DERIVED MODULI OF SECTIONS AND PUSH-FORWARDS

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Abstract. We use the derived moduli of sections $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})$ to give derived enhancements of various moduli spaces, including stable maps and stable quasi-maps, which are compatible with their usual perfect obstruction theories. As an application, we prove that G-theoretic stable map and quasi-map invariants of projective spaces are equal.

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Introduction

Statements of the main results. This work suggests a new approach to virtual push-forward formulae via derived geometry. In the first part we explain how to obtain quasi-smooth derived enhancement for certain moduli spaces of curves.

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These enhancements encode both the classical and the virtual geometry of many well-studied moduli spaces, such as moduli of stable maps and quasi-maps (see Section 2); in the second part we use these derived moduli spaces to give a novel geometric proof of the following theorem.

Theorem 0.0.1. *Let g, n, r and d be positive integers and let $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ denote the derived moduli space of genus g, degree d stable maps, respectively quasi-maps to a projective space $\mathbb{P}^r$.*

(1) We have a derived morphism

$$\overline{c} : \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$$

and an isomorphism

$$\overline{c}_* \mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)} \simeq \mathcal{O}_{\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)} \text{ in } \mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathbb{R}\overline{\mathcal{Q}}(\mathbb{P}^r, d)).$$

(2) There is an equality of virtual structure sheaves

$$t_0(\overline{c})_* \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)} = \mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)} \text{ in } G_0(\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)). \tag{1}$$

Consequently, G-theoretic stable map and quasi-map invariants are the same (See Corollary 5.2.3).

The second part of the theorem generalises the already known cohomological result (see [CFK10], [MOP11, Theorem 3] and [Man12b, Proposition 5.19]):

$$t_0(\overline{c})_*[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} = [\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} \text{ in } A_*(\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)). \tag{2}$$

This shows that our statement is a categorification of (2).

The equality above is part of a family of results on wall-crossing formulae on moduli spaces of quasi-maps. In [CFK20] Ciocan-Fontanine and Kim provide a wall-crossing formula for complete intersections in projective spaces and Zhou generalises this to (certain) GIT quotients [Zho22]. Recently, Zhang and Zhou proved the analogue statement in G-theory [ZZ20]. These proofs rely on the construction of a clever master space and localisation on this space.

Our strategy is new, since endowing the moduli spaces with a derived structure allows us to give local geometric arguments while still carrying the information about the virtual structures of these spaces. We first construct a contraction morphism $\overline{c}$ (see Theorem 0.0.1.(1)) at the derived level. Then, it is enough to prove the isomorphism of the structure sheaves of the derived enhancement locally, which is easier. As the virtual sheaves $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)}$ and $\mathcal{O}^{\mathrm{vir}}_{\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)}$ are shadows (see (4)) of the structure sheaves $\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)}$ and $\mathcal{O}_{\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)}$, our result implies the G-theoretic statement. The main part of our local argument consists in constructing compatible derived atlases on the space of stable maps and on the space of quasi-maps.

The advantage over the classical situation is that local information can now be used to obtain global statements: rather than having external information based on choices of perfect obstruction theories, this data is now encoded in the geometry of the derived moduli spaces. We hope that having a derived enhancement of the quasimap moduli space will give a new perspective on wall-crossing and mirror symmetry.

In the following we introduce the moduli of sections (see Chang–Li [CL12, §2]), which is the central object of study in this paper. Consider an Artin stack $\mathfrak{M}$ with a flat, nodal, projective curve $\mathfrak{C}$ and a morphism of $\mathfrak{M}$-Artin stacks $\pi : \mathfrak{Z} \to \mathfrak{C}$. For any test scheme $S \to \mathfrak{M}$ the moduli of sections is defined as

$$\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})(S \to \mathfrak{M}) = \{f : C_S \to Z_S | \pi_S \circ f = id_{C_S}\},$$

for $\pi_S : Z_S := \mathfrak{Z} \times_{\mathfrak{M}} S \to C_S := \mathfrak{C} \times_{\mathfrak{M}} S$. In order to obtain a representable derived enhancement of this space, we additionally need to require that the Artin stack $\mathfrak{Z}$ is smooth relative to $\mathfrak{C}$.

If $\mathfrak{M} = \mathfrak{M}_{g,n}^{\mathrm{pre}}$ is the moduli space of genus g, n-pointed prestable curves and $\mathfrak{C}$ is its universal curve, we can take $\pi : \mathfrak{Z} \to \mathfrak{C}$ to be a trivial fibration $\mathfrak{Z} := \mathfrak{C} \times_{\mathfrak{M}} X$ where X is a smooth projective variety or DM stack. Then $\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times X/\mathfrak{C})$ is the moduli stack of prestable maps, containing as an open the usual moduli space of stable maps to X (See Example 1.1.3). For nontrivial fibrations, this construction recovers moduli of quasi-maps and twisted theories such as stable maps with fields (see examples in §1.2).

In §1, we recall a construction of Lurie [Lur18, 19.1] which gives a natural derived structure, denoted by $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})$, on the moduli of sections. Its relative tangent complex turns out to be compatible with the perfect obstruction theory defined by [CL12] and vastly generalized by [Web22] (see 1.2). For a precise statement, see Corollary 1.3.2.

The moduli space of curves on projective spaces (or more generally on toric DM stacks) admits various compactifications, which are all substacks of a common moduli of sections. As a map to the projective space $\mathbb{P}^r$ is a line bundle with sections, the (underived) moduli of quasi-maps to $\mathbb{P}^r$, denoted by $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$, and the moduli of stable maps, denoted by $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, are both open substacks of a moduli of sections over $\mathfrak{Pic} = \mathfrak{Pic}_{g,n,d}$ — the moduli space of line bundles over pre-stable curves. We thus obtain derived structures on the moduli space of stable maps and quasi-maps, denoted by $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ respectively. In [STV15] the authors define another derived structure on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, which is induced by a Hom-space over $\mathfrak{M}_{g,n}^{\mathrm{pre}}$. In §2, we prove that the derived structure in [STV15] and the derived structure described above are the same (see Theorem 2.3.2).

Outline of the paper.

In §1 we review the natural derived structure on the moduli of sections and its properties. We also investigate the case when $\mathfrak{Z}$ is a bundle (see Proposition 1.2.3).

In §2 we study in detail the cases of the moduli of stable maps and quasi-maps viewed inside the derived stacks of sections. We prove that the two derived structures – the one coming from the moduli of sections and the one from maps – are equivalent (see Theorem 2.3.2).

In §3 (see Theorem 3.3.10) we construct the derived morphisms

$$\overline{c} : \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}}), \tag{3}$$

where $\mathfrak{Pic}$ denotes the stack parametrising pre-stable curves together with a line bundle and $\widetilde{\mathfrak{Pic}}$ denotes the stack parametrising pre-stable curves without rational tails[1] together with a line bundle. Over $\mathfrak{Pic}$ we have a universal curve and a universal line bundle:

$$\mathfrak{L} \to \mathfrak{C} \to \mathfrak{Pic}.$$

Similarly, over $\widetilde{\mathfrak{Pic}}$ we have a universal family

$$\breve{\mathfrak{L}} \to \breve{\mathfrak{C}} \to \widetilde{\mathfrak{Pic}}.$$

The truncation of morphism (3) recovers the map

$$\overline{c} : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$$

defined in [CFK10], [MOP11, Theorem 3] and [Man12b, Proposition 5.19], which contracts rational tails.

In §4 we prove that the pushforward by $\overline{c}$ of the derived structure sheaf of the moduli space of stable maps is the structure sheaf of the space of quasi-maps (see Theorem 5.2.1). The main idea is to find compatible derived atlases on the two spaces. On $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ each chart consists of a triple (W, F, θ), where W is a smooth stack over $\mathfrak{Pic}$, F is a vector bundle over W and θ is a section of F such that locally $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \simeq Z^h(\theta)$. Here $Z^h(\theta)$ denotes the derived vanishing locus of θ. We construct a similar atlas for $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$.

Historical note on derived algebraic geometry applied to moduli spaces. Moduli spaces appearing in Gromov–Witten theory and, more broadly, in enumerative geometry, are usually singular and they may have irreducible components of different dimensions. To extract information about enumerative problems, such as various types of invariants, one needs to integrate over these moduli spaces. As such spaces do not carry a fundamental class of pure dimension, various techniques have been developed to construct an ersatz.

Historically, Li–Tian [LT98] and Behrend–Fantechi [BF97] have proposed solutions to the integration problem by introducing virtual cycles, which allowed cohomological Gromov–Witten invariants to be formally mathematically defined. Using similar techniques, Lee [Lee04] constructed a virtual structure sheaf, which is key in defining K-theoretical (or in fact G-theoretical) invariants. These constructions formalize the objects used by Kontsevich in [Kon95]. The definitions of these virtual objects are not intrinsic; rather, they depend on the choice of a replacement for the cotangent complex of the singular moduli space. The unworkable cotangent

[1]Rational tails are trees of $\mathbb{P}^1$ that do not have marked point. See Definition 3.3.4 for details.

complex is replaced locally by a 2-term complex of vector bundles: this is the perfect obstruction theory. For many moduli spaces, the choice of this replacements comes from the geometry of the original moduli problem.

In the seminal paper [Kon95], Kontsevich proposed a different approach to solve this problem via the notion of differential graded manifolds (or schemes), in short, *dg-manifolds*. This idea was developed by Kapranov and Ciocan-Fontanine in [CFK01] and[CFK02].

In [STV15], Schürg-Toën-Vezzosi use *derived algebraic geometry* to give a more geometric interpretation of these virtual objects. This idea is one of the numerous applications of the field derived algebraic geometry developed by Toën–Vezzosi (eg. [TV05] and [TV08], see [Toe14, §3.1] for a nice overview) and by Lurie in [Lur18]. The derived and *dg* approach are related, but they are not equivalent (see [Toe14] for the difference).

On the side of differential geometry, Joyce has developed parallel theories of d-manifolds and d-orbifolds and closely related theories of Kuranishi spaces (see [Joy14] for a summary of d-manifolds, [Joy19] for Kuranishi spaces). Central to the study of moduli spaces are the ideas of derived critical loci [Vez20], studied by Vezzosi, and the parallel concept of algebraic d-critical loci introduced by Joyce [Joy15], as well as those of shifted symplectic structures [PTVV13] of Pantev–Toën–Vaquié–Vezzosi, applied to the study of Donaldson–Thomas invariants by Brav–Bussi–Joyce [BBJ19]. Nowadays, many works use derived algebraic geometry to study moduli spaces amongst them we recall [MR18] [Ker20] [PY20] [BZCG$^+$21] [Kha19] [AP19] [AKL$^+$22] [Kha21] [MTFJ19], [JS19]. Just as perfect obstruction theories, derived structures on a scheme (or stacks) are not unique: they depend on a choice. In many cases there are natural ones coming from the geometry.

Virtual structure sheaves via derived algebraic geometry. In this paper, we use derived algebraic geometry to study the moduli space of sections. In the following we sketch the way in which derived algebraic geometry recovers virtual objects. For a derived stack $\mathbb{R}\mathfrak{X}$, its truncation $t_0(\mathbb{R}\mathfrak{X}) = \mathfrak{X}$ has a closed embedding or *derived enhancement*:

$$j : \mathfrak{X} \hookrightarrow \mathbb{R}\mathfrak{X}.$$

Informally, $\mathbb{R}\mathfrak{X}$ and $\mathfrak{X}$ have the same underlying geometric space, but the derived stack is akin to a nilpotent thickening. If the derived stack $\mathbb{R}\mathfrak{X}$ is quasi-smooth, that is its cotangent complex is cohomologically supported in $(-1, \infty]$, its structural sheaf $\mathcal{O}_{\mathbb{R}\mathfrak{X}}$ has only finite cohomology. We can define a sheaf class on $\mathfrak{X}$ via

$$\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir,DAG}} := (j_*)^{-1}\mathcal{O}_{\mathbb{R}\mathfrak{X}} \in G_0(\mathfrak{X}), \tag{4}$$

where j_* is the induced map between G-theory groups, which by dévissage is invertible.

On the other hand, the derived enhancement gives a perfect obstruction theory for $\mathfrak{X}$, as long as $\mathbb{R}\mathfrak{X}$ is quasi-smooth and $\mathfrak{X}$ is a Deligne–Mumford stack. The differential of the inclusion j gives a morphism

$$dj : j^*\mathbb{L}_{\mathbb{R}\mathfrak{X}} \to \mathbb{L}_{\mathfrak{X}},$$

which, under our assumptions, is a perfect obstruction theory [STV15, Proposition 1.2]. Using this perfect obstruction theory, we can follow the recipe of Lee [Lee04] to construct a virtual sheaf $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir,POT}}$ for $\mathfrak{X}$. We get an a priori different sheaf on $\mathfrak{X}$. The equality of $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir,POT}}$ and $\mathcal{O}_{\mathfrak{X}}^{\mathrm{vir,DAG}}$ in the G-theory of $\mathfrak{X}$ is a deep statement, which was proved in [BZCG+21, MR, §5.4 and §5.5] (see also [PY20, §6]).

Historical note on quasi-map wall-crossing. The moduli of quasi-maps for a toric variety was defined in 2010 by Ciocan-Fontanine and Kim in [CFK10]. These moduli spaces carry enumerative information closely related to the one of moduli of stable maps, but with an easier geometry. Previous related spaces appeared in [MM07],[MOP11]. The definition of stable quasi-maps was generalised by many authors in [Tod11], [CFKM14], [CCFK15]. One of the main uses of quasimaps is in mirror symmetry (see [GT14], [Giv15a], [Giv15b], [RZ18], [CJR21], [ZZ20]). A nice overview of the quasi-map theory can be found in [CFK14a].

Wall-crossing between quasi-map spaces has been extensively studied (see [CFK20], [CFK14b], [CCFK15], [TY16], [CJR17], [CJR21],[Zho22]). Wall-crossing is trivial for sufficiently Fano varieties and non-trivial in the non-Fano case. This translates into no-mirror transformation in the first case and a non-trivial mirror transformation in the second. In the case of Grasmannians it is easy to obtain a statement using virtual push-forwards, giving a good geometric understanding. For the more general case, we do not have an analogous proof of the wall-crossing formula in [CFK14b]. We hope that derived geometry will shed light on this case.

Further directions. We believe that our main theorem is part of a new strategy to prove equalities between virtual objects. The strategy is:

(1) to construct a morphism at the derived level so that we have a morphism between virtual structure sheaves, and
(2) to prove locally that this morphism is an isomorphism.

For more general statements, one needs to develop a more general machinery: we expect situations in which we have a simple virtual push-forward theorem, but a more complicated relation between derived structure sheaves.

In terms of applications of such a machinery, it is natural to consider stable maps and quasi-maps to a general toric variety X and to try to derive a relation between (derived) structure sheaves. This is not straight-forward, as for a general X there is no morphism

$$\overline{c} : \mathbb{R}\overline{\mathcal{M}}_{g,n}(X, d) \dashrightarrow \mathbb{R}\overline{\mathcal{Q}}_{g,n}(X, d).$$

On the other hand, it is possible to get an easy local picture.

We will treat these problems in future works.

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Notation

- Everything is over $\mathbb{C}$.
- For locally free sheaf $\mathcal{E}$ on a space X (a scheme, stack, or derived stack) the vector bundle of $\mathcal{E}$ is $\mathbb{V}(\mathcal{E}) := \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}^\vee$.
- Let $\mathfrak{M}$ be an Artin stack with a flat proper family $\pi : \mathfrak{C} \to \mathfrak{M}$ of relative dimension 1. For any morphism $\mathfrak{U} \to \mathfrak{M}$, we denote $\pi_{\mathfrak{U}} : \mathfrak{C}_{\mathfrak{U}} \to \mathfrak{U}$ the pullback of $(\pi, \mathfrak{C})$. The most classical example would be $\mathfrak{M}$ being the moduli of prestable curves, denoted by $\mathfrak{M}_{g,n}$ of genus g with n marked points and $\mathfrak{C}_{g,n}$ its universal curve.
- We use $\mathbb{R}$ to mean a derived structure on a geometric object (for example $\mathbb{R}X$), and $\mathbf{R}$ (respectively $\mathbf{L}$) a right (resp. left) derived functor, for example $\mathbf{R}f_*$ (resp $\mathbf{L}f^*$).
- For X, Y, Z non-derived stacks $\underline{\mathrm{Hom}}_X(Y, Z)$ are Hom-stacks (relative internal hom) whereas $\mathrm{Hom}_X(Y, Z)$ are groupoids.
- For X, Y, Z derived (or non-derived) stacks $\mathbb{R}\underline{\mathrm{Hom}}_X(Y, Z)$ are derived Hom-stacks whereas $\mathbb{R}\mathrm{Hom}_X(Y, Z)$ are simplicial sets.
- For X a non-derived stack, $\mathcal{F}$, $\mathcal{G}$ sheaves on X, $\mathrm{Hom}_{\mathcal{O}_X-\mathrm{mod}}(\mathcal{F}, \mathcal{G})$ is the global Hom of sheaves. For X a derived stack and $\mathcal{F}$, $\mathcal{G}$ complexes of sheaves, $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X-dgm}(\mathcal{F}, \mathcal{G})$ denotes the simplicial set associated by the Dold–Kan correspondence to the complex $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G})$ defined as $\mathrm{Hom}^i(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}^0(\mathcal{F}, \mathcal{G}[i])$.
- $\mathfrak{Pic}_{g,n,d}$ (or $\mathfrak{Pic}$ for short) is the moduli space of prestable curves of genus g with n marked points together with a degree d line bundle, more formally, $\mathfrak{Pic}_{g,n,d} := \underline{\mathrm{Hom}}_{\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, B\mathbb{G}_m \times \mathfrak{M}_{g,n})$. When we impose some stability conditions, we will write $\mathfrak{Pic}^{\mathrm{s}}$ (see Notation §3.1).
- $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ are the (derived) moduli of stable maps of genus g with n marked points to projective space $\mathbb{P}^r$.
- $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ and $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ are the (derived) moduli of quasi-maps of genus g with n marked points to projective space $\mathbb{P}^r$.
- We use the notations $\mathfrak{M}_{g,n}, \mathfrak{C}_{g,n}, \mathfrak{Pic}, \mathfrak{Pic}^{\mathrm{s}}, \mathbb{R}U, \mathbb{R}V, W, \mathcal{W}, ...$ for all the objects related to stable maps (for example prestable curve), that is objects where the curve could have rational tails. We put a "check" on the same kind of objects $\breve{\mathfrak{M}}_{g,n}, \breve{\mathfrak{C}}_{g,n}, \breve{\mathfrak{Pic}}, \breve{\mathfrak{Pic}}^{\mathrm{s}}, \mathbb{R}\breve{U}, \mathbb{R}\breve{V}, \breve{W}, \breve{\mathcal{W}}, ...$ for all the objects related to quasi-maps, that is without rational tails.

