_2 y + \gamma_2), \tag{4.4}$$

in which case the bicircular quartic $E(z) = 0$ degenerates into a pair of circles as for Theorem 3.2.

As an example of a non-degenerate case, let us choose the circles C_0, C_1 and C_2 each of radius r but with centers respectively $c_0 = 2i$, $c_1 = -3 - i$, $c_2 = +3 - i$, thus at the vertices of an isosceles triangle. Then (see Fig. 3) for $r = 0$, the bicircular quartic reduces to the points $z = \pm\sqrt{2}$. For $0 < r < \sqrt{3} - 1$, the quartic consists of two distinct ovals, one surrounding point $z = -\sqrt{2}$ and the other point $z = +\sqrt{2}$. For $r = \sqrt{3} - 1$ the two ovals touch one another at $z = 0$. For $\sqrt{3} - 1 < r < \sqrt{3} + 1$ the quartic is a single oval. For $r = \sqrt{3} + 1$, the quartic consists of a single oval and the isolated point $z = 0$. For $r > \sqrt{3} + 1$, the quartic consists of two ovals, with one enclosed in the other; in this case, any critical point, not in one of the given circular regions C_0, C_1 or C_2, lies in or on the outer oval of the bicircular quartic.

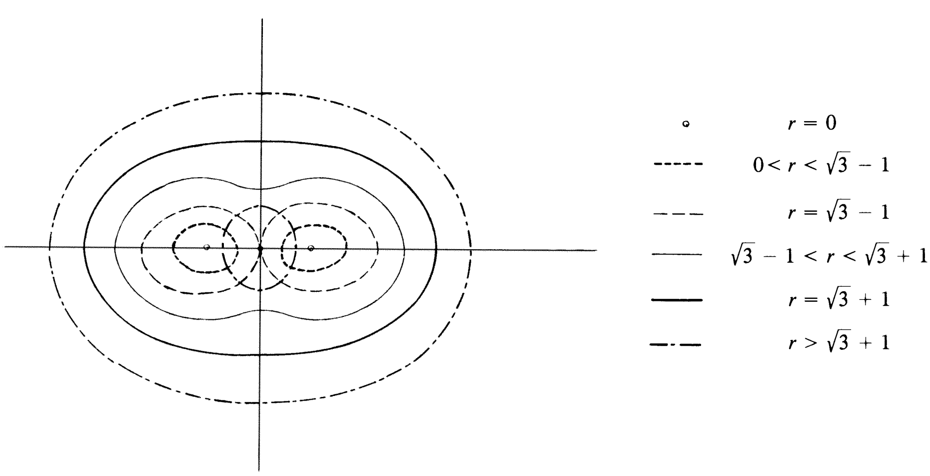

FIG. 3

In the general case equation (4.2) has the form

$$E(z) \equiv n^2(x^2 + y^2)^p + \text{lower degree terms}, \tag{4.5}$$

so that $E(z) = 0$ represents a p-circular $2p$-ic curve. While such curves have been known for some time, very little has been written about their properties. It may happen that the curve $E(z) = 0$ degenerates into a set of circles as in the case of Theorem 3.2.

Another example where curve $E(z) = 0$ degenerates into a set of circles is that in which the circles C_j are all of the same radius r and have their centers at the $p + 1$ vertices of a regular polygon P. The curve $E(z) = 0$ then reduces to a set of concentric circles centered at the center of P.

5. Locating only some critical points. So far we have discussed some complex domain analogues, not to Rolle's Theorem, but to its corollary, Theorem 2.1, that involves all the zeros and all the critical points of a polynomial. However, Rolle's Theorem proper deals with the location of the critical points of a real polynomial in relation only to pairs of real zeros.

A complex domain analogue closer to Rolle's Theorem is the following due to the two British mathematicians J. H. Grace in 1902 and P. J. Heawood in 1907 [**2**, pp. 107-110].

THEOREM 5.1. *If z_1 and z_2 are any two distinct zeros of a polynomial f of degree n, then at least one critical point of f lies on the circular disk $|z - c| \leqslant r$, where*

$$c = (z_1 + z_2)/2 \quad \text{and} \quad r = (|z_1 - z_2|/2)\cot(\pi/n).$$

This result turns out to be the best possible in that the limits given in Theorem 5.1 are attained by the polynomial

$$f(z) = \int_{-1}^{z} [s - i\cot(\pi/n)]^{n-1}\, ds.$$

From Theorem 5.1 one may deduce the following result, discovered independently by the American topologist Alexander in 1915.

