Determinants of Catalan matrices from the cohomology of $G(2, n+1)$

Friedrich HIRZEBRUCH, Claus HERTLING and Etienne MANN

January 2003

Notes of discussions

Max-Planck-Institut für Mathematik Bonn

Contents

These notes are the results of some talks between Friedrich Hirzebruch, Claus Hertling and Etienne Mann in the Max Planck Institute of Mathematics during december 2002 and january 2003.

The second section is devoted to compute the determinant of a matrix composed of Catalan numbers. In order to compute it, we see this matrix as an intersection matrix in the cohomology ring of the Grassmannian $G(2, n+1)$ and then we compute this matrix in another base of the cohomology where the determinant might be easier to compute. This part is based on an old article of Schubert [2] and on a more modern approach in the book [1]. Section 3 is based on the quantum cohomology of the complex projective space. In this part, we show that the WDVV equations for degree 1 are equivalent to the fundamental relations of the first part. Moreover we prove some new formulas.

1 Cohomology of the Grassmannian $G(2, n+1)$

In this section, we express the matrix of the intersection form in a suitable base of the cohomology ring of the Grassmannian $G(2, n+1)$.

1.1 Definitions and notations

We consider the Grassmannian $G(2, n+1)$, which has the complex dimension $2n-2$. We denote the cohomology ring of $G(2, n+1)$ with coefficients in $\mathbb{Z}$ as $A^\star$, i.e. $A^d := H^{2d}(G(2, n+1), \mathbb{Z})$.

Definition 1.1.

1. Let a and b be two integers such that $0 \leq b < a \leq n$. Let $L_1 \subset L_2 \subset \mathbb{C}^{n+1}$ be two hyperplanes of complex codimension a and b. We denote by (a, b) the cohomology class in A^* which corresponds to the subvariety $\{\Lambda \in G(2, n+1)$ such that $\Lambda \cap L_1 \neq \emptyset$ and $\Lambda \subset L_2\}$. We call it a Schubert cocycle.
2. If a and b satisfy the condition $0 \leq a < b \leq n$ then $(a, b) := -(b, a)$.
3. If a or $b \in \mathbb{Z} - \{0, \ldots, n\}$ or $a = b$ then $(a, b) := 0$.
4. We define $(a) := (a, 0)$.

The following is well known.

Proposition 1.2 (e.g. [1] p.196). *$(a, b) \in A^{a+b-1}$ and $A^\star$ is a free $\mathbb{Z}$-module with base the Schubert cocycles (a, b) with $0 \leq b < a \leq n$.*

Remark 1.3. (1) (resp. $(n, n-1)$) generates A^0 (resp. A^{2n-2}).

1.2 The product and the intersection form on the cohomology ring

Proposition 1.4 (Pieri's formula, [1] p 203). *Let $a, b, \beta \in \{0, \ldots, n\}$ with $b > \beta$. Then*

$$(a)(b,\beta) = \sum_{k=0}^{b-\beta-1} (a+b-1-k, \beta+k) \tag{1}$$

Corollary 1.5 (Giambelli's formula, [1] p 205). *Let $a, b \in \{0, \ldots, n\}$.Then*

$$(a,b) = (a)(b+1) - (a+1)(b) \tag{2}$$

Proof. It is an obvious consequence of the formula (1) :

$$\begin{aligned}(a)(b+1) - (a+1)(b) &= \sum_{k=0}^{b}(a+b-1-k,k) - \sum_{k=0}^{b-1}(a+b-k,k) \\ &= (a,b)\end{aligned}$$

□

Finally we get the formula which is the main general result in Schubert's article [2] (§4).

Corollary 1.6 (Schubert's formula, [2] §2). *Let $a, b, \alpha, \beta \in \{0, \ldots, n\}$ with $b > \beta$. We have :*

$$(a,\alpha)(b,\beta) = \sum_{k=0}^{b-\beta-1} (a+b-1-k, \alpha+\beta+k) \tag{3}$$

Remark 1.7. As special cases we obtain the following formulas. In fact, they hold for all $b, \beta \in \mathbb{Z}_{\geq 0}$.