1. Background on the derived moduli of sections

In this section, we recall the definition and basic properties of the derived moduli of sections, also known as Weil restriction, constructed by Lurie in [Lur18, 19.1]).

1.1. **Derived structure of the moduli of sections.** Let $\mathfrak{M}$ be a (possibly derived) Artin stack, $\pi : \mathfrak{C} \to \mathfrak{M}$ a flat, proper morphism of relative dimension 1. Let $\mathfrak{Z}$ be a (possibly derived) Artin stack with a smooth morphism $p : \mathfrak{Z} \to \mathfrak{C}$. We have an ∞-functor π_* called the Weil restriction of scalars, right adjoint to the base-change ∞-functor π^* (and constructed for example in [Lur18, Construction 19.1.2.3]), that will be seen to preserve derived Artin stacks of locally finite presentation as stated in [TV22]:

$$\mathbf{dSt}/\mathfrak{M} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathbf{dSt}/\mathfrak{C}.$$

Definition 1.1.1. [Lur18, §19.1] For a derived Artin stack $\mathfrak{Z} \xrightarrow{p} \mathfrak{C}$, we denote

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C}) \coloneqq \pi_*\mathfrak{Z}.$$

Proposition 1.1.2. [Lur18, §19.1]

(1) The derived moduli of sections $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})$ is the homotopical cartesian product

$$\begin{array}{ccc} \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C}) & \longrightarrow & \mathfrak{M} \\ \downarrow & h & \downarrow i \\ \mathbb{R}\underline{\mathrm{Hom}}_{\mathfrak{M}}(\mathfrak{C},\mathfrak{Z}) & \xrightarrow{q} & \mathbb{R}\underline{\mathrm{Hom}}_{\mathfrak{M}}(\mathfrak{C},\mathfrak{C}). \end{array} \tag{5}$$

where q is induced by composition by $p : \mathfrak{Z} \to \mathfrak{C}$ and i is given by the identity morphism.

(2) If $\mathfrak{Z} \to \mathfrak{M}$ is a locally almost finitely presented (relative) 1-Artin derived stack with quasi-affine diagonal, then $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C}) \to \mathfrak{M}$ is a locally almost finitely presented 1-Artin derived stack, with quasi-affine diagonal.

(3) If $\mathfrak{Z}, \mathfrak{C}, \mathfrak{M}$ are classical (non derived) stacks, the truncation

$$\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C}) \coloneqq t_0\left(\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})\right)$$

is given by the functor $\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C}) : (\mathbf{Sch}/\mathfrak{M})^{op} \to \mathbf{Gpoid}$ taking sections of $\mathfrak{Z}$ over $\mathfrak{C}$, that is:

$$\begin{aligned} \underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})(T \to \mathfrak{M}) &= \{s : C_T \coloneqq T \times_{\mathfrak{M}} \mathfrak{C} \to Z_T \coloneqq C_T \times_{\mathfrak{C}} \mathfrak{Z} | p_T \circ s = \mathrm{id}_{C_T}\} \\ &= \mathrm{Hom}_{C_T}(C_T, Z_T) \end{aligned}$$

where $p_T \colon Z_T \to C_T$ is the projection induced by p.

Example 1.1.3 (Moduli of stable maps). Let $\mathfrak{C} \xrightarrow{\pi} \mathfrak{M}$ be the moduli space of prestable genus g, n-pointed curves with its universal family. Let $\mathfrak{Z} = \mathfrak{C} \times X$ for a smooth projective variety X. Then

$$\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times X/\mathfrak{C}) = \underline{\mathrm{Hom}}_{\mathfrak{M}}(\mathfrak{C}, \mathfrak{M} \times X).$$

For any choice of effective class β, the moduli space $\overline{\mathcal{M}}(X,\beta)$ of stable maps to X is then an open substack of the moduli of sections $\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times X/\mathfrak{C})$. Similarly,

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times X/\mathfrak{C}) = \mathbb{R}\underline{\mathrm{Hom}}_{\mathfrak{M}}(\mathfrak{C}, \mathfrak{M} \times X).$$

The usual derived enhancement of the moduli of stable maps [STV15, Section 2], denoted $\mathbb{R}\overline{\mathcal{M}}(X,\beta)$, is the unique derived structure on $\overline{\mathcal{M}}(X,\beta)$ which makes the following diagram homotopy Cartesian

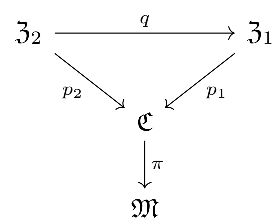
$$\begin{array}{ccc} \overline{\mathcal{M}}(X,\beta) & \hookrightarrow & \mathbb{R}\overline{\mathcal{M}}(X,\beta) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times X/\mathfrak{C}) & \hookrightarrow & \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times X/\mathfrak{C}). \end{array}$$

Remark 1.1.4. The stack $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})$ is in general a derived stack even if $(\mathfrak{M}, \mathfrak{C}, \mathfrak{Z})$ is a triple of classical stacks.

We also record here the following functoriality result, which we will use in §2. If we have stacks $\mathfrak{Z}_2 \to \mathfrak{Z}_1 \to \mathfrak{C} \to \mathfrak{M}$, we can take sections of $\mathfrak{Z}_2 \to \mathfrak{C}$ by passing through sections of $\mathfrak{Z}_1 \to \mathfrak{C}$ first.

Proposition 1.1.5. *Consider*

(6)



with $\mathfrak{Z}_1, \mathfrak{Z}_2$ *as in Definition 1.1.1. For* $i \in \{1, 2\}$*, we form the moduli of sections*

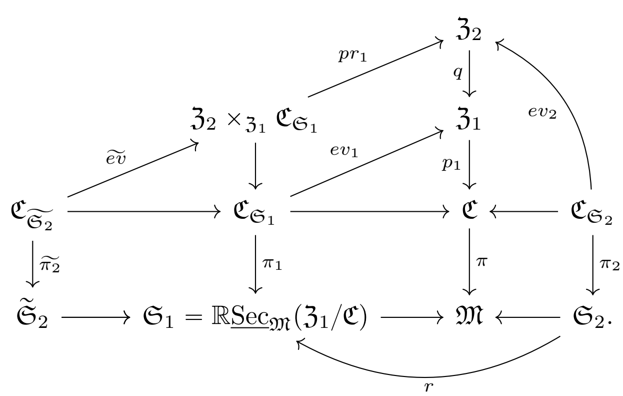
$$\mathfrak{S}_i := \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}_i/\mathfrak{C}) \to \mathfrak{M}$$

with their universal curves $\pi_i : \mathfrak{C}_{\mathfrak{S}_i} \to \mathfrak{S}_i$ *and evaluations* $\mathrm{ev}_i : \mathfrak{C}_{\mathfrak{S}_i} \to \mathfrak{Z}_i$*. We can also form the moduli of sections over* $\mathfrak{S}_1$ *of the morphism* q*:*

$$\widetilde{\mathfrak{S}}_2 := \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{S}_1}(\mathfrak{C}_{\mathfrak{S}_1} \times^h_{\mathfrak{Z}_1} \mathfrak{Z}_2/\mathfrak{C}_{\mathfrak{S}_1}).$$

Then $\mathfrak{S}_2$ *and* $\widetilde{\mathfrak{S}}_2$ *are derived equivalent as derived stacks over* $\mathfrak{M}$*.*

Proof. To fix ideas and notation, consider the following diagram



The composition $pr_1 \circ \widetilde{ev}$ gives an evaluation map $\mathfrak{C}_{\widetilde{\mathfrak{S}}_2} \to \mathfrak{Z}_2$ over $\mathfrak{C}$, which in turn defines a morphism $f : \widetilde{\mathfrak{S}}_2 \to \mathfrak{S}_2$. For simplicity, we consider f a morphism over $\mathfrak{S}_1$, where the map $r : \mathfrak{S}_2 \to \mathfrak{S}_1$ is that corresponding to the evaluation $ev_2 \circ q$. The classical truncation of this morphism is an isomorphism, as proved in [CJW21, Lemma A.1.2]. Moreover, the differential of f induces an equivalence of tangent complexes (we anticipate here the formulae of 1.3). We have a distinguished triangle

$$\mathbb{T}_{\mathfrak{S}_2/\mathfrak{S}_1} \to \mathbb{T}_{\mathfrak{S}_2/\mathfrak{M}} = R^\bullet\pi_{2,*}ev_2^*\mathbb{T}_{\mathfrak{Z}_2/\mathfrak{C}} \to r^*\mathbb{T}_{\mathfrak{S}_1/\mathfrak{M}} = R^\bullet\pi_{2,*}ev_2^*q^*\mathbb{T}_{\mathfrak{Z}_1/\mathfrak{C}}$$

which we can use to identify $\mathbb{T}_{\mathfrak{S}_2/\mathfrak{S}_1}$ with $R^\bullet\pi_{2,*}ev_2^*\mathbb{T}_{\mathfrak{Z}_2/\mathfrak{Z}_1}$. On the other hand,

$$\mathbb{T}_{\widetilde{\mathfrak{S}}_2/\mathfrak{S}_1} = R^\bullet\widetilde{\pi}_{2,*}\widetilde{ev}^*\mathbb{T}_{\mathfrak{Z}_2\times^h_{\mathfrak{Z}_1}\mathfrak{C}_{\mathfrak{S}_1}/\mathfrak{C}_{\mathfrak{S}_1}} = R^\bullet\widetilde{\pi}_{2,*}\widetilde{ev}^*pr_1^*\mathbb{T}_{\mathfrak{Z}_2/\mathfrak{Z}_1}$$

So the result follows by identifying $r \circ f : \widetilde{\mathfrak{S}}_2 \to \mathfrak{S}_1$ with the structure morphism of $\widetilde{\mathfrak{S}}_2 = \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{S}_1}(\mathfrak{C}_{\mathfrak{S}_1} \times^h_{\mathfrak{Z}_1} \mathfrak{Z}_2/\mathfrak{C}_{\mathfrak{S}_1}) \to \mathfrak{S}_1$, which is simply the observation that the square

$$\begin{array}{ccc} \mathfrak{Z}_2 & \xrightarrow{q} & \mathfrak{Z}_1 \\ {\scriptstyle pr_1 \circ \widetilde{ev}} \uparrow & & \uparrow {\scriptstyle ev_1} \\ \mathfrak{C}_{\widetilde{\mathfrak{S}}_2} & \longrightarrow & \mathfrak{C}_{\mathfrak{S}_1} \end{array}$$

of the big diagram is commutative. □

1.2. **The linear case: $\mathfrak{Z}$ is a vector bundle.** We will now consider the important special case where $\mathfrak{Z}$ is a linear stack. As we will see, this case covers many disparate constructions: moduli of stable maps to projective spaces (see Section 2.3), and more generally to varieties which are GIT quotients by linear groups, as well as moduli spaces of quasi-maps (see Section 2.4) and moduli of stable maps with fields (Example 1.2.5). In this case, the derived moduli of sections is an affine stack over its base.

We start with a review of the classical (non derived) construction. Let $\mathfrak{Z} = \mathbb{V}(\mathcal{E}) := \mathrm{Spec}_{\mathfrak{C}}\mathrm{Sym}(\mathcal{E}^\vee)$ for $\mathcal{E}$ a locally-free sheaf over $\mathfrak{C}$. As proved in [CL12], sections of $\mathbb{V}(\mathcal{E})$ over $\mathfrak{M}$ are an affine scheme, in fact an abelian cone:

$$\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathbb{V}(\mathcal{E})/\mathfrak{C}) = \mathrm{Spec}_{\mathfrak{M}}\mathrm{Sym}(\mathbf{R}^1\pi_*\mathcal{E}^\vee \otimes \omega_\pi).$$

Indeed, let $f : T \to \mathfrak{M}$ and $\widehat{f} : C_T \to \mathfrak{C}$, by Serre's duality and flat base change we have

$$\begin{aligned} \underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathbb{V}(\mathcal{E})/\mathfrak{C})(T \to \mathfrak{M}) &= \mathrm{Hom}_{C_T}(C_T, \widehat{f}^*\mathcal{E}) \\ &= \mathrm{Hom}_{\mathcal{O}_T-\mathrm{mod}}(\mathbf{R}^1\pi_{T*}\widehat{f}^*\mathcal{E}^\vee \otimes \omega_{\pi_T}, \mathcal{O}_T) \\ &= \mathrm{Hom}_{\mathcal{O}_T-\mathrm{mod}}(\mathbf{R}^1\pi_{T*}\widehat{f}^*(\mathcal{E}^\vee \otimes \omega_\pi), \mathcal{O}_T) \\ &= \mathrm{Hom}_{\mathcal{O}_T-\mathrm{mod}}(f^*\mathbf{R}^1\pi_*\mathcal{E}^\vee \otimes \omega_\pi, \mathcal{O}_T) \\ &= \mathrm{Spec}_{\mathfrak{M}}\mathrm{Sym}(\mathbf{R}^1\pi_*\mathcal{E}^\vee \otimes \omega_\pi)(T \to \mathfrak{M}). \end{aligned}$$

Example 1.2.1 (Hodge bundle). For $\mathfrak{M} = \mathfrak{M}^{\mathrm{pre}}_{g,n}$, the moduli of pre-stable curves, the Hodge bundle $\mathfrak{H}$ is the cone of sections

$$\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathbb{V}(\omega_\pi)/\mathfrak{C}) = \mathrm{Spec}_{\mathfrak{M}}\mathrm{Sym}(\mathbf{R}^1\pi_*\mathcal{O}_{\mathfrak{C}}).$$

This is a vector bundle of rank g, since $\mathbf{R}^0\pi_*\mathcal{O}_{\mathfrak{C}} \cong \mathcal{O}_{\mathfrak{M}}$.

Example 1.2.2 (Stable maps with fields). Let $\mathfrak{X} = \overline{\mathcal{M}}_{g,n}(X, \beta)$ with its universal family

$$(\pi_{\mathfrak{X}}, \mathrm{ev}_{\mathfrak{X}}) : \mathfrak{C}_{\mathfrak{X}} \to \mathfrak{X} \times X.$$

Let $\mathcal{E}$ be a locally-free sheaf over X. The moduli space of stable maps with fields in $E = \mathbb{V}(\mathcal{E}) \to X$ (see [CL12, CJW21, Pic21]) denoted $\mathfrak{X}^E$ can be seen as

$$\mathfrak{X}^E = \underline{\mathrm{Sec}}_{\mathfrak{X}}(\mathbb{V}(\mathrm{ev}^*_{\mathfrak{X}}\mathcal{E}^\vee \otimes \omega_{\mathfrak{C}_{\mathfrak{X}}/\mathfrak{X}})/\mathfrak{C}_{\mathfrak{X}}) = \mathrm{Spec}_{\mathfrak{X}}\mathrm{Sym}(\mathbf{R}^1\pi_{\mathfrak{X}*}\mathrm{ev}^*_{\mathfrak{X}}\mathcal{E}).$$

We will now cover the general case where $\mathfrak{Z}$ is a derived vector bundle, that is

$$\mathfrak{Z} = \mathbb{V}(\mathcal{E}) := \mathbb{R}\mathrm{Spec}_{\mathfrak{C}}\left(\mathrm{Sym}(\mathcal{E}^\vee)\right)$$

for $\mathcal{E} \in \mathrm{Perf}^{\geqslant 0}(\mathcal{O}_{\mathfrak{C}})$.

In this specific case, the derived space of section is itself a derived vector bundle.