THEOREM 5.2. *If the disk $|z| \leqslant R$ contains two zeros of an nth degree polynomial f, the concentric disk $|z| \leqslant R\csc(\pi/n)$ contains at least one critical point of f.*

In an attempt to generalize Theorem (5.2), the Japanese mathematician Kakeya established in 1917 the following result [see **2**, pp. 113-121].

THEOREM 5.3. *If the disk $|z| \leqslant R$ contains p zeros of an nth degree polynomial where $2 \leqslant p \leqslant n$, then there exists a number $\phi(n, p)$ dependent only upon n and p such that at least $p - 1$ critical points of f lie in the disk $|z| \leqslant R\phi(n, p)$.*

The value $\phi(n, n) = 1$ follows from Lucas' result in Theorem 2.4. Kakeya himself was able to show $\phi(n, 2) = \csc(\pi/n)$ confirming our Theorem 5.2, but admitted an inability to calculate $\phi(n,p)$ for general p. In 1927 the Polish mathematician M. Biernacki calculated the value $\phi(n, n - 1) = [1 + (1/n)]^{1/2}$. In 1936 Marden showed that

$$\phi(n, p) \leqslant \csc(\pi/2q), \tag{5.1}$$

where $q = n - p + 1$, and in (1945) Biernacki showed that

$$\phi(n, p) \leqslant \prod_{k=1}^{n-p} [(n + k)/(n - k)]. \tag{5.2}$$

However, neither (5.1) nor (5.2) provides the least upper bound to $\phi(n, p)$.

In 1972, Marden [**4**] studied the Kakeya problem relative to the special family of polynomials

$$f(z) = (z - z_1)^{\lambda}(z - z_2)^{\mu}(z - z_3)^{\nu},$$

where

$$\lambda + \mu = p, \quad \nu = n - p, \quad |z_1| = |z_2| = R, \quad \text{and} \quad |z_3| \geqslant R.$$

For such polynomials he proved that $\phi(n, p) \leqslant [2 - (p/n)]^{1/2}$ and that this bound is attained when p is an even integer.

An application of the above leads to the result:

THEOREM 5.4 (Marden). *An nth degree polynomial P is at most p-valent* $(2 \leqslant p \leqslant n)$ *on the disk* D: $|z| \leqslant \sin[\pi/2(n-p)]$ *if, in the unit disk* $|z| \leqslant 1$, *P has at most* $p-1$ *critical points.*

For a function f to be at most p-valent in a region S means that, if A is an arbitrary real or complex number, the equality $f(z) = A$ holds on at most p distinct points z within S.

Theorem 5.4 follows almost immediately from Theorem 5.3 and inequality (5.1). For, if for some number A the equation $P(z) = A$ were valid at $p+1$ points in the disk D, then $f(z) = P(z) - A$ would have at least p critical points in the disk

$$|z| \leqslant \phi(n, p+1)\sin[\pi/2(n-p)] \leqslant \csc[\pi/2(n-p)]\sin[\pi/2(n-p)] = 1,$$

in contradiction to the hypothesis of Theorem 5.4. Hence, $P(z) = A$ in at most p points of D.

6. Some recent conjectures. During the past dozen years considerable interest has been aroused regarding the location of the critical points of any polynomial f all of whose zeros lie in the unit disk $|z| \leqslant 1$. By Theorem 2.4 we know that all the critical points of f also lie on the unit disk. The question recently raised is how close to each zero do the critical points lie [see **3**].

The following conjecture was made by the Bulgarian mathematician B. Sendov in 1962 but became later known as the "Ilyeff Conjecture."

CONJECTURE 1. *If all the zeros of an nth degree polynomial lie on the disk* $|z| \leqslant 1$ *and* z_0 *is any one of the zeros, then at least one critical point of f lies within unit distance from* z_0.

This conjecture has been proved in a number of special cases including $n = 2, 3, 4, 5$, but not as yet in general.

Another conjecture was made in 1969 by the American mathematicians Goodman and Roth, the Canadian-Indian mathematician Rahman and the German mathematician Schmeisser.