- For $\alpha = 0$ and $a = 2$:

$$(2)(b,\beta) = (b+1,\beta) + (b,\beta+1) \tag{4}$$

- For $a = 2$ and $\alpha = 1$:

$$(2,1)(b,\beta) = (b+1,\beta+1) \tag{5}$$

Proof. By using (2) and (1), one has :

$$\begin{aligned}
(a,\alpha)(b,\beta) &= ((a)(\alpha+1)-(a+1)(\alpha))(b,\beta) \\
&= \sum_{k=0}^{b-\beta-1}\sum_{l=0}^{b+\alpha-\beta-2k-1}(a+b+\alpha-k-l-1,\beta+k+l) \\
&\quad -\sum_{k=0}^{b-\beta-1}\sum_{l=0}^{b+\alpha-\beta-2k-2}(a+b+\alpha-k-l-1,\beta+k+l) \\
&= \sum_{k=0}^{b-\beta-1}(a+\beta+k,b+\alpha-k-1) \\
&= \sum_{l=0}^{b-\beta-1}(a+b+-l-1,\beta+\alpha+l)
\end{aligned}$$

□

The next proposition explains what happens when the product of two Schubert cocycles is in the top cohomology group.

Proposition 1.8. *Let $a,\alpha,b,\beta\in\{0,\dots,n\}$ such that $a>\alpha$, $b>\beta$ and $a+\alpha+b+\beta=2n$. Then the product $(a,\alpha)(b,\beta)$ is in A^{2n-2} and we have :*

$$(a,\alpha)(b,\beta)=\begin{cases}(n,n-1) & \text{if } a+\beta=b+\alpha=n\\ 0 & \text{else}\end{cases}$$

Proof. We will use the formula (3) :

$$(a,\alpha)(b,\beta)=(a+b-1,\alpha+\beta)+(a+b-2,\alpha+\beta-1)+\ldots+(a+\beta,\alpha+b-1)$$

As $a+b+\alpha+\beta=2n$, there are three possibilities :

- either $a+\beta>n$, then $(a,\alpha)(b,\beta)=0$ because the terms on the right hand side all vanish (see definition 1.1).
- or $\alpha+b>n$, then $(a,\alpha)(b,\beta)=0$ for the same reason.
- or $a+\beta=\alpha+b=n$. Then the only term which does not vanish is the last one, i.e. $(n,n-1)$.

□

Now we will compute the intersection form in the cohomology ring in a suitable base $\underline{e}$.

We denote by $[x]$ the integer part of $x\in\mathbb{R}$.

Let $d\in\{0,\dots,n-1\}$. Let $\underline{e}:=(e_0,\dots,e_{[\frac{d}{2}]})$ a base of A^d where $e_i:=(d+1-i,i)$. Let $\underline{E}:=(E_0,\dots,E_{[\frac{d}{2}]})$ a base of A^{2n-2-d} where $E_i:=(n-j,n-1-d+j)$.

We consider the intersection form :

$$\begin{array}{rccc} I: & A^d \times A^{2n-2-d} & \longrightarrow & \mathbb{Z} \\ & ((a,b),(c,d)) & \longmapsto & \frac{(a,b)(c,d)}{(n,n-1)} \end{array}$$

Corollary 1.9. *The matrix of the intersection form in the bases* $\underline{e}$ *and* $\underline{E}$ *is the identity matrix of rank* $[\frac{d}{2}]+1$.

Remark 1.10. If $d=n-1$, we have $\underline{e}=\underline{E}$. So the intersection matrix for $\underline{e}$ is the identity matrix.

Proof. The matrix of the intersection form is the matrix of coefficients $e_i.E_j$ for $i,j \in \{0,\ldots,[\frac{d}{2}]\}$. Then we apply the proposition 1.8, so $e_i.E_j = (d+1-i,i)(n-j,n-1-d+j) = \delta_{i,j}$. □

2 About determinants of Catalan matrices

2.1 A Catalan triangle

In this section, first we will describe a Catalan triangle and then we will show where these numbers appear in the cohomology ring of the Grassmannian.

We start with a description of a Catalan triangle :

$i\backslash j$	0	1	2	3	4	5	6
0	1	0					
1	1	1	0				
2	1	2	2	0			
3	1	3	5	5	0		
4	1	4	9	14	14	0	
5	1	5	14	28	42	42	0

For $j-i \leq 1$, we denote $c(i,j)$ the number in the ith line and in the jth row.