Proposition 1.2.3. *Let $\mathfrak{Z} = \mathbb{V}(\mathcal{E})$ for $\mathcal{E}$ as above. Then*

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathbb{V}(\mathcal{E})/\mathfrak{C}) = \mathbb{R}\mathrm{Spec}_{\mathfrak{M}}\mathrm{Sym}((\mathbf{R}\pi_{\mathfrak{X}*}\mathcal{E})^\vee).$$

Proof. Let $f : T = \mathbb{R}\mathrm{Spec}A_\bullet \to \mathfrak{M}$ be an affine derived scheme over $\mathfrak{M}$ with $\widehat{f} : C_T := T \times^h_{\mathfrak{M}} \mathfrak{C} \to \mathfrak{C}$ the induced map and $\pi_T : C_T \to T$ the induced projection. From Definition 1.1.1, we have

$$\begin{aligned}\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathbb{V}(\mathcal{E})/\mathfrak{C})(T \to \mathfrak{M}) &= \mathbb{R}\mathrm{Hom}_T(C_T, \mathbb{V}(\mathbf{L}\widehat{f}^*\mathcal{E})) \times^h_{\mathbb{R}\mathrm{Hom}_T(C_T,C_T)} T \\ &= \mathbb{R}\mathrm{Hom}_{C_T}(C_T, \mathbb{V}(\mathbf{L}\widehat{f}^*\mathcal{E})) \\ &= \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{C_T}-\mathrm{dgm}}(\mathcal{O}_{C_T}, \mathbf{L}\widehat{f}^*\mathcal{E}).\end{aligned}$$

The second line follows by [Lur09, 5.5.5.12]. On the other hand,

$$\begin{aligned}\mathbb{R}\mathrm{Spec}_{\mathfrak{X}}\mathrm{Sym}((\mathbf{R}\pi_{\mathfrak{X}*}\mathcal{E})^\vee)(T \to \mathfrak{M}) &= \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{\mathfrak{M}}-\mathrm{cdga}}(\mathrm{Sym}((\mathbf{R}\pi_{\mathfrak{X}*}\mathcal{E})^\vee, \mathbf{R}f_*\mathcal{O}_T) \\ &= \mathbf{R}\mathrm{Hom}_{\mathcal{O}_T-\mathrm{dgm}}((\mathbf{L}f^*R\pi_{\mathfrak{X}*}\mathcal{E})^\vee, \mathcal{O}_T).\end{aligned}$$

By flat base-change,

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_T-\mathrm{dgm}}((\mathbf{L}f^*\mathbf{R}\pi_{\mathfrak{X}*}\mathcal{E})^\vee, \mathcal{O}_T) = \mathbf{R}\mathrm{Hom}_{\mathcal{O}_T-\mathrm{dgm}}((\mathbf{R}\pi_{T*}\mathbf{L}\widehat{f}^*\mathcal{E})^\vee, \mathcal{O}_T).$$

By the sheafified Grothendieck duality statement of [Nee10, Corollary 4.4.2],

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\mathcal{O}_T-\mathrm{dgm}}((\mathbf{R}\pi_{T*}\mathbf{L}\widehat{f}^*\mathcal{E})^\vee, \mathcal{O}_T) &= \mathbf{R}\mathrm{Hom}_{\mathcal{O}_T-\mathrm{dgm}}(\mathbf{R}\pi_{T*}\mathbf{R}\mathcal{H}om_{C_T}(\mathbf{L}\widehat{f}^*\mathcal{E}, \omega_{\pi_T}), \mathcal{O}_T) \\ &= \mathbf{R}\mathrm{Hom}_{\mathcal{O}_T-\mathrm{dgm}}(\mathbf{R}\pi_{T*}(\mathbf{L}\widehat{f}^*\mathcal{E}^\vee \otimes \omega_{\pi_T}), \mathcal{O}_T).\end{aligned}$$

By the global duality statement of [Nee10, Theorem 4.1.1],

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_T-\mathrm{dgm}}(\mathbf{R}\pi_{T*}(\mathbf{L}\widehat{f}^*\mathcal{E}^\vee \otimes \omega_{\pi_T}), \mathcal{O}_T) = \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{C_T}-\mathrm{dgm}}(\mathbf{L}\widehat{f}^*\mathcal{E}^\vee \otimes \omega_{\pi_T}, \pi_T^!\mathcal{O}_T).$$

So finally,

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\mathcal{O}_{C_T}-\mathrm{dgm}}(\mathbf{L}\widehat{f}^*\mathcal{E}^\vee \otimes \omega_{\pi_T}, \pi_T^!\mathcal{O}_T) &= \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{C_T}-\mathrm{dgm}}(\mathbf{L}\widehat{f}^*\mathcal{E}^\vee \otimes \omega_{\pi_T}, \omega_{\pi_T}) \\ &= \mathbf{R}\mathrm{Hom}_{\mathcal{O}_{C_T}-\mathrm{dgm}}(\mathcal{O}_{C_T}, \mathbf{L}\widehat{f}^*\mathcal{E}).\end{aligned}$$

□

Example 1.2.4 (Derived Hodge bundle). The derived version of the Hodge bundle of Example 1.2.1 is

$$\mathbb{R}\mathfrak{H} = \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathbb{V}(\omega_\pi)/\mathfrak{C}).$$

In Theorem 5.4.2 [BZCG+21] (see also [PY20] §8.1), we have a deformation from this derived bundle to

$$\mathfrak{H} \times_{\mathfrak{M}} \mathbb{A}^1_{\mathfrak{M}}[-1]$$

The latter consists of the usual Hodge bundle in degree 0 and a trivial line bundle in degree 1.

Example 1.2.5 (Derived stable maps with fields). Keeping the notation from Example 1.2.2, we define the derived version of the moduli space of stable maps with fields. We have from Example 1.1.3 a derived enhancement of the moduli of stable maps to X, $\mathbb{R}\mathfrak{X} := \mathbb{R}\overline{\mathcal{M}}_{g,n}$ with a universal family $\pi_{\mathbb{R}\mathfrak{X}}, \mathrm{ev}_{\mathbb{R}\mathfrak{X}} : \mathfrak{C}_{\mathbb{R}\mathfrak{X}} \to \mathbb{R}\mathfrak{X} \times X$. The derived enhancement of the moduli of stable maps can be constructed as

$$\mathbb{R}\mathfrak{X}^E = \mathbb{R}\underline{\mathrm{Sec}}_{\mathbb{R}\mathfrak{X}}(\mathbb{V}(ev^*_{\mathbb{R}\mathfrak{X}}\mathcal{E}^\vee \otimes \omega_{\mathfrak{C}_{\mathbb{R}\mathfrak{X}}/\mathbb{R}\mathfrak{X}})/\mathfrak{C}_{\mathbb{R}\mathfrak{X}}) = \mathbb{R}\mathrm{Spec}_{\mathbb{R}\mathfrak{X}}\mathrm{Sym}(\mathbf{R}\pi_{\mathbb{R}\mathfrak{X}_*}ev^*_{\mathbb{R}\mathfrak{X}}\mathcal{E}[1]),$$

the second equality coming from Proposition 1.2.3 and Grothendieck duality.

Remark 1.2.6. The formation of the derived definition of the moduli of sections commutes with flat base-change, so for a flat morphism $\mathfrak{U} \to \mathfrak{M}$ we have

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{U}}(\mathbb{V}(\mathcal{E}_{\mathfrak{U}})/\mathfrak{C}_{\mathfrak{U}}) \simeq \mathfrak{U} \times_{\mathfrak{M}} \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathbb{V}(\mathcal{E})/\mathfrak{C}).$$

1.3. **Tangent complex and perfect obstruction theory.** Recall that for a derived Hom-stack $H := \mathbb{R}\underline{\mathrm{Hom}}_X(Y, Z)$ we have a universal family

$$\begin{array}{ccc} H \times^h_X Y & \xrightarrow{\mathrm{ev}_H} & Z \\ \Big\downarrow{\scriptstyle \pi_H} & & \\ H & & \end{array}$$

and the relative tangent complex $\mathbb{T}_{H/X}$ is given by the following simple expression (see [CFK02, Thm 5.4.8] or [STV15, p.13] or the proof of [MR18, Prop.4.3.1] or [CHS22, Proposition B.10.21]):

$$\mathbb{T}_{H/X} = \mathbf{R}\pi_{H*}\mathbf{L}\mathrm{ev}_H^*\mathbb{T}_{Z/X}. \tag{7}$$

Applying this fact to the diagram in Definition 1.1.1 allows us to compute $\mathbb{T}_{\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})/\mathfrak{M}}$. The cotangent complex of a Weil restriction is also computed in [Lur18, §19.1.4].

Theorem 1.3.1. [Lur18, §19.1.4] *Let* $\mathbb{R}\mathfrak{S} := \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})$*, as per our convention we have* $\pi_{\mathbb{R}\mathfrak{S}} \colon \mathfrak{C}_{\mathbb{R}\mathfrak{S}} = \mathbb{R}\mathfrak{S} \times^h_{\mathfrak{M}} \mathfrak{C} \to \mathbb{R}\mathfrak{S}$ *and* $\mathrm{ev}_{\mathbb{R}\mathfrak{S}} \colon \mathfrak{C}_{\mathbb{R}\mathfrak{S}} \to \mathfrak{Z}$.

$$\mathbb{T}_{\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C})/\mathfrak{M}} = \mathbf{R}\pi_{\mathbb{R}\mathfrak{S}*}\mathbf{L}\mathrm{ev}_{\mathbb{R}\mathfrak{S}}^*\mathbb{T}_{\mathfrak{Z}/\mathfrak{C}}.$$

Using the well-established relationship between quasi-smooth derived enhancements and perfect obstruction theories, we obtain the following.

Corollary 1.3.2. *(c.f.* [STV15, §2.2]*) If* $\mathfrak{Z} \to \mathfrak{C}$ *is a smooth Deligne–Mumford (not derived) stack,*

$$\mathfrak{S} := \underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C}) = t_0(\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{Z}/\mathfrak{C}))$$

has a relative perfect obstruction theory in the sense of [BF97] *given by*

$$\mathbb{T}_{\mathfrak{S}/\mathfrak{M}} \to \mathbb{E}_{\mathfrak{S}/\mathfrak{M}} := \mathbf{R}\pi_{\mathfrak{S}*}\mathrm{ev}_{\mathfrak{S}}^* T_{\mathfrak{Z}/\mathfrak{C}}.$$

2. Derived structure on stable maps and quasi-maps

There are several ways of constructing derived moduli spaces of maps to a quotient. The rest of the paper is concerned with maps to projective space $\mathbb{P}^r$. Below we describe the construction of the stacks of prestable curves with line bundles, stable maps and quasi-maps to projective spaces as particular cases of the moduli space of sections.

2.1. **Background on the classifying stack** $B\mathbb{G}_m$**.** We first recall the following algebro-geometric description of the classifying space of line bundles, which is representable by a smooth algebraic stack of locally finite type by [LMB00]. The classifying stack of line bundles, or equivalently $\mathbb{G}_m$-torsors, is the quotient stack

$B\mathbb{G}_m = [\bullet/\mathbb{G}_m]$. By definition a morphism $T \to B\mathbb{G}_m$ is given by Cartesian diagram

$$\begin{array}{ccc} P & \longrightarrow & \bullet \\ \downarrow & \ulcorner & \downarrow \\ T & \longrightarrow & B\mathbb{G}_m \end{array}$$

where the vertical morphisms are $\mathbb{G}_m$-torsors. The universal $\mathbb{G}_m$-torsor over the classifying stack $B\mathbb{G}_m$ is the quotient morphism $\bullet \to B\mathbb{G}_m$. The associated universal line bundle is $[\mathbb{A}^1/\mathbb{G}_m] = \mathbb{A}^1 \times_{\mathbb{G}_m} \bullet \to B\mathbb{G}_m$, so that a line bundle $L \to T$ is a pullback

$$\begin{array}{ccc} L & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & \ulcorner & \downarrow \\ T & \longrightarrow & B\mathbb{G}_m \end{array}.$$

Finally, as first observed by Lafforgue, and shown in the derived setting in [KR19, Proposition 3.2.6], $[\mathbb{A}^1/\mathbb{G}_m]$ is the classifying stack of line bundles together with a global section, since given $T \to B\mathbb{G}_m$, the dashed arrow in the following Cartesian diagram is equivalent to specifying a section of L:

$$\begin{array}{ccc} L & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & \ulcorner \;\nearrow & \downarrow \\ T & \longrightarrow & B\mathbb{G}_m \end{array}.$$

By the same token, $[\mathbb{A}^r/\mathbb{G}_m]$ is the stack classifying a line bundle together with r global sections.

2.2. **Prestable curves with a line bundle.** The stack parametrizing prestable curves with a line bundle can be viewed as an example of a derived moduli of sections. We require line bundles to be sufficiently ample when restricted to unmarked components, this slight modification simplifies the arguments of §3.

Let $\mathfrak{M} = \mathfrak{M}^{\mathrm{pre}}_{g,n}$ be the moduli space of pre-stable genus g, n-pointed curves with universal curve $\mathfrak{C}$. Consider the moduli space $\mathfrak{Pic}^{\mathrm{s}}{}_d := \mathfrak{Pic}^{\mathrm{s}}{}_{g,n,d}$ parametrizing pairs (C, L) of a pre-stable curve and a line bundle of degree d with the additional "stability" conditions

(1)
$$\omega_C^{\log} \otimes L^{\otimes 3} > 0$$
where $\omega_C^{\log}$ is the canonical bundle of the curve twisted by the sum of the n marked points.

(2)
$$\deg(L)|_{C_i} \geqslant 0$$
on all components C_i of C.

This is an open substack of the usual stack of curves with a degree d line bundle, denoted by $\mathfrak{Pic}^{\mathrm{s}}{}_d$.

Then $\mathfrak{Pic}^{\mathrm{s}}{}_d$ and an open substack of the derived moduli of sections of $\mathfrak{C} \times B\mathbb{G}_m$, that is

$$\mathfrak{Pic}^{\mathrm{s}}{}_d \subset \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times B\mathbb{G}_m/\mathfrak{C}).$$

The pullback of the universal curve over ${\mathfrak{Pic}^{\mathrm{s}}}_d$ is denoted as usual by $\pi_{{\mathfrak{Pic}^{\mathrm{s}}}_d} : \mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d} \to {\mathfrak{Pic}^{\mathrm{s}}}_d$. The universal section induces an evaluation $\ell_d : \mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d} \to B\mathbb{G}_m$. By Theorem 1.3.1, the relative tangent of the morphism ${\mathfrak{Pic}^{\mathrm{s}}}_d \to \mathfrak{M}$ is

$$\begin{aligned}\mathbb{T}_{{\mathfrak{Pic}^{\mathrm{s}}}_d/\mathfrak{M}} &= \mathbf{R}\pi_{{\mathfrak{Pic}^{\mathrm{s}}}_d *}\mathbf{L}\ell_d^*\mathbb{T}_{B\mathbb{G}_m}\\ &= \mathbf{R}\pi_{{\mathfrak{Pic}^{\mathrm{s}}}_d *}\mathcal{O}_{\mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}}[1].\end{aligned}$$

${\mathfrak{Pic}^{\mathrm{s}}}_d \to \mathfrak{M}$ is a smooth Artin stack of relative dimension $g-1$.

2.3. **Stable maps to $\mathbb{P}^r$ as sections.** From example 1.1.3, we can construct $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ as an open substack of $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C}\times\mathbb{P}^r/\mathfrak{C})$. Then theorem 1.3.1 recovers the usual formula

$$\mathbb{T}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)/\mathfrak{M}} = \mathbf{R}\pi_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)*}f^*T_{\mathbb{P}^r}$$

where $f : \mathfrak{C}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)} \to \mathbb{P}^r$ is the second component of the universal evaluation $\mathrm{ev}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)}$.

On the other hand, we may view degree d maps into $\mathbb{P}^r$ as (an open substack of) sections of $r+1$ degree d line bundles over a curve. With notation from example 2.2 we can define the universal bundle of ${\mathfrak{Pic}^{\mathrm{s}}}_d$ as the pullback of the universal bundle $[\mathbb{A}^1/\mathbb{G}_m]$ over the classifying space $B\mathbb{G}_m = [\bullet/\mathbb{G}_m]$:

$$\begin{array}{ccc}\mathfrak{L}_d & \xrightarrow[\mathfrak{r}]{} & [\mathbb{A}^1/\mathbb{G}_m]\\ \downarrow & & \downarrow\\ \mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d} & \xrightarrow{\ell_d} & B\mathbb{G}_m.\end{array}$$

The corresponding locally-free sheaf is denoted by $\mathcal{L}_d$. In the non-derived setting, this is indeed well-known that we can think of stable maps to projective space as an open substack of the moduli of sections of line bundles:

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \subset \underline{\mathrm{Sec}}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}).$$

This description gives rise to a perfect obstruction theory relative to ${\mathfrak{Pic}^{\mathrm{s}}}_d$ which has been proved to be compatible with the usual one (see for example [CL12, CFK10]). In the discussion below, we strengthen previous results by proving a derived statement (our Theorem 2.3.2) which easily implies the classical one (Corollary 2.3.4).

Lemma 2.3.1. *The derived stack of sections $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C}\times\mathbb{P}^r/\mathfrak{C})$ is an open substack of $\mathbb{R}\underline{\mathrm{Sec}}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d})$, the derived stack of $(r+1)$-tuples of sections of the universal bundle of ${\mathfrak{Pic}^{\mathrm{s}}}_d$.*

Proof. By definition, $\mathfrak{L}_d^{\oplus r+1} = \mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d} \times_{[\bullet/\mathbb{G}_m]} [\mathbb{A}^{r+1}/\mathbb{G}_m]$. Since $\mathbb{P}^r = [\mathbb{A}^{r+1}\backslash\{0\}/\mathbb{G}_m]$ is open in the global quotient stack $[\mathbb{A}^{r+1}/\mathbb{G}_m]$, then at the level of derived moduli of sections we obtain an open immersion:

$$\mathbb{R}\underline{\mathrm{Sec}}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}(\mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d} \times_{[\bullet/\mathbb{G}_m]} \mathbb{P}^r/\mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}) \subset \mathbb{R}\underline{\mathrm{Sec}}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}). \tag{8}$$

Finally, can identify the derived stacks of sections $\mathbb{R}\underline{\mathrm{Sec}}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}(\mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d}\times_{[\bullet/\mathbb{G}_m]}\mathbb{P}^r/\mathfrak{C}_{{\mathfrak{Pic}^{\mathrm{s}}}_d})$ and $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C}\times\mathbb{P}^r/\mathfrak{C})$ by applying Proposition 1.1.5 with $\mathfrak{Z}_1 = \mathfrak{C}\times B\mathbb{G}_m$, $\mathfrak{Z}_2 = \mathfrak{C}\times\mathbb{P}^r$. □

So far, we have two ways of obtaining a derived enhancement of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$:

- $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ obtained from the open immersion $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \subset \underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C}\times\mathbb{P}^r/\mathfrak{C})$ and the enhancement $\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C}\times\mathbb{P}^r/\mathfrak{C}) \hookrightarrow \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C}\times\mathbb{P}^r/\mathfrak{C})$, or

- $\mathbb{R}'\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ obtained from the open immersion $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}^s{}_d})$ and the derived enhancement $\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}^s{}_d}) \hookrightarrow \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}^s{}_d})$.

We will see below that these two enhancements are equivalent, thus we use the notation $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ freely for either of them.

Theorem 2.3.2. *The derived enhancement $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\mathbb{R}'\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ described above are equivalent derived stacks.*

Proof. The proof easily follows from Lemma 2.3.1 and the fact that given an open substack $i : \mathfrak{X} \to \mathfrak{Y}$ and a derived enhancement $\widetilde{\mathfrak{Y}}$ of $\mathfrak{Y}$, there exists a unique (up to derived equivalence) derived open substack $\widetilde{i} : \widetilde{\mathfrak{X}} \to \widetilde{\mathfrak{Y}}$ enhancing i.

