

ON KAMEKO'S MORPHISM

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ABSTRACT. This short note gives a presentation of the Kameko morphism on the cohomology of an elementary abelian p -group, using the standard tools of the theory of unstable modules, which allows a straightforward relation of this morphism to the Verschiebung of the Steenrod algebra.

1. INTRODUCTION

The cohomology $\text{Ext}_{\mathcal{A}}^*(\mathbb{F}, \mathbb{F})$ of the Steenrod algebra over the prime field $\mathbb{F}$ with two elements is an object of great interest. For V an $\mathbb{F}$ -vector space, Singer [9] constructed an algebraic version of the transfer

$$\mathbb{F} \otimes_{GL(V)} PH_*(BV) \rightarrow \text{Ext}_{\mathcal{A}}^*(\mathbb{F}, \mathbb{F})$$

which gives important information on the cohomology of the Steenrod algebra. In his thesis [5], Kameko constructed an operation on the left hand side which is a model for the squaring operation on $\text{Ext}_{\mathcal{A}}^*(\mathbb{F}, \mathbb{F})$. (The reader is referred to the presentation of Kameko's construction by Crabb and Hubbuck [3]). This has been exploited by a number of authors [1, 2, 4, 6] in the study both of the module of indecomposables $\mathbb{F} \otimes_{\mathcal{A}} H^*(BV)$ and with respect to the algebraic transfer.

In this note, cohomology is used in place of homology, hence the Kameko morphism is a Verschiebung rather than a squaring operation. In particular, the morphism constructed by Kameko is related to the Verschiebung on the Steenrod algebra. The purpose of this note is to give a formulation of the Kameko morphism using the standard tools of the theory of unstable modules. This leads to a uniqueness statement and an elementary description of the Verschiebung as a morphism $H^*(BV) \rightarrow \Sigma^{\dim V} \Phi H^*(BV)$ in the category of modules over $\Phi H^*(BV)$ in $\mathcal{U}$, where Φ is the doubling operator (see Theorem 2.3 below for the precise statement).

This presentation makes the proof of the compatibility with the squaring operation on the cohomology $\text{Ext}_{\mathcal{A}}^*(\mathbb{F}, \mathbb{F})$ transparent. Namely, following [6, Section 4], it is sufficient to prove the compatibility with the Verschiebung on the Steenrod algebra. This key step is proved below as Proposition 3.3.

2. EXISTENCE AND UNIQUENESS

Unless otherwise stated, the underlying coefficient field, $\mathbb{F}$, is the field $\mathbb{F}_2$ with two elements. Indications are given on the modifications which are necessary in odd characteristic. The reader is referred to [8] for standard facts about the category $\mathcal{U}$ of unstable modules and the category $\mathcal{K}$ of unstable algebras. Recall that $\Phi : \mathcal{U} \rightarrow \mathcal{U}$ denotes the doubling functor and Σ the suspension functor; these are restrictions of functors on the category of graded vector spaces, which are denoted by the same symbols.

Notation 2.1.

- (1) For d a natural number, let P_d denote the unstable algebra $H^*(B\mathbb{F}^d)$.

Date: 30 ix 2007.

- (2) Let $K\text{-}\mathcal{U}$ denote the category of K -modules in $\mathcal{U}$ over the unstable algebra K .
- (3) A morphism which is not necessarily $\mathcal{A}$ -linear will be denoted by $- \text{--} \succ$.

For d a positive integer, $\text{End}(\mathbb{F}^d)$ is a monoid with group of units $GL_d(\mathbb{F})$. There is a morphism of algebras $\mathbb{F}[\text{End}(\mathbb{F}^d)] \rightarrow \mathbb{F}[GL_d(\mathbb{F})]$ which is a retract of the canonical inclusion, induced by the linear morphism which sends singular morphisms to zero. The following statement is clear.

Lemma 2.2. *For d a positive integer,*

- (1) $\Lambda^d(\mathbb{F}^d)$ gives a natural representation of $\text{End}(\mathbb{F}^d)$, which restricts to the trivial representation $\mathbb{F}$ of $GL_d(\mathbb{F})$;
- (2) for any $\mathbb{F}[GL_d(\mathbb{F})]$ -representation M , $M \otimes \Lambda^d(\mathbb{F}^d)$ is naturally a representation of $\mathbb{F}[\text{End}(\mathbb{F}^d)]$, which is equivalent to the restriction of M via the morphism $\mathbb{F}[\text{End}(\mathbb{F}^d)] \rightarrow \mathbb{F}[GL_d(\mathbb{F})]$.