Proposition 2.1. *Let i and j be two integers such that $j-i \leq 1$.*

- *This triangle is characterized by the following relations :*

$$c(i,j) = c(i-1,j) + c(i,j-1),\ c(i,i+1) = 0 \textit{ and } c(i,0) = 1$$

- *We have* $c(i,j) = \binom{i+j}{i}\frac{i-j+1}{i+1} = \binom{i+j}{j} - \binom{i+j}{j-1}$.

Remark 2.2. • The number $c(i,i) = c(i,i-1)$ is the ith Catalan number, that is $\binom{2i}{i}\frac{1}{i+1}$, and we denote it by C_i.

- Schubert proved in his article [2] in §5 the following relation :

$$(i,j)(2)^{2n-i-j-1} = c(n-j-1, n-i)\,(n, n-1) \tag{6}$$

It is a consequence of formula (4) and of the definition of the Catalan triangle numbers. So if $j = 0$, one has :

$$(i)(2)^{2n-i-1}/(n, n-1) = c(n-1, n-i) = \binom{2n-i-1}{n-1}\frac{i-1}{n} \tag{7}$$

And if $i = 1$, one has :

$$(2)^{2n-2}/(n, n-1) = c(n-1, n) = c(n, n) = C_n.$$

The next lemma shows where these numbers appear in the computation of $(2)^k$ in the cohomology ring.

Lemma 2.3. *For any $k \in \mathbb{N}$, we have the following formula :*

$$\begin{aligned}(2)^k &= c(k,0)(k+1) + c(k-1,1)(k,1) + \ldots + \begin{cases} c(\frac{k}{2}, \frac{k}{2})(\frac{k+2}{2}, \frac{k}{2}) & \text{if } k \text{ is even} \\ c(\frac{k+1}{2}, \frac{k-1}{2})(\frac{k+3}{2}, \frac{k-1}{2}) & \text{if } k \text{ is odd} \end{cases} \\ &= c(k,0)(k+1) + c(k-1,1)(k,1) + \ldots + c([\frac{k+1}{2}], [\frac{k}{2}])([\frac{k+3}{2}], [\frac{k}{2}]).\end{aligned}$$

Proof. We will write the proof only for k even. The proof is based on formula (4) and on theorem 2.1. By an induction, it is obvious that :

1. $(2)^k = a_{k,0}(k+1) + a_{k,1}(k,1) + \ldots + a_{k,\frac{k}{2}}(\frac{k}{2}+1, \frac{k}{2})$
2. $a_{k,0} = 1$.
3. $a_{k,\frac{k}{2}+1} = 0$.
4. $a_{k,i} = a_{k-1,i} + a_{k-1,i-1}$

This implies that $a_{k,j} = c(k-j, j)$. □

2.2 The matrix of the intersection form

We will use a different base of the cohomology ring and compare the matrices of the intersection form.

Let $(a, b) \in A^{a+b-1}$. We can prove the following theorem.

Theorem 2.4. *Let d be an integer in $\{0, \ldots, n-1\}$.*

(i) $\underline{f} := \left(f_0, \ldots, f_{[\frac{d}{2}]}\right)$ *where* $f_i := (2)^{d-2i}(2,1)^i$ *is a base of* A^d *and we have the relation that* $\underline{f} = \underline{e}B$.

(ii) $\underline{F} := \left(F_0, \ldots, F_{[\frac{d}{2}]}\right)$ *where* $F_j := (2)^{d-2j}(2,1)^{n-1-d+j}$ *is a base of* A^{2n-2-d} *and we have the relation that* $\underline{F} = \underline{E}B$.

Here B *is the following lower triangular matrix :*

$$\begin{pmatrix} c(d,0) & & & \mathbf{0} \\ c(d-1,1) & c(d-2,0) & & \\ \vdots & \vdots & \ddots & \\ c([\frac{d+1}{2}],[\frac{d}{2}]) & c([\frac{d-1}{2}],[\frac{d-2}{2}]) & \cdots & c(\epsilon,0) \end{pmatrix}$$

where $\epsilon = [\frac{d+1}{2}] - [\frac{d}{2}]$, *i.e.* $\epsilon \in \{0,1\}$ *and* $\epsilon \equiv d$ *mod* 2.

Remark 2.5. The matrix B has only 1 on its diagonal.

Proof. We will just write the proof for d even.

In order to prove (i) and (ii), we will use the same strategy. First, we will express f_i (resp. F_i) in terms of e_j (resp. E_j); then we will see that the passing matrix is invertible which proves that $\underline{f}$ (resp. $\underline{F}$) is a basis.