More explicitly, observe that the open immersion

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}^s{}_d})$$

factors as

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{C}_{\mathfrak{Pic}^s{}_d} \times_{[\bullet/\mathbb{G}_m]} \mathbb{P}^r/\mathfrak{C}_{\mathfrak{Pic}^s{}_d}) \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}^s})$$

and by the proof of Lemma 2.3.1 the middle space has equivalent derived enhancements $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{C}_{\mathfrak{Pic}^s{}_d} \times_{[\bullet/\mathbb{G}_m]} \mathbb{P}^r/\mathfrak{C}_{\mathfrak{Pic}^s{}_d})$ and $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times \mathbb{P}^r/\mathfrak{C})$. Moreover, $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{C}_{\mathfrak{Pic}^s{}_d} \times_{[\bullet/\mathbb{G}_m]} \mathbb{P}^r/\mathfrak{C}_{\mathfrak{Pic}^s{}_d})$ is also equivalent to the enhancement of its classical truncation coming from

$$\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{C}_{\mathfrak{Pic}^s{}_d} \times_{[\bullet/\mathbb{G}_m]} \mathbb{P}^r/\mathfrak{C}_{\mathfrak{Pic}^s{}_d}) \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}^s}) \hookrightarrow \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}^s})$$

by Equation (8). This shows that there is a unique derived enhancement $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ that is open in both $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{M}}(\mathfrak{C} \times \mathbb{P}^r/\mathfrak{C})$ and $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s{}_d}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}^s})$. □

From this discussion, we can write a point in $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ as $(C, L, s_0, \ldots, s_r)$ where C is a genus g, n-marked prestable curve (we suppress the notation for the marked points), L is a degree d line bundle on C and $s_0, \ldots, s_r$ are sections. In this notation, the stability conditions of stable maps translate to the following.

Definition 2.3.3. [Stability conditions of stable maps as sections]

1. The bundle $\omega_C^{\log} \otimes L^{\otimes 3}$ is ample, which is a condition on the pair (C, L) already present in $\mathfrak{Pic}^s$,
2. The sections $(s_0, \ldots, s_r)$ have no common zeros.

Corollary 2.3.4. *There is a forgetful morphism $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathfrak{Pic}^s{}_d$ sending*

$$(C, L, s_0, \ldots, s_r) \mapsto (C, L).$$

The morphism is quasi-smooth with dual perfect obstruction theory

$$\mathbb{T}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)/\mathfrak{Pic}^s{}_d} \to \mathbb{E}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)/\mathfrak{Pic}^s{}_d} = \mathbf{R}^\bullet \pi_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)*} \mathcal{L}^{\oplus r+1}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)}$$

where $\mathcal{L}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)}$ is the locally-free sheaf on $\mathfrak{C}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)}$ obtained from the map into $\mathfrak{Pic}^s{}_d$. This perfect obstruction theory is compatible with the usual perfect obstruction theory of stable maps in the sense of [Man12a].

Proof. This follows from the discussion above and Corollary 1.3.2. □

2.4. **Quasi-maps to $\mathbb{P}^r$ as sections.** We have another way of understanding maps of curves to $\mathbb{P}^r$, by relaxing the concept of map and allowing a linear system $(L, s_0, \ldots, s_r)$ on a curve C to develop some base points. Consider $\mathfrak{Pic}_d = \mathfrak{Pic}_{g,n,d}$ the usual stack parametrizing genus g, n-marked pre-stable curves with a degree d bundle without stability conditions. Let $\mathfrak{C}_{\mathfrak{Pic}_d}, \mathfrak{L}_d$ denote the universal curve and universal bundle respectively.

Definition 2.4.1 (Stable quasi-maps[CFK10, Definition 3.1.1])**.**

$$\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}_d})$$

is the open substack defined by imposing following conditions on each geometric fiber

(1) (non-degeneracy)The linear system $(L, s_0, \ldots, s_r)$ has finitely many base points away from the nodes and the markings of C.
(2) (stability) The line bundle $\omega_C^{\log} \otimes L^{\otimes \epsilon} > 0$ for any $\epsilon \in \mathbb{Q}_{>0}$.

The derived enhancement $\underline{\mathrm{Sec}}_{\mathfrak{Pic}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}_d}) \hookrightarrow \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}_d}(\mathfrak{L}_d^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}_d})$ gives a derived enhancement $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \overset{j}{\hookrightarrow} \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$. The usual perfect obstruction for the moduli of quasi-maps (eg. [CFK10]) comes from this derived extension. Indeed, the computation in Theorem 1.3.1 shows that

$$\mathbb{T}_{\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)/\mathfrak{Pic}} \to j^*\mathbb{T}_{\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)/\mathfrak{Pic}} = \mathbf{R}\pi_* f^*\mathcal{O}_{\mathbb{P}^r}(1),$$

where as usual π and f are the universal projection and evaluation respectively from $\mathfrak{C}_{\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)}$. We will see in the next section a slightly different construction of $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ that yields an equivalent derived enhancement.

3. Stable maps and quasi-maps to $\mathbb{P}^r$

In this section, we construct a morphism between the derived enhancement of the moduli space of stable maps to $\mathbb{P}^r$ and of quasi maps that is

$$\overline{c} : \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d).$$

We prove that

$$\overline{c}_*\mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)} = \mathcal{O}_{\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)} \text{ in } \mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)).$$

3.1. **Revised notation.** From here, we will adopt a slightly different notation from that of the preceding sections in the interest of clarity. Let $\mathfrak{M} := \mathfrak{M}_{g,n}$ denote the moduli space of genus g pre-stable curves with n marked points and let $\underline{\pi} : \underline{\mathfrak{C}} \to \mathfrak{M}$ denote its universal curve. Let $\mathfrak{Pic}^{\mathrm{s}} := \mathfrak{Pic}^{\mathrm{s}}{}_{g,n,d}$ denote the moduli space defined in Example 2.2 and let $\underline{\pi} : \mathfrak{C} \to \mathfrak{Pic}^{\mathrm{s}}$ denote its universal curve. Recall, that $\mathfrak{Pic}^{\mathrm{s}}$ parametrizes pairs (C, L), with C a prestable curve in $\mathfrak{M}$ and L is a line bundle of fixed degree d over C subject to the stability conditions in 2.2. Then we define $\mathfrak{C}$ by the following cartesian diagram

$$\begin{array}{ccc} \mathfrak{C} & \longrightarrow & \underline{\mathfrak{C}} \\ \downarrow{\scriptstyle \pi} & \ulcorner & \downarrow{\scriptstyle \underline{\pi}} \\ \mathfrak{Pic}^{\mathrm{s}} & \longrightarrow & \mathfrak{M}. \end{array}$$

Notice that $\mathfrak{C} \to \mathfrak{M}$ is flat so that the stack fiber product is also the homotopical fiber product. Let $\mathfrak{L}$ over $\mathfrak{C}$ denote the universal bundle so we have

$$\mathfrak{L} \longrightarrow \mathfrak{C} \xrightarrow{\pi} \mathfrak{Pic}^{\mathrm{s}} \longrightarrow \mathfrak{M}. \tag{9}$$

Definition 3.1.1. Let C be a point in $\mathfrak{M}$. A *rational tail* Γ in C is a maximal tree of rational components without marked points and such that $\Gamma \cap \overline{C \backslash \Gamma}$ is a point.

Let $\breve{\mathfrak{M}}$ denote the moduli space of pre-stable curves of genus g with n marked points without rational tails. Let $\underline{\pi} : \underline{\breve{\mathfrak{C}}} \to \breve{\mathfrak{M}}$ denote the universal curve. In [CFK10, p.12], the authors prove that $\breve{\mathfrak{M}}$ is an open substack of finite type in $\mathfrak{M}$, with universal curve isomorphic to the restriction of $\mathfrak{C}$ to $\breve{\mathfrak{M}}$.

Let $\breve{\mathfrak{Pic}}$ denote the Cartesian product

$$\begin{array}{ccc} \breve{\mathfrak{Pic}} & \longrightarrow & \mathfrak{Pic} \\ \downarrow & \ulcorner & \downarrow \\ \breve{\mathfrak{M}} & \longrightarrow & \mathfrak{M}. \end{array}$$

A closed point in $\breve{\mathfrak{Pic}}$ is a pair $(\breve{C}, \breve{L})$ of a marked pre-stable curve with no rational tails and a line bundle.

Definition 3.1.2. As in the case of $\mathfrak{Pic}^{\mathrm{s}}$ let $\breve{\mathfrak{Pic}}^{\mathrm{s}}$ denote the substack of $\breve{\mathfrak{Pic}}$ with the additional stability conditions:

(1) for any $\epsilon \in \mathbb{Q}_{>0}$, we have

$$\omega_{\breve{C}}^{\log} \otimes \breve{L}^{\otimes \epsilon} > 0,$$

Here $\omega_{\breve{C}}^{\log}$ denotes the dualizing sheaf of the curve twisted by the sum of the n marked points.

(2) on all components $\breve{C}_i$ of $\breve{C}$, we have

$$\deg(\breve{L})|_{\breve{C}_i} \geqslant 0.$$

We have that $\breve{\mathfrak{Pic}}^{\mathrm{s}}$ is an open substack of $\breve{\mathfrak{Pic}}$.

We define $\breve{\mathfrak{C}}$ as the following fiber product

$$\begin{array}{ccc} \breve{\mathfrak{C}} & \longrightarrow & \underline{\breve{\mathfrak{C}}} \\ \downarrow \breve{\pi} & \ulcorner & \downarrow \underline{\breve{\pi}} \\ \breve{\mathfrak{Pic}}^{\mathrm{s}} & \longrightarrow & \breve{\mathfrak{M}}. \end{array}$$

We have a universal line bundle, denoted by $\breve{\mathfrak{L}}$ over $\breve{\mathfrak{C}}$. As in (9), we have the following morphisms.

$$\breve{\mathfrak{L}} \longrightarrow \breve{\mathfrak{C}} \xrightarrow{\breve{\pi}} \breve{\mathfrak{Pic}}^{\mathrm{s}} \longrightarrow \breve{\mathfrak{M}}. \tag{10}$$

3.2. **Quasi-maps are defined over $\breve{\mathfrak{Pic}}^{\mathrm{s}}$.** In Section 2.4 we defined

$$\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \subset \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}}). \tag{11}$$

Given the definitions of this sections, we have a new substack $\breve{\mathfrak{Pic}}^{\mathrm{s}} \subset \mathfrak{Pic}$. The stability conditions of quasi-maps imply that the source curve cannot have rational tails. So the morphism $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \to \mathfrak{Pic}$ factors through $\breve{\mathfrak{Pic}}$. Moreover, the stability conditions of $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ imply those of Definition 3.1.2, so we obtain a morphism $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \to \breve{\mathfrak{Pic}}^{\mathrm{s}} \subset \mathfrak{Pic}$.

By Remark 1.2.6, we have that

$$\mathbb{R}\underline{\mathrm{Sec}}_{\breve{\mathfrak{Pic}}^{\mathrm{s}}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}}) = \breve{\mathfrak{Pic}}^{\mathrm{s}} \times^h_{\mathfrak{Pic}} \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{Pic}}).$$

So the open embedding in (11) factors through an open embedding $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \subset \mathbb{R}\underline{\mathrm{Sec}}_{\breve{\mathfrak{Pic}}^{\mathrm{s}}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}})$. We state the implications of this below.

Proposition 3.2.1. *The moduli space of quasi-maps $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ has a forgetful morphism to the stack $\breve{\mathfrak{Pic}}^{\mathrm{s}}$ of pre-stable curves with no rational tails with a stable line bundle (Definition 3.1.2). Over $\breve{\mathfrak{Pic}}^{\mathrm{s}}$ it admits an open embedding into the derived stack of sections $\mathbb{R}\underline{\mathrm{Sec}}_{\breve{\mathfrak{Pic}}^{\mathrm{s}}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}})$. This endows $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ with a derived enhancement $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ which is compatible with the derived enhancement from Section 2.4.*

In particular, this derived enhancement recovers the canonical perfect obstruction theory of the moduli space of quasi-maps.

3.3. **Derived morphism between stable maps and quasi-maps.** In this subsection, we want to construct the following.

(1) A commutative diagram

$$\begin{array}{ccc} \mathfrak{L} & \longrightarrow & \breve{\mathfrak{L}} \\ \downarrow & & \downarrow \\ \mathfrak{C} & \xrightarrow{k} & \breve{\mathfrak{C}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \breve{\pi}} \\ \mathfrak{Pic}^{\mathrm{s}} & \xrightarrow{c} & \breve{\mathfrak{Pic}}^{\mathrm{s}} \\ \downarrow & & \downarrow \\ \mathfrak{M} & \xrightarrow{\underline{c}} & \breve{\mathfrak{M}} \end{array} \tag{12}$$

which relates (9) and (10).

(2) A morphism (see Proposition 3.3.10 below)

$$\overline{c} : \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\breve{\mathfrak{Pic}}^{\mathrm{s}}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}})$$

that restricts to a morphism $\mathbb{R}\overline{\mathcal{M}}(\mathbb{P}^r, d) \to \mathbb{R}\overline{\mathcal{Q}}(\mathbb{P}^r, d)$.

3.3.1. *Construction of contraction morphism $\underline{c} : \mathfrak{M} \to \breve{\mathfrak{M}}$.* Recall that in [PR03, proof of Thm 7.1] or [Man14, Prop 2.3]) one can construct a non separated[2] morphism

$$\underline{c} : \mathfrak{M} \to \breve{\mathfrak{M}}$$

which contracts the rational tails. For $S \to \mathfrak{M}$,

[2]This morphism is not separated because, the trivial family $\mathbb{P}^1 \times \mathbb{A}^1 \backslash \{0\} \to \mathbb{A}^1 \backslash \{0\}$ can be completed at $\{0\}$ by $\mathbb{P}^1$ or the blowup of the trivial family in any number of points in the special fibre.

$$\underline{c} : \mathfrak{M} \to \widetilde{\mathfrak{M}}$$
$$(C_S, S) \mapsto (\check{C}_S, S)$$

where $\check{C}_S$ is the family C_S with rational tails contracted in each fiber. Recall that $\underline{\mathfrak{C}}$ (resp. $\check{\underline{\mathfrak{C}}}$) is the universal curve of $\mathfrak{M}$ (resp. $\widetilde{\mathfrak{M}}$). Moreover, we have a commutative diagram which is not Cartesian

$$\begin{array}{ccc} \underline{\mathfrak{C}} & \xrightarrow{\underline{k}} & \check{\underline{\mathfrak{C}}} \\ \downarrow{\scriptstyle \underline{\pi}} & & \downarrow{\scriptstyle \check{\underline{\pi}}} \\ \mathfrak{M} & \xrightarrow{\underline{c}} & \widetilde{\mathfrak{M}}. \end{array} \tag{13}$$

Notice that $\underline{c}$ is a birational morphism.

3.3.2. *Construction of contraction of tails morphism $c : \mathfrak{Pic}^{\mathrm{s}} \to \widetilde{\mathfrak{Pic}}^{\mathrm{s}}$.* Recall that we have (9),

$$\mathfrak{L} \longrightarrow \mathfrak{C} \xrightarrow{\pi} \mathfrak{Pic}^{\mathrm{s}} \longrightarrow \mathfrak{M}.$$

Let $\mathfrak{M}^{\mathrm{rt}}$ be the divisor in $\mathfrak{M}$ where the curve has rational tails. We obtain a divisor $\mathfrak{D}$ on $\mathfrak{C}$ by pulling back $\mathfrak{M}^{\mathrm{rt}}$ to $\mathfrak{C}$ and taking the irreducible components of each fiber which correspond to rational tails.

Example 3.3.3. Consider a trivial family of smooth curves $C \times \mathbb{A}^1 \to \mathbb{A}^1$, and let $p \in C$ a closed point. We can obtain a family of curves with rational tails $\mathcal{C} = \mathrm{Bl}_{p \times \{0\}}(C \times \mathbb{A}^1) \to \mathbb{A}^1$. Then the divisor of rational tails on $\mathcal{C}$ is the exceptional divisor E, which is an irreducible component of the pullback of the divisor $0 \in \mathbb{A}^1$.

Definition 3.3.4. Let denote $\mathfrak{D}$ be the divisor in $\mathfrak{C}$ described above. By considering the restriction of the universal bundle $\mathfrak{L}$ on $\mathfrak{D}$ we can split the divisor into

$$\mathfrak{D} = \bigsqcup_{i=1}^{d} \mathfrak{D}_i$$

such that $\mathfrak{L}|_{\mathfrak{D}_i}$ has degree δ_i. We write $\delta\mathfrak{D}$ for $\sum_i \delta_i \mathfrak{D}$.

We first define $c : \mathfrak{Pic}^{\mathrm{s}} \to \widetilde{\mathfrak{Pic}}^{\mathrm{s}}$ at the level of points (see [Man14, §2.2]). Let $(C, L) \in \mathfrak{Pic}^{\mathrm{s}}$. Let T_i be the rational tails of C and let δ_i denote the total degree of L on T_i. Notice that $\sum_i \delta_i = \deg(L|_{\sqcup_i T_i})$. Let $\check{C}$ be the closure of $C \backslash \bigcup_i T_i$. Let Q_i denote the point $T_i \cap \check{C}$. We define

$$\check{L} := L|_{\check{C}}(\sum_i \delta_i Q_i).$$

In families we proceed similarly: let $S \to \mathfrak{Pic}^{\mathrm{s}}$ with a family of curves $C_S \to S$ and a line bundle $\mathcal{L}_S$. We define $\check{C}_S := \underline{c}(C_S)$ contracting rational tails. We put

$$\check{\mathcal{L}}_S := \mathcal{L}|_{\widetilde{C_S}}(\delta\mathfrak{D}),$$

and we obtain a morphism

$$c : \mathfrak{Pic}^{\mathrm{s}} \to \widetilde{\mathfrak{Pic}}^{\mathrm{s}}$$
$$(C_S, \mathcal{L}_S) \mapsto (\check{C}_S, \check{\mathcal{L}}_S).$$

To show it factors as the required morphism

$$c : \mathfrak{Pic}^{\mathrm{s}} \to \widecheck{\mathfrak{Pic}}^{\mathrm{s}}$$

we need to check that the following are true:

(1) If L has non-negative degree on each component of C, then $\check{L}$ has non-negative degree on each component of $\check{C}$.