CONJECTURE 2. *Under the same hypotheses as for Conjecture* 1, *at least one critical point of f lies on the disk* $|z - (z_0/2)| \leqslant 1 - (|z_0|/2)$.

This has been proved when $|z_0| = 1$, but some counterexamples have been devised for the case $|z_0| < 1$ by M. J. Miller [see **5**]. One of these is the sixth degree polynomial $P(z) = (z - 0.84)Q(z)$, where

$$Q(z) = z^5 + 1.182303196z^4 + 1.34007004z^3 + 1.34007004z^2 + 1.182303196z + 1.$$

Miller shows that the six zeros of $P(z)$ are approximately the following quantities:

$$z_0 = 0.84, \quad z_1 = -1, \quad z_2 = 0.415548 + 0.909571i,$$
$$z_3 = -0.506700 + 0.862122i, \quad z_4 = \bar{z}_2, \quad z_5 = \bar{z}_3.$$

Obviously $|z_0| < 1$. His calculations show that

$$|z_1| = |z_2| = |z_3| = |z_4| = |z_5| = 1.$$

On the other hand, the five zeros of $P'(z)$ are computed by him to be approximately the following:

$$\zeta_1 = -0.162505, \quad \zeta_2 = -0.159819 + 0.055911i, \quad \zeta_3 = 0.098445 + 0.485713i,$$
$$\zeta_4 = \bar{\zeta}_2, \quad \zeta_5 = \bar{\zeta}_3.$$

If Conjecture 2 were valid for polynomial P and for the above zero z_0, then the inequality

$$S_k = |\zeta_k - 0.42| \leqslant 0.58$$

would hold for at least one $k, k = 1, 2, \ldots, 5$. However, his calculations show that $S_k \geqslant 0.582505$ for all k, thus refuting Conjecture 2.

7. Critical points of non-polynomials. So far we have surveyed the results aimed at extending Rolle's Theorem to polynomials in a complex variable. We now shall consider the corresponding work with entire functions, rational functions and more general functions of a complex variable.

The names of Laguerre, Cesàro, and Pólya are associated with the earlier work on entire functions of a complex variable. By an entire function is, of course, meant one for which the Maclaurin series

$$f(z) = b_0 + b_1 z + b_2 z^2 + \cdots$$

converges for all values of z. Of special interest are entire functions of finite order ρ, meaning

$$\rho = \limsup_{k\to\infty} \frac{k \log k}{\log(1/|b_k|)} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} < \infty,$$

where $M(r,f) = \{\max|f(z)|, |z| = r\}$.

The following is a result about the critical points of entire functions of finite order ρ.

THEOREM 7.1 (Marden). *If f, an entire function of finite order ρ, has all its zeros in a semi-infinite strip T, at most n critical points of f lie in sector S (see* Fig. 4), *where n is the largest positive integer less than or equal to ρ.*

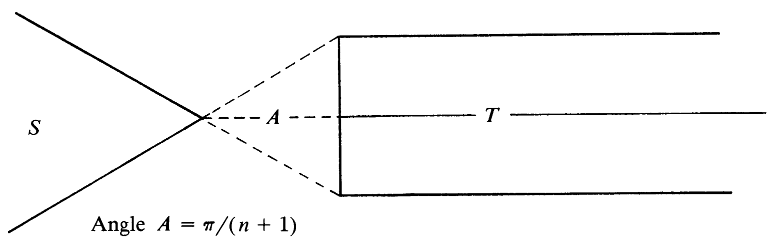

FIG. 4

For example, let ρ be a non-negative integer and let

$$f(z) = e^{\lambda z^\rho} \prod_{k=0}^{m} \left[(z-k)^2 + 1\right].$$

This entire function of order ρ has all its zeros in a semi-infinite strip T: $0 \leqslant x$, $-1 \leqslant y \leqslant 1$. Hence, at most $n = \rho$ critical points lie in the sector S.

Another example

$$f(z) = \prod_{k=-p}^{k=p} \cos(z - ki)^{1/2}$$

is that of an entire function of order $\rho = 1/2$. Here one finds as the semi infinite strip T, $x \geqslant \pi^2/4, |y| \leqslant p$, whereas $n = 0$. Thus no critical points lie in S. In fact, one can show that all the critical points lie in T.