There is a natural (mono)morphism $\Phi P_d \xrightarrow{Sq_0} P_d$ in $\mathcal{K}$, with respect to which P_d becomes an object of $\Phi P_d\text{-}\mathcal{U}$.

Theorem 2.3. *For d a natural number, there is a unique surjective morphism*

$$P_d \text{--} \overset{\psi_d}{\succ} \Sigma^d \Phi P_d$$

of graded ΦP_d -modules.

This satisfies the following properties:

- (1) ψ_d is $\mathcal{A}$ -linear, hence is a morphism in the category $\Phi P_d\text{-}\mathcal{U}$;
- (2) ψ_d induces an $\text{End}(\mathbb{F}^d)$ -equivariant morphism

$$P_d \rightarrow \Sigma^d \Phi P_d \otimes \Lambda^d(\mathbb{F}^d).$$

Proof. The theorem has a sense for sense for $d = 0$, it suffices to take $\psi_0 : \mathbb{F} \xrightarrow{\cong} \mathbb{F}$ and uniqueness is clear. Henceforth, we suppose $d > 0$.

The key point of the Theorem is the uniqueness statement, which makes essential use of the ΦP_d -module structure. The algebra P_d identifies with the symmetric algebra $S^*(\mathbb{F}^d)$, where $\mathbb{F}^d$ is considered as being in degree one. There is a canonical surjection of graded algebras $S^*(\mathbb{F}^d) \rightarrow \Lambda^*(\mathbb{F}^d)$; together with the diagonal morphism $P_d \rightarrow P_d \otimes P_d$, this induces an isomorphism of ΦP_d -modules $P_d \cong \Phi P_d \otimes \Lambda^*(\mathbb{F}^d)$. In particular, P_d is a free ΦP_d -module on the space of indecomposables $\Lambda^*(\mathbb{F}^d)$.

There is a unique surjection $\Lambda^*(\mathbb{F}^d) \rightarrow \Sigma^d \mathbb{F}$ of graded vector spaces which induces a canonical surjection $P_d \rightarrow \Sigma^d \Phi P_d$ of graded ΦP_d -modules, for grading reasons. It is straightforward to deduce that this morphism is $\text{End}(\mathbb{F}^d)$ -equivariant after tensoring the range with $\Lambda^d(\mathbb{F}^d)$. It remains to establish the fact that the morphism is $\mathcal{A}$ -linear; for this it suffices to exhibit an explicit construction.

For $d = 1$, there is a unique surjective morphism of unstable modules $P_1 \rightarrow \Sigma \Phi P_1$, which corresponds to the short exact sequence

$$0 \rightarrow \Phi P_1 \rightarrow P_1 \rightarrow \Sigma \Omega P_1 \rightarrow 0,$$

via the isomorphism $\Omega P_1 \cong \Phi P_1$. This is a morphism of ΦP_1 -modules, hence satisfies the properties of the statement of Theorem 2.3.

For $d > 1$, use the Künneth isomorphism $P_d \cong P_1^{\otimes d}$ in $\mathcal{K}$ to define ψ_d to be the tensor product $P_d \cong (P_1)^{\otimes d} \xrightarrow{\psi_1^{\otimes d}} (\Sigma \Phi P_1)^{\otimes d} \cong \Sigma^d \Phi P_d$. Using the fact that Φ commutes naturally with tensor products, this induces a surjective morphism ψ_d in $\Phi P_d\text{-}\mathcal{U}$ as required. $\square$

Remark 2.4. The result has an obvious generalization to the odd prime case, taking P_d to be the polynomial algebra $\mathbb{F}[y_1, \dots, y_d]$ on generators of degree two (see [6]). For this, the exterior algebra has to be replaced by the truncated symmetric algebra

$\overline{S}^*$, where the quotient is taken by the ideal generated by the image of the Frobenius. Correspondingly, the top class lies in degree $2d(p-1)$ (note the grading places generators in degree two). Uniqueness is only up to non-zero scalar in $\mathbb{F}_p^\times$.