(i) Let $i \in \{0, \ldots, \frac{d}{2}\}$. By using the lemma 2.3 and the formula (5), we obtain:

$$\begin{aligned} f_i &= (2)^{d-2i}(2,1)^i \\ &= (2,1)^i(c(d-2i,0)(d-2i+1) + c(d-2i-1,1)(d-2i,1) + \ldots \\ &\quad +c(\tfrac{d}{2}-i,\tfrac{d}{2}-i)(\tfrac{d}{2}-i+1,\tfrac{d}{2}-i)) \\ &= c(d-2i,0)(d-i+1,i) + c(d-2i-1,1)(d-i,1+i) + \ldots \\ &\quad +c(\tfrac{d}{2}-i,\tfrac{d}{2}-i)(\tfrac{d}{2}+1,\tfrac{d}{2}) \\ &= c(d-2i,0)e_i + c(d-2i-1,1)e_{i+1} + \ldots + c(\tfrac{d}{2}-i,\tfrac{d}{2}-i)e_{\frac{d}{2}} \end{aligned}$$

(ii) Let $j \in \{0, \ldots, \frac{d}{2}\}$. By using the lemma 2.3 and the formula (5), one has:

$$\begin{aligned} F_j &= (2)^{d-2j}(2,1)^{n-1-d+j} \\ &= (2,1)^{n-1-d+j}(c(d-2j,0)(d-2j+1) + c(d-2j-1,1)(d-2j,1) \\ &\quad + \ldots + c(\tfrac{d}{2}-j,\tfrac{d}{2}-j)(\tfrac{d}{2}-j+1,\tfrac{d}{2}-j)) \\ &= c(d-2j,0)(n-j,n-1-d+j) + \\ &\quad c(d-2j-1,1)(n-j+1,n-d+j) + \ldots \\ &\quad +c(\tfrac{d}{2}-j,\tfrac{d}{2}-j)(n-\tfrac{d}{2}+1,n-\tfrac{d}{2}-1) \\ &= c(d-2j,0)E_j + c(d-2j-1,1)E_{j+1} + \ldots + c(\tfrac{d}{2}-j,\tfrac{d}{2}-j)E_{\frac{d}{2}} \end{aligned}$$

In both cases, we have that $\underline{f} = \underline{e}B$ and $\underline{F} = \underline{E}B$ where B is an invertible matrix.

□

Corollary 2.6. *Consider the base* $\underline{f}$ *of* A^d *and the base* $\underline{F}$ *of* A^{2n-2-d}.

(i) The intersection form $A^d \times A^{2n-2-d} \longrightarrow \mathbb{Z}$ has in these bases the matrix :

$$\begin{pmatrix} C_d & C_{d-1} & \cdots & C_{d-[\frac{d}{2}]} \\ C_{d-1} & & & \vdots \\ \vdots & & & \vdots \\ C_{d-[\frac{d}{2}]} & C_{d-1-[\frac{d}{2}]} & \cdots & C_\epsilon \end{pmatrix}$$

We call this matrix the Catalan matrix which ends with C_ϵ.

(ii) The intersection form can also be expressed as B^tB.

Remark 2.7. For $d = n-1$, we have $\underline{e} = \underline{E}$, $\underline{f} = \underline{F}$, and B is invertible with integer coefficients. Then this corollary proves that such a Catalan matrix is equivalent, in the meaning of quadratic forms, over the integers to the identity matrix.

Proof. (i) We use the formulas (5) and (6) and compute the intersection form in these bases :

$$\begin{aligned} f_i \cdot F_j &= (2)^{d-2i}(2,1)^i \cdot (2)^{d-2j}(2,1)^{n-1-d+j} \\ &= (2)^{2d-2(i+j)}(2,1)^{n-1-d+i+j} \\ &= (2)^{2d-2(i+j)}(n-d+i+j, n-1-d+i+j) \\ &= c(n-(n-1-d+i+j)-1, n-(n-d+i+j)))\,(n,n-1) \\ &= c(d-i-j, d-i-j)\,(n,n-1) \\ &= C_{d-i-j}\,(n,n-1). \end{aligned}$$

(ii) We just apply the base change from $\underline{e}$ to $\underline{f}$ and from $\underline{E}$ to $\underline{F}$.