More generally, it may be proved that in the case of any entire function f with an order ρ, $0 \leqslant \rho < 1$, any convex polygon H which contains all the zeros of f also contains all the critical points of f, just as in Theorem 2.3. However, H will ordinarily be unbounded, since f may have an infinite number of zeros.

For rational and meromorphic functions, there are theorems similar to those on polynomials and entire functions respectively.

For a more general function f which is analytic in a region T, some results have been obtained by the noted French mathematician Jean Dieudonné in 1930 [see **1**]. Supposing that f has in T a zero at $z = z_1$ and one at $z = z_2$, and that the circle C with the line segment from z_1 to z_2 as diameter is contained in T, he then determines conditions on f such that at least one critical point of f lies in C. However, these conditions are too complicated to be stated here.*

*If z_1 and z_2 are sufficiently close, there should be a critical point in their immediate vicinity, since the critical points are continuous functions of the zeros. For, $z_2 \to z_1$ makes z_1 a double zero and thus a critical point.

8. Concluding remarks. In summary, we have surveyed the efforts made towards devising some complex plane counterparts to Rolle's Theorem. These attempts began about one hundred and fifty years ago with Gauss and have continued to the present day. None of the results so far have had the generality and simplicity of Rolle's Theorem. Hence, it remains a challenge for the future to find a true analogue of Rolle's Theorem in the complex domain.

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References

1. Jean Dieudonné, Sur une généralisation du théorème de Rolle aux fonctions d'une variable complexe, Ann. of Math., 31 (1930) 79–116.
2. Morris Marden, Geometry of Polynomials, Math. Survey no. 3, Amer. Math. Soc., Providence, 1966.
3. ______, Conjectures on the critical points of a polynomial, this MONTHLY, 90 (1983) 267–276.
4. ______, Kakeya's problem for polynomials, Tensor, N.S. 25 (1972) 365–371.
5. Michael J. Miller, On Ilieff's Conjecture and the Geometry of Polynomials, Doctoral Thesis, University of Rochester, 1984.

ON ALL SORTS OF AUTOMORPHISMS*

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0. Introduction. If ϕ is a linear transformation of a finite-dimensional vector space into itself, then there is a simple criterion that ϕ be invertible, that is, an automorphism, given by $\det \phi \neq 0$. However, this criterion fails, as we shall see, in general for an endomorphism of a finitely-generated module over a commutative unitary ring. In this note we find a criterion for such an endomorphism ϕ to be an automorphism in terms of the existence of certain polynomial annihilators of ϕ, basing ourselves on a Cayley-Hamilton Theorem for such endomorphisms. We then examine the very different situation which occurs if we drop the condition that the module be finitely-generated; and this in turn leads us to criteria for special classes of automorphisms of (non-commutative) groups.

The criterion in terms of polynomial annihilators, and the extension to the non-finitely-generated case, arose, in the context of homotopy theory, in trying to understand some interesting work of J. M. Cohen **[2]**, **[3]** on self-maps of fiber spaces. Cohen, in fact, used one aspect of the criterion in **[2]**, and gave an ad hoc proof, in the case of abelian groups, in **[3]**. However, the problem of understanding Cohen's work was exacerbated by a defect in his definition of a *pseudo-identity* of an abelian group as a special type of automorphism. Nevertheless, it became plain, from a study of **[3]**, that a simplification of a basic tool used by Cohen could be achieved by using a

Peter Hilton was born in London, England, on April 7, 1923. He was educated at Oxford University and spent the years 1942–5 of the Second World War as a cryptanalyst at Bletchley Park, where he became very friendly with J. H. C. (Henry) Whitehead. On demobilization he returned to Oxford to obtain a D.Phil. in algebraic topology under Whitehead's supervision. After an academic career in England, culminating in his holding the Mason Chair of Pure Mathematics at the University of Birmingham, he came to the U.S. in 1962 to become Professor of Mathematics at Cornell University. He is now Distinguished Professor of Mathematics at SUNY-Binghamton.

Hilton's research interests are, as they have always been, in algebraic topology and homological algebra, but he has taken an increasing interest in mathematics education, at all levels, since coming to the U.S. He has been Chairman of the U.S. Commission on Mathematical Instruction and Secretary of the International Commission. He was First Vice-President of the MAA, 1978–80.

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