(2) If $\omega_C^{\log} \otimes L^{\otimes 3} > 0$, then $\omega_{\check{C}}^{\log} \otimes \check{L}^{\otimes \epsilon} > 0$ for all $\epsilon \in \mathbb{Q}_{>0}$.

The first statement is clear. The only case where the first condition does not immediately imply the second is that of a genus 0 component C_i with less than two marked points. The first condition then requires that the degree of $L|_{C_i}$ is at least 1. Note that any component of $\check{C}$ has at least one marked point, so the degree of $\omega_{\check{C}}^{\log}$ is greater or equal than -1. This shows that the degree of $\check{L}|_{\underline{c}(C_i)}$ is at least 1 and thus the claim.

We thus get the following commutative diagram

$$\begin{array}{ccc} \mathfrak{Pic}^{\mathrm{s}} & \xrightarrow{c} & \widecheck{\mathfrak{Pic}}^{\mathrm{s}} \\ \downarrow & & \downarrow \\ \mathfrak{M} & \xrightarrow{\underline{c}} & \widecheck{\mathfrak{M}}. \end{array} \tag{14}$$

3.3.5. *Construction of the morphism* $k : \mathfrak{C} \to \check{\mathfrak{C}}$. As $\mathfrak{C} := \underline{\mathfrak{C}} \times_{\mathfrak{M}} \mathfrak{Pic}^{\mathrm{s}}$ (resp. $\check{\mathfrak{C}} := \check{\underline{\mathfrak{C}}} \times_{\widecheck{\mathfrak{M}}} \widecheck{\mathfrak{Pic}}^{\mathrm{s}}$) and the Cartesian diagram (13), writing all the diagrams, we get the morphism $k : \mathfrak{C} \to \check{\mathfrak{C}}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{k} & \check{\mathfrak{C}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \check{\pi}} \\ \mathfrak{Pic}^{\mathrm{s}} & \xrightarrow{c} & \widecheck{\mathfrak{Pic}}^{\mathrm{s}} \\ \downarrow & & \downarrow \\ \mathfrak{M} & \xrightarrow{\underline{c}} & \widecheck{\mathfrak{M}}. \end{array} \tag{15}$$

3.3.6. *Construction of the morphism* $\mathfrak{L} \to \check{\mathfrak{L}}$. We can decompose the morphism k as $\ell \circ \kappa$ as in the following diagram:

$$\begin{array}{ccccc} \mathfrak{L} & & \ell^*\check{\mathfrak{L}} & \xrightarrow{\quad} & \check{\mathfrak{L}} \\ \downarrow & & \downarrow & \ulcorner & \downarrow \\ \mathfrak{C} & \xrightarrow{\kappa} & c^*\check{\mathfrak{C}} & \xrightarrow{\ell} & \check{\mathfrak{C}} \\ & \searrow^{\pi} & \downarrow{\scriptstyle c^*\check{\pi}} & \ulcorner & \downarrow{\scriptstyle \check{\pi}} \\ & & \mathfrak{Pic}^{\mathrm{s}} & \xrightarrow{c} & \widecheck{\mathfrak{Pic}}^{\mathrm{s}} \\ & & \downarrow & & \downarrow \\ & & \mathfrak{M} & \xrightarrow{\underline{c}} & \widecheck{\mathfrak{M}}. \end{array} \tag{16}$$

Notice that $\check{\pi}, \pi$ and $c^*\check{\pi}$ are projective, so κ is projective. As κ is birational and $\mathcal{L}(\delta\mathfrak{D})$ is trivial on rational tails, we have that

$$\mathbf{R}^1\kappa_*\mathcal{L}(\delta\mathfrak{D}) = 0.$$

Then $\mathbf{R}^0\kappa_*\mathcal{L}(\delta\mathfrak{D})$ is a line bundle on $c^*\check{\mathfrak{C}}$. (See [PR03, Lemma 7.1 and p.652-654].

Claim 3.3.7. We have that

$$\mathbf{R}^0\kappa_*\mathcal{L}(\delta\mathfrak{D}) = \ell^*\check{\mathcal{L}}.$$

Proof of the claim 3.3.7. As κ is birational, the two sheaves are isomorphic away from the tails. On the tails, both are trivial. On a smooth atlas of $c^*\check{\mathfrak{C}}$, they are isomorphic away from the locus where the tails are attached to the curve which is of codimension 2. We deduce the statement. □

Remark 3.3.8. At the level of sheaves we have :

$$\mathcal{L} \to \mathcal{L}(\delta\mathfrak{D}) \text{ and by adjunction } \kappa^*\kappa_*\mathcal{L}(\delta\mathfrak{D}) \to \mathcal{L}(\delta\mathfrak{D}).$$

Notice that $\kappa^*\kappa_*\mathcal{L}(\delta\mathfrak{D}) = \mathcal{L}(\delta\mathfrak{D})$ because both are isomorphic outside tails and trivial on tails. Finally, we get a morphism from

$$\mathcal{L} \to \mathcal{L}(\delta\mathfrak{D}) = \kappa^*\kappa_*\mathcal{L}(\delta\mathfrak{D}) = \kappa^*\ell^*\check{\mathcal{L}},$$

which leads to a morphism $\mathfrak{L} \to \kappa_*\mathfrak{L}(\delta\mathfrak{D}) = \ell^*\check{\mathfrak{L}}$ that fills the diagram (16).

3.3.9. *Construction of the morphism* $\overline{c} : \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^s}}(\check{\mathfrak{L}}^{\oplus r+1}/\check{\mathfrak{C}})$.

Theorem 3.3.10. *We have a morphism*

$$\overline{c} : \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^s}}(\check{\mathfrak{L}}^{\oplus r+1}/\check{\mathfrak{C}}).$$

Moreover, the restriction of $\overline{c}$ *to* $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ *factors through*

$$\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d) \subset \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^s}}(\check{\mathfrak{L}}^{\oplus r+1}/\check{\mathfrak{C}}),$$

giving a morphism, denoted by the same name,

$$\overline{c} : \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d).$$

Proof. Multiplication by the canonical section gives a morphism $a : \mathcal{L} \to \mathcal{L}(\delta\mathfrak{D})$. We have the divisor exact sequence

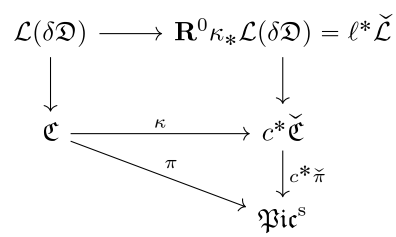
$$0 \to \mathcal{L} \to \mathcal{L}(\delta\mathfrak{D}) \to \mathcal{L}(\delta\mathfrak{D})|_{\delta\mathfrak{D}} \to 0 \tag{17}$$

over $\mathfrak{Pic}^s$.

The morphism a of sheaves induces a morphism $\mathfrak{L} \to \mathfrak{L}(\delta\mathfrak{D})$ of total spaces, which induces

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}/\mathfrak{C}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}(\delta\mathfrak{D})/\mathfrak{C}). \tag{18}$$

Now recall the locally-free sheaves



and let $\kappa_*\mathfrak{L}(\delta\mathfrak{D})$ denote the total space of $\mathbf{R}^0\kappa_*\mathcal{L}(\delta\mathfrak{D})$. There is an equivalence

$$\begin{aligned}\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\mathfrak{L}(\delta\mathfrak{D})/\mathfrak{C}) &= \mathrm{Spec}_{\mathfrak{Pic}^{\mathrm{s}}}\mathrm{Sym}((\mathbf{R}\pi_*\mathcal{L}(\delta\mathfrak{D}))^\vee) \\ &\simeq \mathrm{Spec}_{\mathfrak{Pic}^{\mathrm{s}}}\mathrm{Sym}((\mathbf{R}(c^*\breve{\pi})_*\kappa_*\mathcal{L}(\delta\mathfrak{D}))^\vee) \\ &= \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\kappa_*\mathfrak{L}(\delta\mathfrak{D})/c^*\breve{\mathfrak{C}}). \qquad (19)\end{aligned}$$

This equivalence is simply a restatement of the fact that the sections of the push-forward of a sheaf on an open are sections of the original sheaf on the preimage. Then by claim 3.3.7 and (19), we have

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\mathfrak{L}(\delta\mathfrak{D})/\mathfrak{C}) \simeq \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\kappa_*\mathfrak{L}(\delta\mathfrak{D})/c^*\breve{\mathfrak{C}}) \simeq \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\ell^*\breve{\mathfrak{L}}/c^*\breve{\mathfrak{C}}). \qquad (20)$$

Now we just have to construct a morphism

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\ell^*\breve{\mathfrak{L}}/c^*\breve{\mathfrak{C}}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^{\mathrm{s}}}}(\breve{\mathfrak{L}}/\breve{\mathfrak{C}}).$$

Let us consider the cartesian diagram

$$\begin{array}{ccc} c^*\breve{\mathfrak{C}} & \xrightarrow{\ \ell\ } & \breve{\mathfrak{C}} \\ {\scriptstyle \rho=c^*\breve{\pi}}\downarrow & & \downarrow{\scriptstyle \breve{\pi}} \\ \mathfrak{Pic}^{\mathrm{s}} & \xrightarrow{\ c\ } & \widetilde{\mathfrak{Pic}^{\mathrm{s}}}. \end{array}$$

By cohomology and base change, we get an isomorphism

$$\mathbf{R}\rho_*\ell^*\breve{\mathcal{L}} \to c^*\mathbf{R}\breve{\pi}_*\breve{\mathcal{L}},$$

that is we deduce that at the level of spaces, we have

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}\left(\mathfrak{L}\left(\delta\mathfrak{D}\right)/\mathfrak{C}\right) \simeq c^*\mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^{\mathrm{s}}}}(\breve{\mathfrak{L}}/\breve{\mathfrak{C}}) = \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\ell^*\breve{\mathfrak{L}}/c^*\breve{\mathfrak{C}}).$$

By composing, we deduce a morphism

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\mathfrak{L}(\delta\mathfrak{D})/\mathfrak{C}) \simeq c^*\mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^{\mathrm{s}}}}(\breve{\mathfrak{L}}/\breve{\mathfrak{C}}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^{\mathrm{s}}}}(\breve{\mathfrak{L}}/\breve{\mathfrak{C}}). \qquad (21)$$

Composing (18) with (21) we get a morphism

$$\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\mathfrak{L}/\mathfrak{C}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\mathfrak{L}(\delta\mathfrak{D})/\mathfrak{C}) \to \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^{\mathrm{s}}}}(\breve{\mathfrak{L}}/\breve{\mathfrak{C}}).$$

By applying the same argument to $\mathfrak{L}^{\oplus r+1}$, we deduce the desired morphism $\overline{c}$

$$\begin{array}{ccc} \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}) & \xrightarrow{\ \overline{c}\ } & \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^{\mathrm{s}}}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}}) \\ \downarrow & & \downarrow \\ \mathfrak{Pic}^{\mathrm{s}} & \xrightarrow{\ c\ } & \widetilde{\mathfrak{Pic}^{\mathrm{s}}}. \end{array} \qquad (22)$$

Now we are left to check that the restriction of $\overline{c}$ to $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ takes image in $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$. We can check this on points, let $(C, L, s_0, \dots, s_r) \in \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. We need to see that stability conditions of stable maps on $(C, L, s_0, \dots, s_r)$ imply those of quasi-maps on $\overline{c}(C, L, s_0, \dots, s_r)$. The conditions about the ampleness of the bundles, are already checked at the level of $c : \mathfrak{Pic}^{\mathrm{s}} \to \widetilde{\mathfrak{Pic}^{\mathrm{s}}}$. We only need to show that if $(L, s_0, \dots, s_r)$ has no base points, then $(\breve{L}, \breve{s}_0, \dots, \breve{s}_r)$ has finitely many base points away from markings and nodes. Let Q_i be the attaching nodes of the rational tail T_i on C. The only base points that are acquired by applying $\overline{c}$ are on the images of the Q_is in $\breve{C}$, but these are smooth and unmarked points of $\breve{C}$.

Thus we have a well-defined map given by the restriction of 22, which we will still denote by the same name:

$$\overline{c} : \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d).$$

□

4. Local embeddings

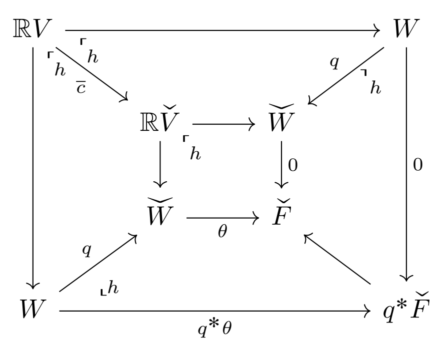
The idea of this section is to control the map $\overline{c}$ locally. In this section, by a slight abuse of notation we also denote by $\overline{c}$ the restriction/ base-change of $\overline{c}$ to various charts of $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.

For any point $\xi \in \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ we construct

(1) $\mathbb{R}V \to \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ an étale neighbourhood of ξ,
(2) $\mathbb{R}\check{V} \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ an étale neighbourhood of $\check{\xi} := \overline{c}(\xi)$, where the map $\overline{c} : \mathbb{R}V \to \mathbb{R}\check{V}$ is the base change of $\overline{c} : \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$,
(3) A smooth Deligne–Mumford stack W and a smooth scheme $\breve{W}$ with a morphism $q : W \to \breve{W}$ which is proper and birational,
(4) a vector bundle $\check{F}$ on $\breve{W}$ together with a section θ such that
   - the homotopical zero locus of θ is $\mathbb{R}\check{V}$,
   - the homotopical zero locus of $q^*\theta$ is $\mathbb{R}V$.

Let us sum up the situation in the following diagram, where each square is Cartesian.

(23)



Practically, we have that

$$\mathbb{R}\check{V} = Z^h(\theta) \text{ and } \mathbb{R}V = Z^h(q^*\theta).$$

Notice that the right and bottom squares are homotopically Cartesian by [Sta22, Lemma 08I6].

We will construct a different collection of open stacks: $\mathbb{R}U \subset \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C})$, and $\mathbb{R}\check{U} \subset \mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^s}}(\check{\mathfrak{L}}^{\oplus r+1}/\check{\mathfrak{C}})$. We will also construct $\mathcal{W}$, $\breve{\mathcal{W}}$ smooth stacks such that $\mathbb{R}U$ and $\mathbb{R}\check{U}$ sit inside them as a derived vanishing locus. These are all moduli of sections with a minor stability condition. Later, by imposing the full stability conditions for stable maps and quasi-maps respectively, we will obtain schemes $\mathbb{R}V$, W and $\mathbb{R}\check{V}$, $\breve{W}$.

4.1. **Constructions.** For any point $(\breve{C}, \breve{L}, \breve{s}_0 \dots, \breve{s}_r) \in \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$, let $BL(\breve{s}) = \cap_{i=0}^r Z(\breve{s}_i)$ be the base locus of $(\breve{s}_0, \dots, \breve{s}_r)$. By construction, $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ comes with a universal curve, a universal line bundle over it and a universal $(r+1)$-tuple of sections $\breve{\sigma}$. We have that $BL(\breve{\sigma}) \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ is a finite morphism.

Fix a closed point $\xi \in \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\breve{\xi} = (\breve{C}', \breve{L}', \breve{s}')$ its image in $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$. We construct here some open substacks of the moduli of sections $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C})$ and $\mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^{\mathrm{s}}}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}})$ containing ξ and $\breve{\xi}$ in respectively. Later on, we will impose stability conditions on these opens.

Construction 4.1.1 (Construction of the DM stacks W, $\mathbb{R}U$ and $\breve{W}$, $\mathbb{R}\breve{U}$). The first step will be to choose an open substack $\breve{\mathfrak{U}} \subset \widetilde{\mathfrak{Pic}^{\mathrm{s}}}$ containing the point $(\breve{C}', \breve{L}')$ and a divisor $\breve{A}$ on the universal curve $\breve{\mathfrak{C}}$ that behaves nicely over $\breve{\mathfrak{U}}$. These choices will depend on the choice of the quasi-map $\breve{\xi}$ and not just on its source curve.

By the stability conditions on $\widetilde{\mathfrak{Pic}^{\mathrm{s}}}$, the line bundle $\omega_{\breve{\pi}} \otimes \breve{\mathfrak{L}}$ is $\breve{\pi}$-relatively ample. After replacing $\omega_{\breve{\pi}} \otimes \breve{\mathfrak{L}}$ by an appropriate multiple, we may assume we have a very ample line bundle with vanishing $\mathbf{R}^1\breve{\pi}_*$. The divisor $\breve{A}$ is given by a choice of a section of this very ample line bundle, i.e. a hyperplane on the projective space of sections of this bundle. We can choose one such hyperplane that intersects the image of $(\breve{C}', \breve{L}')$ transversally at non-special points, and we can restrict to the complement $\breve{\mathfrak{U}}$ of the closed substack where $\breve{A}$ intersects the curves in the fiber at special points or is ramified. We can moreover guarantee by a change of coordinates that $\breve{A} \cap \breve{C}'$ consists of points disjoint from $BL(\breve{s}')$. Recall that $BL(\breve{s}')$ comes from the choice of $\breve{\xi}$. By construction,

$$\mathbf{R}^1\breve{\pi}_*\breve{L}(\breve{A}) = 0 \tag{24}$$

on all curves in this chosen neighborhood. To sum up, by our choice of hyperplane we have that

1. (1) $\breve{A}$ does not contain 1-dimensional fibers of the restriction of $\breve{\mathfrak{C}}_{\breve{\mathfrak{U}}} \to \breve{\mathfrak{U}}$ and
2. (2) $\breve{A}$ on the chosen curve $\breve{C}'$ is disjoint from the base locus of $\breve{s}'$ of the fixed quasi-map $\breve{\xi}$.
3. (3) $\breve{A}$ is disjoint from the special points of $(\breve{C}, \breve{L})$ (i.e. nodes and marked points) for all points $(\breve{C}, \breve{L}) \in \breve{\mathfrak{U}}$.