□

The last result shows that all determinants of Catalan matrices which end with C_0 or C_1 are equal to 1 whatever the size.

Question :

What can we say about the determinant of a Catalan matrix which ends with C_r (whatever the size)?

We will give some results in the next section.

2.3 About the determinant of a Catalan matrix

Let r be in the set $\{0, \ldots, n-1\}$. We generalize the intersection form in the following way :

$$\begin{aligned} I^r : \quad A^{n-1-r} \times A^{n-1-r} &\longrightarrow \mathbb{Z} \\ ((a,b),(c,d)) &\longmapsto \frac{(a,b)(c,d)(2)^{2r}}{(n,n-1)} \end{aligned}$$

This comes from the Lefschtz's theorem because (2) corresponds to the first Chern class of a line bundle.

Proposition 2.8. *The matrix of* I^r *in the base* $\underline{f} := (f_0, \ldots, f_{[\frac{n-1-r}{2}]})$*, where* $f_i := (2,1)^i(2)^{n-r-2i-1}$*, depends on the parity of* $n-1-r$*, namely :*

1. *If* $n-1-r$ *is even the matrix of* I^r *is the Catalan matrix of size* $\frac{n+1-r}{2}$ *which ends with* C_r *i.e.*

$$\begin{pmatrix} C_{n-1} & C_{n-2} & \cdots & C_{\frac{n-1+r}{2}} \\ C_{n-2} & & & C_{\frac{n+1+r}{2}} \\ \vdots & & & \vdots \\ C_{\frac{n-1+r}{2}} & C_{\frac{n+1+r}{2}} & \cdots & C_r \end{pmatrix}$$

2. *If* $n-1-r$ *is odd the matrix of* I^r *is the Catalan matrix of size* $\frac{n-r}{2}$ *which ends with* C_{r+1} *i.e.*

$$\begin{pmatrix} C_{n-1} & C_{n-2} & \cdots & C_{\frac{n+r}{2}} \\ C_{n-2} & & & C_{\frac{n+2+r}{2}} \\ \vdots & & & \vdots \\ C_{\frac{n+r}{2}} & C_{\frac{n+2+r}{2}} & \cdots & C_{r+1} \end{pmatrix}$$

Proof. In both cases, it is enough to compute $f_i f_j (2)^{2r}$. We just obtain these numbers by (6).

$$\begin{aligned} f_i f_j (2)^{2r} &= (2,1)^i(2)^{n-r-2i-1}(2,1)^j(2)^{n-r-2j-1} \\ &= (2,1)^{i+j}(2)^{2n-2i-2j-2} \\ &= c(n-i-j-1, n-i-j-1)\,(n, n-1) \\ &= C_{n-i-j-1}\,(n, n-1) \end{aligned}$$

□

Now we will compute the matrix of I^r in the base $\underline{e} := (e_0, \ldots, e_{[\frac{n-1-r}{2}]})$ where $e_i := (n-r-i, i)$. But first we state the following lemma.

Lemma 2.9. *For all* k*, we have :*

$$(2)^k(a,b) = \sum_{l=0}^{k} \binom{k}{l} (a+k-l, b+l)$$

Proof. It is obvious by an induction on k □

Remark 2.10. Later, we will only use the formula for $k := 2r$, namely :

$$(2)^{2r}(a,b) = \sum_{l=-r}^{r} \binom{2r}{l+r} (a+r-l, b+l+r) \tag{8}$$

Proposition 2.11. *The coefficient in position (i, j) of the matrix of I^r in the base $\underline{e}$ is :*

$$\binom{2r}{r-i+j} - \binom{2r}{n-i-j}$$

for all i, j in $\{0, \ldots, [\frac{n-1-r}{2}]\}$.

We denote this matrix by $B(n, r)$.

Remark 2.12. We use now the same argument as in corollary 2.6. If $n-1-r$ is even (resp. odd) then the determinant of the Catalan matrix which ends with C_r (resp. C_{r+1}) is the same as the determinant of the matrix $B(n, r)$.

Moreover, if $n-r-1$ is even (resp. odd) then the proposition 2.8 implies that the derminant of $B(n, r)$ is equal to the determinant of $B(n, r-1)$ (resp. $B(n, r+1)$).