3. THE VERSHIEBUNG AND THE TRANSFER

The unstable algebra P_1 identifies with the polynomial algebra $\mathbb{F}[u]$ on a generator of degree one, with the unique unstable algebra structure. The localization $\mathbb{F}[u^{\pm 1}]$ has the structure of an algebra over the Steenrod algebra and of a module over $\mathbb{F}[u]$ in the category $\mathcal{M}$ of (graded) modules over the Steenrod algebra; the category of modules in $\mathcal{M}$ over $\mathbb{F}[u]$ is denoted $\mathbb{F}[u]-\mathcal{M}$. There is a subobject $\mathbb{F}[u]u^{-1} \subset \mathbb{F}[u^{\pm 1}]$ which is the $\mathbb{F}[u]$ -submodule generated by the class u^{-1} . The action of the Steenrod algebra is determined by the total Steenrod square $Sq^T(u^{-1})(t) = u^{-1}(1+tu)^{-1}$.

Observe that there is a canonical isomorphism of $\Phi\mathbb{F}[u]$ with $\mathbb{F}[u^2]$. The augmentation ideal of the Steenrod algebra will be denoted by $\overline{\mathcal{A}}$.

Lemma 3.1.

- (1) *There is a canonical surjective morphism in $\mathcal{M}$:*

$$\tau : \mathcal{A} \twoheadrightarrow \Sigma\mathbb{F}[u]u^{-1}.$$

This morphism induces a surjection $\bar{\tau} : \overline{\mathcal{A}} \rightarrow \Sigma\mathbb{F}[u]$ in $\mathcal{M}$.

- (2) *There is a commutative diagram in the category $\Phi\mathbb{F}[u]-\mathcal{M}$, in which the rows and columns are short exact sequences*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Phi\mathbb{F}[u] & \longrightarrow & \mathbb{F}[u] & \longrightarrow & \Sigma\Phi\mathbb{F}[u] \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Phi\mathbb{F}[u] & \longrightarrow & \mathbb{F}[u]u^{-1} & \longrightarrow & (\Phi\mathbb{F}[u])u^{-1} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \Sigma^{-1}\mathbb{F} & \xlongequal{\quad} & \Sigma^{-1}\mathbb{F} \end{array}$$

in which $(\Phi\mathbb{F}[u])u^{-1}$ is considered as a quotient $\Phi\mathbb{F}[u]$ -module of $\mathbb{F}[u]u^{-1}$.

- (3) *As an object of $\Phi\mathbb{F}[u]-\mathcal{M}$, $(\Phi\mathbb{F}[u])u^{-1} \cong \Sigma\Phi(\mathbb{F}[u]u^{-1})$.*

Proof. Straightforward. □

Notation 3.2. Let $\mathbb{V} : \mathcal{A} \rightarrow \Phi\mathcal{A}$ denote the Verschiebung morphism, which restricts to a morphism $\overline{\mathbb{V}} : \overline{\mathcal{A}} \rightarrow \Phi\overline{\mathcal{A}}$.

Proposition 3.3. (Cf. [6, Lemma 4.3]) *There is a commutative diagram in $\mathcal{M}$:*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\tau} & \Sigma\mathbb{F}[u]u^{-1} & \longleftarrow & \Sigma\mathbb{F}[u] \\ \downarrow \mathbb{V} & & \downarrow & & \downarrow \Sigma\psi_1 \\ & & \Sigma(\Phi\mathbb{F}[u])u^{-1} & & \\ & & \downarrow \cong & & \\ \Phi\mathcal{A} & \xrightarrow{\Phi\tau} & \Phi(\Sigma\mathbb{F}[u]u^{-1}) & \longleftarrow & \Sigma^2\Phi\mathbb{F}[u]. \end{array}$$

This induces a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{A}} & \xrightarrow{\bar{\tau}} & \Sigma\mathbb{F}[u] \\ \downarrow \overline{\mathbb{V}} & & \downarrow \Sigma\psi_1 \\ \Phi\overline{\mathcal{A}} & \xrightarrow{\Phi\bar{\tau}} & \Sigma^2\Phi\mathbb{F}[u]. \end{array}$$

Proof. Straightforward using the properties stated in Lemma 3.1. $\square$

Remark 3.4. From certain viewpoints on the Steenrod algebra, this result is tautological. Namely, the Verschiebung on the Steenrod algebra is *induced* by the Kameko morphism ψ_1 . The reader is encouraged to make sense of this statement by considering tensor products of $\mathbb{F}[u]u^{-1}$ as in the work of Mitchell and Priddy [7].

Remark 3.5. These results generalize to the odd prime case, following similar modifications as indicated for the definition of the Kameko morphism.

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