We fix the notation $\breve{\mathfrak{U}} \subset \widetilde{\mathfrak{Pic}^{\mathrm{s}}}$ for this substack, which depends on a choice of a point $\xi \in \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$.

Let $\mathfrak{U} := c^{-1}(\breve{\mathfrak{U}})$ and $A := k^*\breve{A}$. Since L has positive degree on rational tails (see Section 2.2 for the definition of $\mathfrak{Pic}^{\mathrm{s}}$), on $\mathfrak{U}$, we have

$$\mathbf{R}^1\pi_*L(A) = 0. \tag{25}$$

We define

$$\mathbb{R}U := \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{U}}(\mathfrak{L}_{\mathfrak{U}}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{U}}) \qquad W := \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{U}}(\mathfrak{L}_{\mathfrak{U}}(A)^{\oplus r+1}/\mathfrak{C}_{\mathfrak{U}})$$

$$\mathbb{R}\breve{U} := \mathbb{R}\underline{\mathrm{Sec}}_{\breve{\mathfrak{U}}}(\breve{\mathfrak{L}}_{\breve{\mathfrak{U}}}^{\oplus r+1}/\breve{\mathfrak{C}}_{\breve{\mathfrak{U}}}) \qquad \breve{W} := \mathbb{R}\underline{\mathrm{Sec}}_{\breve{\mathfrak{U}}}(\breve{\mathfrak{L}}_{\breve{\mathfrak{U}}}(\breve{A})^{\oplus r+1}/\mathfrak{C}_{\breve{\mathfrak{U}}}).$$

Note that $\mathbb{R}U$ and $\mathbb{R}\breve{U}$ are open in $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C})$ and $\mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^s}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}})$ respectively by Remark 1.2.6. Moreover, we have $\xi \in \mathbb{R}U$ and $c(\xi) \in \mathbb{R}\breve{U}$. By (24) and (25) we see that $\mathcal{W}$ and $\breve{\mathcal{W}}$ are smooth Artin stacks and have no derived structure. Multiplication by the defining equations of the divisor A gives a morphism of sheaves $\mathcal{L}_{\mathfrak{U}} \to \mathcal{L}_{\mathfrak{U}}(A)$ that gives a morphism $\mathbb{R}U \to \mathcal{W}$. Similarly, we have $\mathbb{R}\breve{U} \to \breve{\mathcal{W}}$. As in Theorem 3.3.10 we have a morphism $\tilde{q} : \mathcal{W} \to \breve{\mathcal{W}}$

Construction 4.1.2 (Labelling of base points)*.* Fix $\xi := (C', L', s'_0, \dots, s'_r) \in \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ a closed point and $\breve{\xi} := \overline{c}(\xi) = (\breve{C}', \breve{L}', \breve{s}'_0, \dots, \breve{s}'_r) \in \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ and construct the neighborhoods described above.

Let $\breve{\zeta} = (\breve{C}', \breve{L}', \breve{w}'_0, \dots, \breve{w}'_r)$ be the image of $\breve{\xi}$ in $\breve{\mathcal{W}}$. Let $BL(\breve{s}')$ be the base locus of $(\breve{s}'_0, \dots, \breve{s}'_r)$, and $BL(\breve{w}')$ be the base locus of $(\breve{w}'_0, \dots, \breve{w}'_r)$.

By construction, we have that the base locus $BL(\breve{w}') = BL(\breve{s}') \sqcup \breve{A}$. This follows because $BL(\breve{s}')$ and $\breve{A}$ are disjoint by construction and for each i, $\breve{w}'_i$ is obtained by multiplying $\breve{s}'_i$ by the local defining equation of $\breve{A}$. Then we have a labelling

$$BL(\breve{w}') = \underbrace{\{\breve{w}'_0 = \cdots = \breve{w}'_r = 0\} \cap \breve{A}}_{BL(\breve{w}')_{\breve{A}}} \sqcup \underbrace{\{\breve{s}'_0 = \cdots = \breve{s}'_r\}}_{BL(\breve{w}')_{\breve{L}}}.$$

By Section 4.1 we have that the pull-back of $\breve{A}$ on the universal curve over $\breve{\mathcal{W}}$ is finite and étale over $\breve{\mathcal{W}}$. The chosen point $\breve{\zeta}$ has the sections $\breve{s}_i$ generically non-degenerate —i.e. they do not all vanish on any component of $\breve{C}'$. By passing to the open substack inside $\breve{\mathcal{W}}$ where the sections are generically non-degenerate, we may assume $\breve{\mathcal{W}}$ is itself a Deligne–Mumford stack (see [CFK10, Lemma 3.1.6]).With this assumption, $\breve{\mathcal{W}}$ admits an étale chart, which is a scheme. On this chart we consider a lift of $\breve{\zeta}$, which by abuse of notation we denote $\breve{\zeta}$. Passing to an étale cover of this chart as in [Sta22, Lemma 04HL], we obtain an étale neighborhood $\breve{\mathcal{W}}'$ of $\breve{\zeta}$, i.e. a scheme, such that on this neighborhood we have that $BL(\breve{w}')_{\breve{A}}$ and $BL(\breve{w}')_{\breve{L}}$ lie on different connected components. This shows that the base-change of $BL(\breve{w})$ to $\breve{\mathcal{W}}'$ can be written as a union of disconnected components $BL(\breve{w})_{\breve{A}}$ and $BL(\breve{w})_{\breve{L}}$, which contain $BL(\breve{w}')_{\breve{A}}$ and $BL(\breve{w}')_{\breve{L}}$ respectively.

This means that for a point $(\breve{C}, \breve{L}, \breve{w}_0, \dots, \breve{w}_r) \in \breve{\mathcal{W}}'$ the base points $BL(\breve{w})$ of $(\breve{L}, \breve{w}_0, \dots, \breve{w}_r)$ are labelled by the connected components of the base locus

$$BL(\breve{w}) = BL(\breve{w})_{\breve{A}} \sqcup BL(\breve{w})_{\breve{L}}.$$

We define the smooth Artin stack

$$\mathcal{W}' := \breve{\mathcal{W}}' \times_{\breve{\mathcal{W}}} \mathcal{W}.$$

Now we define a smooth scheme $\breve{W} \subset \breve{\mathcal{W}}'$ by imposing stability conditions.

Construction 4.1.3 (Construction of schemes $\breve{W}$ and $\mathbb{R}\breve{V}$)*.* Let

$$(\breve{C}, \breve{L}, \breve{w}_0, \dots, \breve{w}_r) \in \breve{\mathcal{W}}' \to \underline{\mathrm{Sec}}_{\breve{\mathfrak{U}}}(\breve{\mathfrak{L}}_{\breve{\mathfrak{U}}}(\breve{A})/\breve{\mathfrak{C}}_{\breve{\mathfrak{U}}}).$$

This point is in $\breve{W}$ if

(i) the base locus of $\breve{w}_0, \dots, \breve{w}_r$ is discrete and disjoint from all the special points of $\breve{C}$,

(ii) for any $\epsilon \in \mathbb{Q}_{>0}$,
$$\omega_{\breve{C}}^{\log} \otimes \breve{L}(\breve{A})^{\otimes \epsilon} > 0.$$
Note that the base locus is labeled in the sense of construction 4.1.2, because we are in $\breve{\mathcal{W}}'$. Finally, we define
$$\mathbb{R}\breve{V} := \mathbb{R}\breve{U} \times_{\breve{\mathcal{W}}} \breve{W}.$$

Remark 4.1.4. We have that $\breve{W}$ is a smooth scheme as it is étale over an an open substack of $\mathbb{V}(\mathbf{R}\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A))$ and $\mathbb{V}(\mathbf{R}\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A))$ is smooth as $\mathbf{R}^1\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A) = 0$. By possibly shrinking $\breve{W}$ we may assume it is an affine scheme.

Construction 4.1.5 (Construction of W and $\mathbb{R}V$). Let $\mathcal{W}' = \breve{\mathcal{W}}' \times_{\breve{\mathcal{W}}} \mathcal{W}$. It is a smooth Artin stack. We have that $\mathcal{W}'$ contains the point ζ, the image of ξ in $\mathcal{W}$. By construction, we have a map $\mathfrak{q} : \mathcal{W}' \to \breve{\mathcal{W}}'$ and an induced map between universal curves $\mathfrak{k}$. If for any point $(C, L, w) \in \mathcal{W}'$, we denote its image under $\mathfrak{q}$ by $(\breve{C}, \breve{L}, \breve{w})$, then we have that $\mathfrak{k}$ maps $BL(w)$ to $BL(\breve{w})$. Since the base locus in $\breve{\mathcal{W}}'$ is labelled, we have that the base locus in $\mathcal{W}'$ is labelled:
$$BL(w) = BL(w)_A \sqcup BL(w)_L.$$
Let
$$(C, L, w_0, \ldots, w_r) \in \mathcal{W}' \subset \underline{\mathrm{Sec}}_{\mathfrak{U}}(\mathfrak{L}_{\mathfrak{U}}(A)/\mathfrak{C}_{\mathfrak{U}}).$$
This point is in W if

(i) the base locus $BL(w)$ of $w_0, \ldots, w_r$ is discrete and disjoint from the special points of C,
(ii) the subset $BL(w)_L$ of the base locus is empty and
(iii) the line bundle $\omega_C^{\log} \otimes L^{\otimes 3}$ is ample.

Notice that by the definition of $\mathcal{W}'$ we have that the base locus is labelled and thus condition (ii) makes sense. Finally, we define
$$\mathbb{R}V := \mathbb{R}U \times_{\mathcal{W}} W.$$

Remark 4.1.6. Notice that for any $(C, L, w_0, \ldots, w_r) \in W$, the choice of $\breve{A}$ and the stability condition imply that $A|_C$ does not intersect rational tails for any C.

This was not the case for points in $\mathcal{W}'$ without the stability condition in (i) in Construction 4.1.5.

Remark 4.1.7. The idea behind the construction is to define compatible atlases on $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$, in the sense that we want charts $\mathbb{R}V$ and $\mathbb{R}\breve{V}$ respectively such that
$$\begin{array}{ccc} \mathbb{R}V & \xrightarrow{\quad} & \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \\ \big\downarrow & \ulcorner h & \big\downarrow c \\ \mathbb{R}\breve{V} & \xrightarrow{\quad} & \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d). \end{array}$$
In addition, we want $\mathbb{R}V$ and $\mathbb{R}\breve{V}$ to be derived vanishing loci of triples (W, F, θ) and $(\breve{W}, \breve{F}, \breve{\theta})$ where the first family is a pullback of the second. These are triples of a smooth scheme, a vector bundle and a section. To achieve this, we start by covering $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ by sets open in $\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C})$ and $\mathbb{R}\underline{\mathrm{Sec}}_{\widetilde{\mathfrak{Pic}^s}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}})$. These sets will be of the form $\mathbb{R}U = \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{U}}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{U}})$,

$\mathbb{R}\breve{U} = \mathbb{R}\underline{\mathrm{Sec}}_{\breve{\mathfrak{U}}}(\breve{\mathfrak{L}}^{\oplus r+1}/\breve{\mathfrak{C}}_{\breve{\mathfrak{U}}})$. They are chosen so that it is possible to pick sufficiently ample divisors on $\breve{A}$ and A on $\breve{\mathfrak{C}}_{\breve{\mathfrak{U}}}$ and $\mathfrak{C}_{\mathfrak{U}}$ which are away from the rational tails and base points and give $\mathbb{R}U$ and $\mathbb{R}\breve{U}$ smooth embeddings (see Proposition 4.2.1 and Lemma 4.2.2 for more details).

We end up with a closed embedding $m_A : \mathbb{R}U \to \mathcal{W} = \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{U}}(\mathfrak{L}(A)^{\oplus r+1}/\mathfrak{C}_{\mathfrak{U}})$ and a similar one for $\mathbb{R}\breve{U}$. Here, $m_A(\mathbb{R}U)$ is the space of $(r+1)$-tuples of sections of $\mathfrak{L}(A)$ which are all divisible by the local equation of A. Now we could define $\mathbb{R}\breve{V} = \mathbb{R}\breve{U} \cap \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$ and $\mathbb{R}V = \mathbb{R}U \cap \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, but more care is needed at this stage.

Here W and $\breve{W}$ are necessary to define "non-degeneracy conditions" on $\mathcal{W}$ that will restrict to those of stable maps when restricted to the subvariety $\mathbb{R}U$. This is the reason we pass to different (étale) neighbourhoods in the construction.

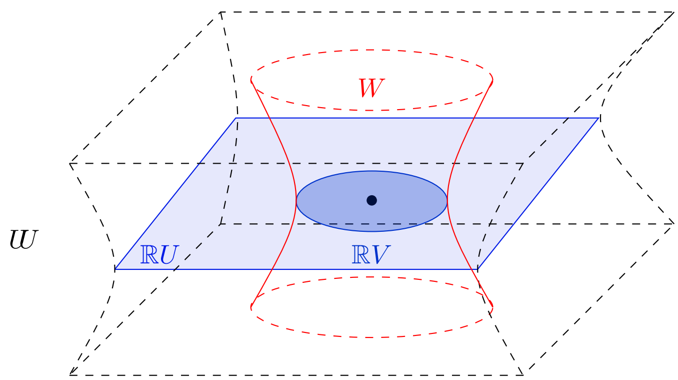


FIGURE 1. The ambient space is $\mathcal{W}$ which is an open in the moduli of sections of $\mathfrak{L}(A)^{r+1}$, $\mathbb{R}U$ is an open in the moduli of sections of $\mathfrak{L}$, $\mathbb{R}V$ is $\mathbb{R}U \cap \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. We draw this picture for stable maps (in $\mathcal{W}$ and not $\breve{\mathcal{W}}$) but we should imagine the same for quasi-maps in a compatible way.

4.2. **Properties.** Recall that $\mathcal{W} := \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{U}}(\mathfrak{L}_{\mathfrak{U}}(A)^{\oplus r+1}/\mathfrak{C}_{\mathfrak{U}})$ and $\breve{\mathcal{W}} := \mathbb{R}\underline{\mathrm{Sec}}_{\breve{\mathfrak{U}}}(\breve{\mathfrak{L}}_{\breve{\mathfrak{U}}}(\breve{A})^{\oplus r+1}/\mathfrak{C}_{\breve{\mathfrak{U}}})$ are smooth (thus only trivially derived) Artin stacks.

Proposition 4.2.1. *There exists a vector bundle E on $\mathcal{W}$ and a section σ such that $\mathbb{R}U$ is the derived zero locus of σ.*

Similarly, there exists a vector bundle $\breve{E}$ on $\breve{\mathcal{W}}$ and a section $\breve{\sigma}$ such that $\mathbb{R}\breve{U}$ is the derived zero locus of $\breve{\sigma}$.

Proof. Recall from (25) and (26) that we have $\mathbf{R}^1\pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}}(A) = \mathbf{R}^1\breve{\pi}_{\mathfrak{U}*}\breve{\mathcal{L}}_{\mathfrak{U}}(\breve{A}) = 0$. Multiplying by a local equation of A and pushing forward gives a distinguished triangle of sheaves on $\mathfrak{U}$.

$$\mathbf{R}\pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}} \to \mathbf{R}\pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}}(A) \xrightarrow{s} \mathbf{R}\pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}}(A)|_A \xrightarrow{+1} \tag{26}$$

Observe that $\mathbb{R}U = \mathbb{V}(\mathbf{R}\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}})$, and

$$\mathcal{W} = \mathbb{V}(\mathbf{R}\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A)) = \mathbb{V}(\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A)).$$

We also have $\mathbf{R}^1\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A)|_A = 0$, forced by the long exact sequence of (26) and (25). Then $\mathfrak{E} := \mathbb{V}(\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A)|_A)$ is a non-derived vector bundle on $\mathfrak{U}$. The distinguished triangle in (26) can be thus written as a fibered and cofibered diagram of derived complexes

$$\begin{array}{ccc} \mathbf{R}\pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}} & \longrightarrow & \pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A) \\ \big\downarrow & & \big\downarrow s \\ 0 & \longrightarrow & \pi_{\mathfrak{U}*}\mathfrak{L}_{\mathfrak{U}}(A)|_A. \end{array}$$

Taking the total space $\mathbb{V}(-) = \mathbb{R}\mathrm{Spec}_{\mathfrak{U}}\mathrm{Sym}^\bullet(-)^\vee$ functor gives us a homotopical fibered product

$$\begin{array}{ccc} \mathbb{R}U & \longrightarrow & \mathcal{W} \\ \big\downarrow & h & \big\downarrow s \\ \mathfrak{U} & \xrightarrow{0} & \mathfrak{E}. \end{array} \tag{27}$$

Let E be the pullback of the bundle $\mathfrak{E}$ by the projection $\mathcal{W} \to \mathfrak{U}$, and σ be the section induced by s. We claim that the homotopical fibered square above implies that the square below is also homotopically fibered

$$\begin{array}{ccc} \mathbb{R}U & \longrightarrow & \mathcal{W} \\ \big\downarrow & & \big\downarrow \sigma \\ \mathcal{W} & \xrightarrow{0} & E. \end{array} \tag{28}$$

To see this, consider

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{0} & E \\ \big\downarrow & & \big\downarrow \\ \mathfrak{U} & \xrightarrow{0} & \mathfrak{E} \end{array} \tag{29}$$

which is obviously fibered. Stacking (28) and (29) yields (27). Since (29) and (27) are homotopical fibered products, (28) must also be a homotopical fibered product.