Proof.

$$\begin{aligned}
(2)^{2r} e_i e_j &= (2)^{2r}(n-r-i, i)(n-r-j, j) \\
&= (2)^{2r} \sum_{k=0}^{n-r-2i-1} (2n-2r-i-j-k-1, i+j+k) \\
&= \sum_{k=0}^{n-r-2i-1} (2)^{2r}(2n-2r-i-j-k-1, i+j+k) \\
&= \sum_{k=0}^{n-r-2i-1} \sum_{l=-r}^{r} \binom{2r}{l+r} (2n-r-i-j-k-l-1, r+i+j+k+l)
\end{aligned}$$

If one looks at the terms in front of $(n, n-1)$ and $(n-1, n)$, they cancel each other except : $\binom{2r}{r+i-j}$ in front of $(n, n-1)$ and $\binom{2r}{n-i-j}$ in front of $(n-1, n)$. So this sum is equal to the difference between these two coefficients. □

Corollary 2.13. *The determinant of the Catalan matrix of size k which ends with $C_2 = 2$ is $k+1$.*

Proof. Let $r = 1$ and $n := 2k+1$. The propositions 2.8 and 2.11 imply that the determinant of the Catalan matrix of size k which ends with $C_2 = 2$ is equal to the determinant of the following matrix which is well known from the root system of type A_k.

$$\begin{pmatrix}
2 & 1 & 0 & \cdots & 0 \\
1 & 2 & 1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & 1 & 2 & 1 \\
0 & \cdots & 0 & 1 & 2
\end{pmatrix}$$

Its determinant is obviously $k+1$. □

Corollary 2.14. *The determinant of the Catalan matrix of size k which ends with $C_3 = 5$ is $\frac{(k+1)(k+2)(2k+3)}{6}$.*

Proof. Let $r = 2$ and $n := 2k + 1$. The propositions 2.8 and 2.11 imply that the determinant of the Catalan matrix of size k which ends with $C_3 = 5$ is equal to the determinant of the following matrix :

$$\begin{pmatrix} 6 & 4 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 4 & 6 & 4 & \ddots & \ddots & & & \vdots \\ 1 & 4 & 6 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 6 & 4 & 1 \\ \vdots & & & \ddots & \ddots & 4 & 6 & 4 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 4 & 5 \end{pmatrix}$$

We can prove that this determinant satisfies a recursive formula which is also satisfied by $\frac{(k+1)(k+2)(2k+3)}{6}$. ☐

Conjecture 2.15. *The determinant of the Catalan matrix of size k which ends with C_r is :*

$$\prod_{1 \leq i \leq j \leq r-1} \frac{2k+i+j}{i+j}$$

3 Formulas from quantum cohomology

In this section, after some reminiscences about quantum cohomology, we will explain some links between the article of Schubert [2] and this new approach. Finally, we will use this modern approach to compute some terms of type $(a)(b)(2)^{2n-a-b}$. For $b = 2$ we recover the same formula (7) that Schubert proved in [2] §5.

3.1 Some reminiscences about quantum cohomology of the complex projective space

To define properly the Gromov Witten invariants, we refer to [3] chap III. Here we just give an interpretation of these numbers.

Let $d \geq 1$ and $i_2, \ldots, i_n \geq 0$ be integers. We denote by $a_k^1, \ldots, a_k^{i_k}$ some hyperplanes of $\mathbb{P}^n$ of codimension k in generic position. The Gromov-Witten invariant $N(d, i_2, ..., i_n)$ is the number of rational curves in $\mathbb{P}^n$ of degree d which intersect all hyperplanes a_k^l, if this number is finite. This requires

$$\sum_{k=2}^{n} (k-1)i_2 = (n+1)d + n - 3. \tag{9}$$

If (9) does not hold we set $N(d, i_2, ..., i_n) := 0$. In the notation of the first part, we have that $N(d, i_2, \ldots, i_n)(n, n-1) = (2)^{i_2} \ldots (n)^{i_n}$.

The Gromov-Witten potential is defined as follows:

$$\Phi := \Phi_{\text{cl}} + \sum_{\substack{d\geq 1, i_2,\ldots,i_n\geq 0 \\ (9)}} N(d, i_2, \ldots, i_n) \frac{t_2^{i_2}}{i_2!} \ldots \frac{t_n^{i_n}}{i_n!} e^{dt_1}$$

where

$$\Phi_{\text{cl}} := \frac{1}{6} \sum_{\substack{i,j,k\geq 0 \\ i+j+k=n}} t_i t_j t_k$$

Let us consider the cohomology of $\mathbb{P}^n$. We have $H := H^\star(P^n, \mathbb{C}) := \sum_{k=0}^n \mathbb{C}\Delta_k$

where Δ_k is the dual class of $\mathbb{P}^{n-k} \subset \mathbb{P}^n$. Let $t_0, \ldots, t_n$ be the coordinates of H corresponding to this base and let $\partial_i := \frac{\partial}{\partial t_i}$ be the coordinate vector fields. We define a bilinear form on this cohomology by :

$$g(\partial_i, \partial_j) = \delta_{i,n-j}$$

This form is nondegenerate (it is the Poincaré form). We call it the metric.

Then we define a product $\circ$ by the formula :

$$g(\partial_i \circ \partial_j, \partial_k) = \frac{\partial^3 \Phi}{\partial_i \partial_j \partial_k} =: \Phi_{ijk}$$

Let $(a, b, c, d) \in \{0, \ldots, n\}^4$. We denote by (a, b, c, d)-WDVV the following equation :

$$\sum_{k=0}^{n} \Phi_{abk} \Phi_{(n-k)cd} = \sum_{k=0}^{n} \Phi_{ack} \Phi_{(n-k)bd}$$

The product $\circ$ is associative if and only if the Gromov-Witten potential satisfy the (a, b, c, d)-WDVV equations for all a, b, c, d.

Theorem 3.1 ([3]). *The product $\circ$ is associative.*

In the rest of the notes, we will only care about Gromov Witten invariants of degree 1.

Proposition 3.2. *The coefficients in front of $\frac{t_2^{i_2}}{i_2!} \ldots \frac{t_n^{i_n}}{i_n!} e^{t_1}$ in the (a, b, c, d)-WDVV equation satisfy :*

$$N_{ab(c+d)}(1, \tilde{i}) + N_{cd(a+b)}(1, \tilde{i}) = N_{ac(b+d)}(1, \tilde{i}) + N_{bd(a+c)}(1, \tilde{i}) \tag{10}$$

where $N_{ab(c+d)}(1, \tilde{i}) := N(1, i_2, \ldots, i_a + 1, \ldots, i_b + 1, \ldots, i_{c+d} + 1, \ldots, i_n)$.

3.2 Relations from quantum cohomology

Here, in the two next propositions, we give an equivalence between the formula (3) and WDVV equation with Giambelli's formula.

If we look at the formula (10) with Schubert notations we have :

$$(a)(b)(c+d)+(c)(d)(a+b)=(a)(c)(b+d)+(b)(d)(a+c) \tag{11}$$

Question :
The equations (10), are they equivalent to the equations (11) ?

Remark 3.3. The formula (11) can be seen as the WDVV equations for degree one in Schubert notations.

If $a=1$ then we have :

$$(b+1)(c)(d)-(b)(c+1)(d)=(c)(b+d)-(b)(c+d) \tag{12}$$

If $a=b=1$ then we have :

$$(2)(c)(d)=(c)(1+d)+(d)(1+c)-(c+d) \tag{13}$$

If $a=b=1$ and $c=2$ then we have :

$$(2)(2)(d)=(2)(1+d)+(d)(3)-(2+d) \tag{14}$$

Proposition 3.4. *1. All these equations are equivalent.*

2. Schubert's relation (3) implies (11)

Proof. 1. • Show that (12) implies (11). We multiply by (a) the equation (12) :

$$(a)(c)(b+d)-(a)(b)(c+d)=(a)(d)((b+1)(c)-(b)(c+1))$$

We remark that the right hand side is symetric in a and d. To conclude, one can see that (11) means that the left hand side of the equation above is symmetric in a and d.

• Show that (13) implies (12). We multiply by (b) the equation (13) :

$$(b)(c+d)-(b)(d)(c+1)=(b)(c)((d+1)-(2)(d))$$

Again, we remark that the right hand side is symmetric in b and c. To conclude, one can see that (12) means that the left hand side of the equation above is symmetric in b and c.

- Show that (14) implies (13). We will proceed by an inductive proof on c. For $c = 2$, it is just (14).