The second part of the statement is proved in the same way, with $\breve{E}$ and $\breve{\sigma}$ coming from the following triangle over $\breve{\mathfrak{U}}$:

$$\mathbf{R}\breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}} \to \breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}}(\breve{A}) \xrightarrow{\breve{s}} \breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}}(\breve{A})|_{\breve{A}} \xrightarrow{+1} . \tag{30}$$

□

The contraction $c : \mathfrak{Pic}^{\mathrm{s}} \to \widetilde{\mathfrak{Pic}}^{\mathrm{s}}$ restricts to $c : \mathfrak{U} \to \breve{\mathfrak{U}}$ and induces maps $\widetilde{c} : \mathbb{R}U \to \mathbb{R}\breve{U}$ and $\widetilde{q} : \mathcal{W} \to \breve{\mathcal{W}}$ by the same construction as 3.3.6, 3.3.10. All these maps are in particular birational.

Lemma 4.2.2. *We have a homotopically cartesian diagram*

$$\begin{array}{ccc} \mathbb{R}U & \xrightarrow{i} & \mathcal{W} \\ \widetilde{c}\big\downarrow & h & \big\downarrow \widetilde{q} \\ \mathbb{R}\breve{U} & \xrightarrow{\breve{i}} & \breve{\mathcal{W}}. \end{array} \tag{31}$$

Proof. By Proposition 4.2.1, it suffices to show that $E = \widetilde{q}^*\breve{E}$ and $\sigma = \widetilde{q}^*\breve{\sigma}$. Recall that (E, σ) is defined by the following diagram coming from (26).

$$\begin{array}{ccc} E & \xrightarrow{\ \ulcorner\ } & \mathrm{Spec}_{\mathfrak{U}}\mathrm{Sym}(\pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}}(A)|_A) \\ {\scriptstyle\sigma}\uparrow\downarrow & \overset{s}{\nearrow} & \downarrow \\ W & \longrightarrow & \mathfrak{U}. \end{array}$$

Similarly $(\breve{E}, \breve{\sigma})$ is defined by the diagram below coming from (30).

$$\begin{array}{ccc} \breve{E} & \xrightarrow{\ \ulcorner\ } & \mathrm{Spec}_{\breve{\mathfrak{U}}}\mathrm{Sym}(\breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}}(\breve{A})|_{\breve{A}}) \\ {\scriptstyle\breve{\sigma}}\uparrow\downarrow & \overset{\breve{s}}{\nearrow} & \downarrow \\ \breve{W} & \longrightarrow & \breve{\mathfrak{U}}. \end{array}$$

For $c : \mathfrak{U} \to \breve{\mathfrak{U}}$ the usual contraction, we need to show that $c^*\breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}}(\breve{A})|_{\breve{A}} \cong \pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}}(A)|_A$ and that $\breve{s} \circ \widetilde{q} = s$.

To see these, start with the triangle (30) defining $\breve{E}$ and $\breve{\sigma}$.

$$\begin{array}{ccccccc} c^*\mathbf{R}\breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}} & \longrightarrow & c^*\breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}}(\breve{A}) & \xrightarrow{c^*\breve{s}} & c^*\breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}}(\breve{A})|_{\breve{A}} & \xrightarrow{+1} & \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ \mathbf{R}\pi_{\mathfrak{U}*}(k^*\breve{\mathcal{L}}_{\mathfrak{U}}) & \longrightarrow & \pi_{\mathfrak{U}*}\Big((k^*\breve{\mathcal{L}}_{\mathfrak{U}}) \otimes \mathcal{O}_{\mathfrak{U}}(A)\Big) & \longrightarrow & (\pi_{\mathfrak{U}*}\Big((k^*\breve{\mathcal{L}}_{\mathfrak{U}}) \otimes \mathcal{O}_{\mathfrak{U}}(A)\Big)|_A & \xrightarrow{+1} & \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ \mathbf{R}\pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D})) & \longrightarrow & \pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D} + A)) & \longrightarrow & \pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(A))|_A & \xrightarrow{+1} & \end{array}$$

The first set of vertical isomorphisms are by cohomology and base-change and the fact that $k^*\breve{A} = A$. The following are given by $k^*\breve{\mathcal{L}} = \kappa^*\ell^*\breve{\mathcal{L}} = \kappa^*\kappa_*\mathfrak{L}(\delta\mathfrak{D}) \simeq \mathcal{L}(\delta\mathfrak{D})$ (see Remark 3.3.8).

By the requirements of our construction, A does not meet $\mathfrak{D}$. Then for the last term we have $\pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(A))|_A \cong \pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D} + A))|_A$. We conclude that $c^*(30)$ is isomorphic to the following triangle:

$$\mathbf{R}\pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D})) \to \pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D} + A)) \xrightarrow{c^*\breve{s}} \pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D} + A))|_A \xrightarrow{+1} . \tag{32}$$

Now we compare $c^*(30)$ =(32) to (26). Twisting by $\delta\mathfrak{D}$ induces a map between them

$$\begin{array}{ccccccc} \mathbf{R}\pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}} & \longrightarrow & \pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}}(A) & \xrightarrow{s} & \pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}}(A)|_A & \xrightarrow{+1} & \\ \downarrow & & \downarrow f & & \downarrow\cong & & \\ \mathbf{R}\pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D})) & \longrightarrow & \pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D} + A)) & \xrightarrow{c^*\breve{s}} & \pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D} + A))|_A & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{R}\pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D}))|_{\delta\mathfrak{D}} & \xrightarrow{\cong} & \pi_{\mathfrak{U}*}(\mathcal{L}_{\mathfrak{U}}(\delta\mathfrak{D} + A))|_{\delta\mathfrak{D}} & \longrightarrow & 0. & & \end{array}$$

The vertical map f above is as follows:

$$\widetilde{q} : \mathfrak{W} \xrightarrow{f} c^*\breve{\mathfrak{W}} \to \breve{\mathfrak{W}}.$$

This shows that $c^*\breve{\pi}_{\breve{\mathfrak{U}}*}\breve{\mathcal{L}}_{\breve{\mathfrak{U}}}(\breve{A})|_{\breve{A}} \cong \pi_{\mathfrak{U}*}\mathcal{L}_{\mathfrak{U}}(A)|_A$ and that $\breve{s} \circ \widetilde{q} = s$, completing the proof.

∎

Recall that we have that W is a DM stack and $\mathfrak{W}'$ and $\mathfrak{W}$ are Artin stacks such that

$$W \xhookrightarrow{\text{Zariski open}} \mathfrak{W}' \xrightarrow{\text{étale}} \mathfrak{W}.$$

We also have

$$\breve{W} \xhookrightarrow{\text{Zariski open}} \breve{\mathfrak{W}}' \xrightarrow{\text{étale}} \breve{\mathfrak{W}}$$

where $\breve{\mathfrak{W}}$ is a smooth Artin stacks and the étale map factors through an open Deligne–Mumford substack. Also, $\breve{W}$ and $\breve{\mathfrak{W}}'$ are smooth affine schemes.

From the definitions of W and $\breve{W}$ we see that $\widetilde{q}$ restricts to a map $q : W \to \breve{W}$.

Lemma 4.2.3. *We have a commutative diagram*

$$\begin{array}{ccc} W & \longrightarrow & \mathfrak{W} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \widetilde{q}} \\ \breve{W} & \longrightarrow & \breve{\mathfrak{W}}. \end{array}$$

Proof. We only need to check that the image of W is contained in $\breve{W}$. This follows by comparing the stability conditions in Construction 4.1.5 and Construction 4.1.3. ∎

Lemma 4.2.4. *We have that $\mathbb{R}V$ is an étale neighbourhood of ξ in $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.*

Proof. Recall that $\mathbb{R}V$ is defined by

$$\begin{array}{ccc} \mathbb{R}V & \xrightarrow{\ulcorner h} & W \\ \downarrow & & \downarrow{\scriptstyle \text{étale}} \\ \mathbb{R}U & \xrightarrow[m_A]{} & \mathfrak{W}. \end{array}$$

The first observation is that W is étale over $\mathfrak{W}$. Indeed, we have defined an étale neighborhood $\mathfrak{W}' \to \mathfrak{W}$ around ζ, the image of ξ under the morphism m_A induced by tensoring with $\mathcal{O}(A)$. In this, W is cut out in Construction 4.1.5 by imposing open stability conditions. Since the point ζ was the image of a stable point ξ, the stability conditions hold for it. So $W \to \mathfrak{W}$ is an étale neighborhood of ζ.

Thus, $\mathbb{R}V \to \mathbb{R}U$ is also étale, and $\xi \in \mathbb{R}V$. On the other hand, we have an open subset $\mathcal{V}$ of $\mathbb{R}U = \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{U}}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}_{\mathfrak{U}})$ given by

$$\mathcal{V} = \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \times_{\mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C})} \left(\mathbb{R}U \times_{\mathfrak{W}} \mathfrak{W}'\right),$$

that is fits in the cartesian diagram

$$\begin{array}{ccccc}
 & & \mathbb{R}V & \xrightarrow[\ulcorner h]{} & W \\
 & & \big\downarrow & & \big\downarrow \\
\mathcal{V} & \xhookrightarrow[\ulcorner h]{} & \mathbb{R}U \times_{\mathcal{W}} \mathcal{W}' & \xrightarrow[\ulcorner h]{} & \mathcal{W}' \\
\big\downarrow \text{ét.} & & \big\downarrow \text{ét.} & & \big\downarrow \text{ét.} \\
 & & \mathbb{R}U & \longrightarrow & \mathcal{W} \\
 & & \big\downarrow & & \\
\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) & \hookrightarrow & \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C}). & &
\end{array}$$

We want to show that $\mathbb{R}V$ and $\mathcal{V}$ are equivalent. First, we show that their truncations are isomorphic, that is $t_0(\mathcal{V}) =: V' \simeq t_0(\mathbb{R}V) = V$. It suffices to show that the conditions of Construction 4.1.5 are equivalent to the stability conditions of stable maps on points in the image of (the truncation of) m_A. We recall them here for the reader's convenience. At any point $(C, L, s_0, \dots, s_r)$ of $U \times_{\mathcal{W}} \mathcal{W}'$ Definition 2.3.3 state it is in V' iff the following hold:

(1) the bundle $\omega_C^{\log} \otimes L^{\otimes 3}$ is ample and
(2) the linear system $(L, s_0, \dots, s_r)$ has no base points.

On the other hand, at any point $(C, L(A), w_0, \dots, w_r)$ of $\mathcal{W}'$, Construction 4.1.5 states it is in W iff:

(i) the base locus $BL(w)$ of $w_0, \dots, w_r$ is discrete and disjoint from the special points of C,
(ii) the subset $BL(w)_L$ of the base locus is empty and
(iii) the line bundle $\omega_C^{\log} \otimes L^{\otimes 3}$ is ample.

Conditions (1) and (iii) are clearly equivalent. We want to show that for points in $m_A(U)$ condition (ii) implies condition (2), and that (2) holding for points of U implies (i) and (ii), that is: their image under m_A lies in W.

Let $(C, L, s_0, \dots, s_r)$ be a point in U,

$$m_A(C, L, s_0, \dots, s_r) = (C, L(A), w_0, \dots, w_r)$$

where w_i is the image of s_i under $H^0(C, L) \to H^0(C, L(A))$. Condition (ii) implies that the $BL(w) \subset A$. On the other hand, $\breve{A}$ and $\breve{\mathfrak{U}}$ were chosen so that $BL(\breve{s})$ does not intersect $\breve{A}$ on the open $\breve{U}$, then also $BL(s)$ does not intersect A in U. Then we see that $BL(s)$ must be empty.

Conversely, if $BL(s)$ is empty, $BL(w)$ must be contained in A, which implies both condition (i) and (ii).

We have that $V \simeq V' = \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \times_{\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C})} (U \times_{\mathcal{W}} \mathcal{W}')$.

Now the two maps $\mathcal{V} \to \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C})$ and $\mathbb{R}V \to \mathbb{R}\underline{\mathrm{Sec}}_{\mathfrak{Pic}^s}(\mathfrak{L}^{\oplus r+1}/\mathfrak{C})$ are étale maps having the same truncation, so by [TV05, Corollary 2.2.2.9] $\mathcal{V}$ and $\mathbb{R}V$ are equivalent. □

Lemma 4.2.5. *We have that $\mathbb{R}\breve{V}$ is a neighbourhood of $\breve{\xi}$ in $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$.*

Proof. This is similar to the proof of 4.2.4.

□

Lemma 4.2.6. *We have a homotopical cartesian diagram*

$$\begin{array}{ccc} \mathbb{R}V & \xrightarrow{\ulcorner h} & W \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{R}\check{V} & \longrightarrow & \breve{W}. \end{array}$$

Proof. We denote the restriction of (E, σ) to W by (F, θ). By Proposition 4.2.1 we had $\mathbb{R}U = \mathbb{R}Z^h(\sigma)$. Since $\mathbb{R}V = \mathbb{R}U \times_{\mathcal{U}} W$, we have

$$\mathbb{R}V = \mathbb{R}Z^h(\theta). \tag{33}$$

By Lemma 4.2.3 and Lemma 4.2.2 the restriction of $(\check{E}, \check{\sigma})$ from $\check{\mathcal{U}}$ to $\breve{W}$ is $(q^*F, q^*\theta)$. From Proposition 4.2.1 we had that $\mathbb{R}\check{U} = \mathbb{R}Z^h(\check{\sigma})$. Then by the definition of $\mathbb{R}\check{V}$ we have

$$\mathbb{R}\check{V} = \mathbb{R}Z^h(q^*\theta). \tag{34}$$

□

5. Main theorem

We are now ready to prove our main theorem on the derived push-forward of the structure sheaf of $\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. Just recall that contracting rational tails gives a morphism

$$\overline{c} : \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$$

To prove our main theorem (See Theorem 5.2.1) that is

$$\overline{c}_* \mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)} = \mathcal{O}_{\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)}.$$

it is enough to do it locally. That's why §4 is useful as we have a local picture for $\overline{c}$. In Section 5.1, we will prove that $q : W \to \breve{W}$ is proper and birational (See Proposition 5.1.1. In 5.2, we use the Zariski Main theorem to prove that

$$q_* \mathcal{O}_W = \mathcal{O}_{\breve{W}} \tag{35}$$

(see proof of Lemma 5.2.6). Then by cohomology and base change, we prove our main theorem (see Theorem 5.2.1).

5.1. **Properness of q.** Recall that W is a smooth DM stack and $\breve{W}$ is smooth affine scheme of finite type.

Proposition 5.1.1. *The morphism $q : W \to \breve{W}$ is proper and birational.*

Proof. Birationality follows from the fact that $\widetilde{q} : \mathcal{U} \to \check{\mathcal{U}}$ is birational and W, $\breve{W}$ are open subsets of $\mathcal{U}$ and $\check{\mathcal{U}}$ respectively.

We use the valuative criterion to prove properness. Let R be a valuation ring with K its field of fractions.

Consider the following diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\varphi^\circ} & W \\ \downarrow & \overset{\exists !}{\nearrow} & \downarrow{\scriptstyle q} \\ \operatorname{Spec} R & \xrightarrow[\check{\varphi}]{} & \breve{W}. \end{array} \tag{36}$$

The morphism $\breve{\varphi}$ above is given by a family $\check{\mathcal{C}} \to \mathrm{Spec}R$, together with a line bundle $\check{\mathcal{G}}$ and sections $(\breve{w}_0, \dots, \breve{w}_r)$. We denote by $\check{\mathcal{C}}^\circ$ the restriction of $\check{\mathcal{C}}$ to $\mathrm{Spec}K$. The morphism φ° gives a family $(\mathcal{C}^\circ, \mathcal{G}^\circ, w_0^\circ, \dots, w_r^\circ)$ such that

$$q(\mathcal{C}^\circ, \mathcal{G}^\circ, w_0^\circ, \dots, w_r^\circ) = (\check{\mathcal{C}}^\circ, \check{\mathcal{G}}^\circ, \breve{w}_0^\circ, \dots, \breve{w}_r^\circ).$$

Here, by abuse of notation we denoted by q the map induced by $q : W \to \breve{W}$.

In the following we show that there exists a unique morphism φ which extends φ° and makes diagram (36) commute. In concrete terms, this amounts to finding $(\mathcal{C}, \mathcal{G}, w_0, \dots, w_r)$ a family over $\mathrm{Spec}R$ which extends $(\mathcal{C}^\circ, \mathcal{G}^\circ, w_0^\circ, \dots, w_r^\circ)$ and such that

$$q(\mathcal{C}, \mathcal{G}, w_0, \dots, w_r) = (\check{\mathcal{C}}, \check{\mathcal{G}}, \breve{w}_0, \dots, \breve{w}_r).$$

Existence.

By definition, $\breve{W}$ parameterises tuples $(\check{C}, \check{L}(\check{A}), \breve{w}_0, \dots \breve{w}_r)$, subject to the non-degeneracy condition in 4.1.3. This shows that $\breve{W}$ is a subset of $\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d+a)$.

Let $\widetilde{W}$ be the fibre product

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\quad\ulcorner\quad} & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d+a) \\ \big\downarrow & & \big\downarrow \bar{c} \\ \breve{W} & \longrightarrow & \overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d+a). \end{array}$$

In the following we construct a morphism $\widetilde{W} \to W$ such that $\widetilde{W} \to \breve{W}$ factors through $\widetilde{W} \to W$. The construction is the one in Theorem 3.3.10 with minor modifications. Let $(\tilde{\mathfrak{C}}, \tilde{\mathfrak{G}})$ denote the universal curve and universal bundle on $\mathfrak{Pic}^{\mathrm{s}}{}_{d+a}$. We have

$$\widetilde{W} \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}{}_{d+a}}(\tilde{\mathfrak{G}}^{\oplus r+1}/\tilde{\mathfrak{C}})$$
$$W \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}{}_{d}}(\mathfrak{L}(A)^{\oplus r+1}/\mathfrak{C}) = W \subset \underline{\mathrm{Sec}}_{\mathfrak{Pic}^{\mathrm{s}}{}_{d}}(\mathfrak{G}^{\oplus r+1}/\mathfrak{C})$$
$$\breve{W} \subset \underline{\mathrm{Sec}}_{\breve{\mathfrak{Pic}}^{\mathrm{s}}{}_{d}}(\check{\mathfrak{L}}(\check{A})^{\oplus r+1}/\check{\mathfrak{C}}) = \underline{\mathrm{Sec}}_{\breve{\mathfrak{Pic}}^{\mathrm{s}}{}_{d+a}}(\check{\mathfrak{G}}_{d+a}^{\oplus r+1}/\check{\mathfrak{C}}_{d+a}) \simeq \underline{\mathrm{Sec}}_{\breve{\mathfrak{Pic}}^{\mathrm{s}}{}_{d+a}}(\check{\mathfrak{L}}_{d+a}^{\oplus r+1}/\check{\mathfrak{C}}_{d+a})$$

where $\check{\mathfrak{C}}_{d+a}, \check{\mathfrak{L}}_{d+a}$ are the universal curve and line bundle over $\breve{\mathfrak{Pic}}^{\mathrm{s}}{}_{d+a} \cong \breve{\mathfrak{Pic}}^{\mathrm{s}}{}_{d}$. The isomorphism $\breve{\mathfrak{Pic}}^{\mathrm{s}}{}_{d} \to \breve{\mathfrak{Pic}}^{\mathrm{s}}{}_{d+a}$ is given by $(\check{C}, \check{L}) \mapsto (\check{C}, \check{L} \otimes \mathcal{O}_{\check{C}}(\check{A}))$.

Claim. Let $(\widetilde{C}, \widetilde{G}, \widetilde{w}) \in \widetilde{W}$ and let RT be a rational tail of $\widetilde{C}$. We have

$$RT = RT_{\tilde{\mathcal{L}}} \sqcup RT_{\tilde{A}}.$$

The claim follows from the fact that $\underline{\widetilde{w}} = \underline{\breve{w}}$ outside the exceptional locus of $\tilde{\mathcal{C}} \to \check{\mathcal{C}}$ and the fact that $BL(\underline{w}) = BL(\underline{w})_L \sqcup BL(\underline{w})_A$. We need to show that the labelling on $\breve{W}$ lifts to a labelling of the rational tails of the universal curve of $\widetilde{W}$. We have that $p : \tilde{\mathfrak{C}} \to \mathfrak{C}$ contracts rational tails. Since the base loci of w_L and w_A are disjoint we get that the base loci of $p^{-1}w_L$ and $p^{-1}w_{\mathcal{A}}$ are disjoint. Moreover, since the base loci of w_L and w_A form disconnected components, the same holds about their inverse images. This proves the claim.

By possibly shrinking $\breve{W}$ and changing the basis of $\mathbb{P}^r$, we have a divisor $Z(\underline{\tilde{w}})_{\tilde{A}}$ on $\tilde{\mathcal{C}}$, which we denote by $\widetilde{A}$. Let $\tilde{\mathcal{L}}$ denote $\tilde{\mathcal{G}} \otimes \mathcal{O}(-\widetilde{A})$.

Let S be a scheme. In the following we contract rational tails of $\widetilde{\mathcal{C}}_S$ which intersect $\widetilde{A}$. Let D_A be the divisor of $\widetilde{\mathcal{C}}_S$, which consists of rational tails $RT_{\widetilde{A}}$ and let δ_A be the degree of $\widetilde{\mathcal{L}}_S$ restricted to the tails. We have that $\mathcal{L}'_S := \widetilde{\mathcal{L}}_S(\delta_A D_A)$ is trivial along the exceptional locus D_A and base point free. Let us define

$$\mathcal{C} = \operatorname{Proj} \sum_n H^0(\widetilde{\mathcal{C}}_S, (\mathcal{L}'_S)^{\otimes n}).$$

Let $\kappa : \widetilde{\mathcal{C}}_S \to \mathcal{C}_S$ and let $\mathcal{L}_S := \kappa_* \mathcal{L}'_S$. Since $\mathcal{L}'_S$ is trivial along D_A, Lemma 7.1 in [PR03] implies that $\mathcal{L}_S$ is a line bundle. In the same way as we did in the proof of Theorem 3.3.10, we construct $(w_0, \ldots, w_r)$ sections of $\mathcal{L}$. We thus obtain a surjective morphism $\widetilde{W} \to W$. It can be seen that $\widetilde{W} \to \widecheck{W}$ factors through $\widetilde{W} \to W$.

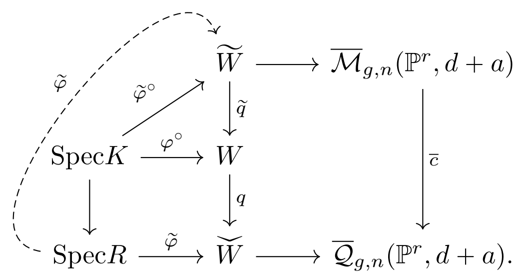
Since $\widetilde{W} \to W$ is surjective, there exists a (non unique) family of maps

$$(\widetilde{\mathcal{C}}^\circ, \widetilde{\mathcal{G}}^\circ, \widetilde{w}_0^\circ, \ldots, \widetilde{w}_r^\circ) \in \widetilde{W}$$

such that

$$q \circ \widetilde{q}(\widetilde{\mathcal{C}}^\circ, \widetilde{\mathcal{G}}^\circ, \widetilde{w}_0^\circ, \ldots, \widetilde{w}_r^\circ) = (\mathcal{C}^\circ, \mathcal{G}^\circ, w_0^\circ, \ldots, w_r^\circ).$$

Equivalently, we have a family $\widetilde{\varphi}^\circ : \operatorname{Spec} K \to \widetilde{W}$ which commutes with φ°. Hence we have the following diagram



Since $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is proper, we have that $\overline{c}$ is proper. This implies that $\widetilde{W} \to \widecheck{W}$ is proper. This shows that $\widetilde{\varphi}^\circ$ extends (uniquely) to $\widetilde{\varphi} : \operatorname{Spec} R \to \widetilde{W}$, and thus the morphism $\widetilde{q} \circ \widetilde{\varphi} : \operatorname{Spec} R \to W$ proves the existence.

we consider the image of $(\widetilde{\mathcal{C}}, \widetilde{\mathcal{L}}, \widetilde{w}_0, \ldots \widetilde{w}_r)$ in W.

Uniqueness. In notations as before, we have morphisms $\widetilde{W} \to W \to \widecheck{W}$. We have that $\widetilde{W} \to \widecheck{W}$ is separated, because by construction it is proper. The map $\widetilde{W} \to W$ is surjective and proper by the discussion above. With this, we are under the assumptions of [Sta22, Tag 09MQ]. This shows that $W \to \widecheck{W}$ is separated. □

As W is a smooth DM stacks, we denote its coarse moduli space by $|W|$. Recall that $\widecheck{W}$ is a smooth scheme and that we have a morphism $q : W \to \widecheck{W}$.

Lemma 5.1.2. *The morphism $|q| : |W| \to \widecheck{W}$ is projective.*

Proof. In the following we show that $|W|$ is projective. This implies that the morphism $|q| : |W| \to \widecheck{W}$ is projective.

Recall that W is open in a DM stack $\pi_* \mathfrak{L}(A)$ defined by the following stability conditions, for a point $(C, L(A), w_0, \ldots w_r)$

(1) the bundle $\omega_C^{\log} \otimes L^{\otimes 3}$ is ample
(2) The base locus $BL(w) = \bigcap_{i=0}^r Z(w_i)$ has dimension 0 and is distinct from marked points and nodes.

To show projectivity, we can follow the proof of [Cor95]. We sketch here the necessary modifications, trying to adhere to the notation of the original proof as much as possible.

A family $F : S \to W$ consists of a pre-stable curve $\pi_S : C_S \to S$ with n marked points $(x_1, \dots x_n) : S \to C_S$, a distinguished divisor A_S of degree a, a line bundle L_S of degree d and sections $(w_0, \dots w_r)$ of $L_S(A_S)$. We can define define a line bundle on S by

$$\mathcal{V}_F = \langle \omega_{\mathfrak{C}_S}^{\log} \otimes \mathcal{L}_S(A_S)^{\otimes 3}, \omega_{\mathfrak{C}_S}^{\log} \otimes \mathcal{L}_S(A_S)^{\otimes 3} \rangle$$

using Deligne's bilinear pairing, explicitly for $V_F = \omega_{\mathfrak{C}_S}^{\log} \otimes \mathcal{L}_S(A_S)^{\otimes 3}$, we have

$$\mathcal{V}_F = \det \mathbf{R}\pi_{S*}\mathcal{O}_S \otimes (\det \mathbf{R}\pi_{S*}V_F)^{\otimes -2} \otimes \det \mathbf{R}\pi_{S*}(V_F \otimes V_F).$$

We want to show this bundle $\mathcal{V}_F$ is ample. Following Cornalba's approach, which relies on Seshadri's criterion, it suffices to show that there exists a constant $\alpha = \alpha(g, n, r, d) > 0$ such that for any non-isotrivial family F over an integral complete curve S, since we have already proved that $|q| : |W| \to \breve{W}$ is proper.

$$(V_F \cdot V_F) \geqslant \alpha m(S) \tag{37}$$

where $m(S)$ denotes the maximum multiplicity of points in S.

Since the number of nodes of the curve C_S is bounded in terms of (g, n, d, r) for any family, we may reduce to the case of a family F whose generic curve is smooth, as in the original proof. Now the idea is to add marked points to C_S to obtain a stable domain curve. Since we do not have a well-defined map to $\mathbb{P}^r$, we can use the sections to add $3(d + a)$ marked points. Indeed, by taking linear combinations of the sections $(w_0, \dots, w_r)$ we may assume that we have a linearly independent set (w_0, w_1, w_2) such that for $i \in \{0, 1, 2\}$ the following conditions hold (c.f. [Cor95, Lemma 2] note that our condition (iii) is equivalent to (ii), (iii) and (v)):

(i) $Z(w_i)$ does not contain components of the fiber of π_S
(ii) $Z(w_i)$ does not contain x_j for $j = 1, \dots, n$
(iii) $Z(w_i)$ consists of $d + a$ distinct, non-special points on all the fibers of π_S which are singular or lie over singular points of S.

We take

$$\begin{aligned} Z(w_0) &= x_{n+1} \cdots x_{n+d+a} \\ Z(w_1) &= x_{n+d+a+1} \cdots x_{n+2(d+a)} \\ Z(w_2) &= x_{n+2(d+a)+1} \cdots x_{n+3(d+a)} \end{aligned}$$

where we may assume, up to some finite base change of bounded degree, that $(x_{n+1}, \dots, x_{n+3(d+a)})$ are distinct as sections of π_S and distinct from the original sections $(x_1, \dots, x_n)$. Now, on smooth fibers of π_S, some of the x_i's may still meet, indeed they will if the w's defined a linear system with non-empty base-locus. We may proceed to resolve them as in [Cor95, Proof of Lemma 2] and obtain a family of stable curves

$$F' = \left\{ C'_S \to S, x'_1, \dots, x'_{n+3(d+a)} \right\}$$

with

$$(V_F \cdot V_F) = (\omega_{C_S}(D) \cdot \omega_{C_S}(D)) \geqslant (\omega_{C'_S}(D') \cdot \omega_{C'_S}(D')). \tag{38}$$

where $D = \sum_{i=1}^{n+3(d+a)} x_i$ and $D' = \sum_{i=1}^{n+3(d+a)} x'_i$. We may assume F' is a non-isotrivial family, otherwise we proceed as in [Cor95, Lemma 3]. Now, (S, C'_S, D') is a non-isotrivial stable family, so $\kappa_1 = \pi_{S'*}(\omega_{C'_S}(D')^{\otimes 2})$ is ample on S, thus by Seshadri's criterion and (38) we have the required α to conclude that (37) holds.

□

5.2. **Derived push-forward.** In this subsection, we will prove the main theorem of this paper that is:

Theorem 5.2.1. *For any, g, n and d, we have that*

$$\overline{c}_* \mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)} = \mathcal{O}_{\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)} \text{ in } \mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathbb{R}\overline{\mathcal{Q}}(\mathbb{P}^r, d)).$$

Remark 5.2.2. At the level of virtual classes, we have that

$$\overline{c}_*[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} = [\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}}.$$

This was proven in [CFK10], [MOP11] and [Man14].

We deduce the following corollary.

Corollary 5.2.3. *The G-theoretic Gromov-Witten invariants and the G-theoretic quasimaps invariants are equal.*

Remark 5.2.4. Let X be a Noetherian derived Artin stack. Recall that by definition (cf. [Kha22] for this definition for derived stacks)

$$K(X) := K(\mathrm{Perf}(X)) \text{ and } G(X) := K(\mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(X)).$$

If X is smooth, $\mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(X)$ and $\mathrm{Perf}(X)$ coincide. When X is a scheme, Lee (see [Lee04]) denotes them respectively $K^\circ(X)$ and $K_\circ(X)$. Our G-theoretic Gromov–Witten invariants are often called K-theoretic invariants by other authors.

We prove the theorem by using the étale neighborhoods $\mathbb{R}V$ and $\mathbb{R}\breve{V}$ constructed in the previous section. With Lemma 4.2.6 in mind, we first want to study the morphism $q : W \to \breve{W}$.

Proposition 5.2.5. *We have*

$$\mathbf{R}^0 q_* \mathcal{O}_W = \mathcal{O}_{\breve{W}} \text{ in } \mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(\breve{W}) \tag{39}$$

$$\mathbf{R}^i q_* \mathcal{O}_W = 0 \text{ for } i > 0. \tag{40}$$

Proof of Proposition 5.2.5. Recall that W is a smooth DM stack. Denote its coarse moduli space by $W \xrightarrow{\alpha} |W|$. The scheme $|W|$ is normal with rational singularities (see [Vie77], Proposition 1), since it is locally the quotient of a smooth scheme by a finite group. Since $\breve{W}$ is a (smooth) scheme, q factors as $W \xrightarrow{\alpha} |W| \xrightarrow{|q|} \breve{W}$, with α a finite morphism and $|q|$ a projective birational morphism. As $|W|$ is also a good moduli space (see [Alp13] or [AOV08]), we have

$$\mathbf{R}^0 \alpha_* \mathcal{O}_W = \mathcal{O}_{|W|} \text{ in } \mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(|W|) \tag{41}$$

$$\mathbf{R}^i \alpha_* \mathcal{O}_W = 0 \text{ for } i > 0.$$

Now $|W|$ admits, by Hironaka's work [Hir64], a projective resolution of singularities and since the singularities were rational we have that

$$p : A \to |W|$$

with

$$\begin{aligned} \mathbf{R}^0 p_* \mathcal{O}_A &= \mathcal{O}_{\breve{W}} \text{ in } \mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(\breve{W}) \\ \mathbf{R}^i p_* \mathcal{O}_A &= 0 \text{ for } i > 0. \end{aligned} \tag{42}$$

The composite map

$$f : A \xrightarrow{p} |W| \xrightarrow{|q|} \breve{W}$$

is a projective birational map between smooth schemes, then by [CR15, Theorem 1.1] we have

$$\begin{aligned} \mathbf{R}^0 f_* \mathcal{O}_A &= \mathcal{O}_{\breve{W}} \text{ in } \mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(\breve{W}) \\ \mathbf{R}^i f_* \mathcal{O}_A &= 0 \text{ for } i > 0. \end{aligned} \tag{43}$$

The relative Leray spectral sequence, defined by

$$E_2^{i,j} = \mathbf{R}^i |q|_* (\mathbf{R}^j p_* \mathcal{O}_A)$$

converges to $\mathbf{R}^{i+j} f_* \mathcal{O}_A$. By eq. (42), the spectral sequence degenerates on the second page and

$$\mathbf{R}^i f_* \mathcal{O}_A = \mathbf{R}^i |q|_* (\mathbf{R}^0 p_* \mathcal{O}_A) = \mathbf{R}^i |q|_* \mathcal{O}_{|W|}$$

and the result follows from combining this with eq. (43) and eq. (41). $\square$

Recall from Lemma 4.2.6, we have the homotopically Cartesian diagram

$$\begin{array}{ccc} \mathbb{R}V & \xrightarrow{\;i\;} & W \\ {\scriptstyle \bar{c}}\downarrow & \ulcorner h & \downarrow {\scriptstyle q} \\ \mathbb{R}\breve{V} & \xrightarrow{\;\breve{i}\;} & \breve{W}. \end{array} \tag{44}$$

Lemma 5.2.6. *We have* $\mathbf{R}\bar{c}_* \mathcal{O}_{\mathbb{R}V} = \mathcal{O}_{\mathbb{R}\breve{V}}$.

Proof. This follows from derived base change, which works by Lemma A.1.3 in [HLP23] as $\breve{i}$ is of finite Tor amplitude thanks to Proposition 4.2.1 and $\mathcal{O}_W$ is cohomologically bounded below as W is smooth. We thus get:

$$\begin{aligned} \mathbf{R}\bar{c}_* \mathcal{O}_{\mathbb{R}V} &= \mathbf{R}\bar{c}_* \mathbf{L} i^* \mathcal{O}_W \\ &= \mathbf{L}\breve{i}^* \mathbf{R} q_* \mathcal{O}_W \\ &= \mathbf{L}\breve{i}^* \mathcal{O}_{\breve{W}} \text{ by (39) and (40)} \\ &= \mathcal{O}_{\mathbb{R}\breve{V}}. \end{aligned}$$

$\square$

Proof of Theorem 5.2.1. The morphism $\bar{c} : \mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r, d)$. Gives a morphism of structure sheaves

$$c : \mathcal{O}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)} \to \bar{c}_* \mathcal{O}_{\mathbb{R}\overline{\mathcal{Q}}_{g,n}(\mathbb{P}^r,d)}.$$

To prove that it is an isomorphism, it is enough to prove it étale locally. That's exactly what we have done in §4. Hence we are in the situation of diagram (44), the Lemma 5.2.6 finishes the proof. $\square$

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