  Suppose that for all d we have :

  $$(2)(c)(d) = (c)(d+1) + (d)(c+1) - (c+d) \tag{15}$$

  We will prove that

  $$(2)(c+1)(d-1) = (c+1)(d) + (d-1)(c+2) - (c+d)$$

  We multiply (14) with $d-1$ by (c). Then we have :

  $$(2)(c)(d) \quad = \quad (2)(2)(c)(d-1) - (c)(d-1)(3) + (1+d)(c)$$

  Then we apply (14) with c :

  $$(2)(c)(d) \quad = \quad (2)(d-1)(c+1) - (d-1)(c+2) + (c)(d+1)$$

  Then we obtain :

  $$(2)(c+1)(d-1) = (2)(c)(d) + (c+2)(d-1) - (1+d)(c)$$

  Finally, we apply (15) with c and d :

  $$(2)(c+1)(d-1) \quad = \quad (c+1)(d) + (d-1)(c+2) - (c+d).$$

2. It is enough to prove that (3) implies (14) and this is easy to check.

□

Proposition 3.5. *If we consider Giambelli's formula as a definition for (a,b), then the equation (11) implies (3).*

Remark 3.6. The propositions 3.4 and 3.5 show us that all the results in the first section are implied by the equation (14) and the Giambelli's formula.

Proof. It is enough to prove that equation (12) implies (1) because of the proposition 1.4. On one hand Giambelli's formula (2) shows that :

$$(a)(b,\beta) = (a)(b)(\beta+1) - (a)(b+1)(\beta)$$

On the other hand, by using the formulas (2) and (12) again, we have :

$$\begin{aligned} \sum_{k=0}^{b-\beta-1} (a+b-k-1, \beta+k) &= \sum_{k=0}^{b-\beta-1} ((a+b-k-1)(\beta+k+1) \\ &\quad -(a+b-k)(\beta+k)) \\ &= (a+\beta)(b) - (a+b)(\beta) \\ &= (a)(b)(\beta+1) - (a)(b+1)(\beta) \end{aligned}$$

□

The last theorem is motivated by the next remark 3.8

Theorem 3.7. *Every Gromov-Witten invariant of degree* 1 *can be calculated inductively with formula (12) from* $(n)(n)$.

Proof. Let $(2)^{i_2}\dots(n)^{i_n}$ be a Gromov Witten invariant, i.e. we have $\sum_{k=2}^{n}(k-1)i_k = 2n-2$. If we use the formula (12) many times, we can express this invariant with a product of two conditions. But the only product of two conditions such that $\sum_{k=2}^{n}(k-1)i_k = 2n-2$ is when $i_n = 2$ and the other are zero. That is exactly the condition $(n)(n)$, i.e. the number of line through two points. □

Remark 3.8. 1. If one applies Giambelli's formula to $(n)(n)$, one sees:

$$(n)(n) = (n, n-1).$$

2. In Quantum Cohomology of the projective space $\mathbb{P}^n$, Manin and Kontsevich prove a much stronger result : 'any Gromov-Witten invariant of any degree can be calculated inductively with the WDVV-equations from $(n)(n)$'.

3.3 Some other formulas

When we study the relation (10) we have the following diamond :

$$\begin{array}{ccccccc}
 & & & (n)(n) & & & \\
 & & (n-1)(n) & & (n)(n-1) & & \\
 & & & & & \ddots & \\
(2)(n) & & & & & & (n)(2) \\
 & (2)(n-1) & & & & (n-1)(2) & \\
\hline
 & & \ddots & & & & \\
 & & & (2)(2) & & &
\end{array}$$

The results are :

1. The numbers in the diamond above the line that is : $(a)(b)$ with $a+b>n$ satisfy the binomial recursive formula. So we have :

$$(a)(b)(2)^{2n-a-b} = \binom{a+b}{a}(n, n-1)$$

2. If $a=2$ or $b=2$ we have the same formula (7) as Schubert, namely:

$$(a)(2)^{2n-a-1} = \binom{2n-a-1}{n-1}\frac{a-1}{n}(n, n-1)$$

3. The other terms can be written as a sum of numbers in the Catalan triangle but we do not have a closed formula.

References

[1] Griffiths, Harris : Principles of algebraic geometry. Wiley Classics Library, John Wiley & Sons Inc., 1994.

[2] Schubert : Die n-dimensionalen Verallgemeinerungen der fundamentalen Anzahlen unseres Raums. Mathematische Annalen **26** (1885), 26-51.

[3] Yu. Manin: Frobenius manifolds, quantum cohomology, and moduli spaces. American Math. Society, Colloquium Publ. v. **47**, 1999.