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THE COHOMOLOGICAL CREPANT RESOLUTION CONJECTURE FOR $\mathbb{P}(1,3,4,4)$

S. BOISSIÈRE

Laboratoire J.A.Dieudonné, Université de Nice Sophia-Antipolis
Parc Valrose, F-06108 Nice, France
sb@math.unice.fr

E. MANN

Département de Mathématiques, Université de Montpellier
Place Eugène Bataillon, F-34 095 Montpellier, France
emann@math.univ-montp2.fr

F. PERRONI

Institut für Mathematik, Universität Zürich
Winterthurerstrasse 190, CH-8057 Zürich
fabio.perroni@math.unizh.ch

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We prove the cohomological crepant resolution conjecture of Ruan for the weighted projective space $\mathbb{P}(1,3,4,4)$. To compute the quantum corrected cohomology ring, we combine the results of Coates–Corti–Iritani–Tseng on $\mathbb{P}(1,1,1,3)$ and our previous results.

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1. Introduction

The *Cohomological Crepant Resolution Conjecture*, as proposed by Ruan, predicts the existence of an isomorphism between the Chen–Ruan cohomology ring of a complex Gorenstein orbifold $\mathcal{X}$ and the quantum corrected cohomology ring of any crepant resolution $\rho : Z \to X$ of the coarse moduli space X of $\mathcal{X}$ [15]. The quantum corrected cohomology ring of Z is the ring obtained from the small quantum cohomology of Z after specialization of the quantum parameters corresponding to the rational exceptional curves to $c_1, \ldots, c_m$ and the remaining parameters to zero, it is denoted by $H^\star_\rho(Z)(c_1, \ldots, c_m)$ (see Sec. 2.3). It is an important issue (see [15]) to determine the $c_1, \ldots, c_m$. Examples suggest that there can be different choices (see e.g. [4, 6, 13]); however, if one assumes the validity of [8, Conjecture 4.1], then

Ruan's conjecture follows ([8, Theorem 7.2]) and the $c_1, \ldots, c_m$ acquire a precise meaning (see [8, Sec. 11]).

In this paper, we consider the weighted projective space $\mathcal{X} = \mathbb{P}(1,3,4,4)$. The quantum corrected cohomology ring depends on four quantum parameters: q_1, q_2, q_3 and q_4; the first three correspond to the components of the exceptional divisor over the transverse singularity, while the fourth corresponds to the component of the exceptional divisor over the isolated singularity (see Sec. 2.3). Our main result, Theorem 4.1, states that, for $(c_1, \ldots, c_4) \in \{(\mathrm{i},\mathrm{i},\mathrm{i},1), (-\mathrm{i},-\mathrm{i},-\mathrm{i},1)\}$, the quantum corrected cohomology ring of Z is isomorphic to the Chen–Ruan cohomology ring of $\mathbb{P}(1,3,4,4)$. These values of the quantum parameters are relevant for the cohomological crepant resolution conjecture and its generalization, [8, Conjecture 4.1]. In the previous paper [1], we proved that $H^\star_\rho(Z)(q_1, \ldots, q_4)$ becomes isomorphic to $H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4))$ when we set the quantum parameters to $(\mathrm{i},\mathrm{i},\mathrm{i},0)$ or $(-\mathrm{i},-\mathrm{i},-\mathrm{i},0)$, which we found strange in regards to the conjecture, where the value $c_4 = 0$ is not considered. A motivation for this work was that to clarify this point.

To prove the theorem, we compute explicitly a presentation of $H^\star_\rho(Z)(c_1, \ldots, c_4)$ with quantum parameters equal to $(\mathrm{i},\mathrm{i},\mathrm{i},1)$ and $(-\mathrm{i},-\mathrm{i},-\mathrm{i},1)$, then we give an explicit isomorphism $H^\star_\rho(Z;\mathbb{C})(c_1, \ldots, c_4) \to H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4))$. The quantum corrections coming from the exceptional divisor over the transverse singularity have been computed in [1]. To compute the quantum corrections coming from the component of the exceptional divisor over the isolated singularity, we use results from [5].

The paper is organized as follows. In Sec. 2, we collect some background material: we first give a presentation of the Chen–Ruan cohomology ring of $\mathbb{P}(1,3,4,4)$; then we describe the crepant resolution of $|\mathbb{P}(1,3,4,4)|$; finally, we write a presentation of $H^\star_\rho(Z)(c_1, \ldots, c_4)$ with $(c_1, \ldots, c_4) = (\mathrm{i},\mathrm{i},\mathrm{i},c_4)$, here, c_4 is a generic complex number and we assume the convergence of the power series (2.6). The fact that (2.6) converges in a neighborhood of the origin to an analytic function that admits analytic continuation in $1 \in \mathbb{C}$ is proved in Sec. 3; more precisely, we show that (2.6) is equal to a constant structure of the small quantum cohomology of the crepant resolution of $|\mathbb{P}(1,1,1,3)|$ and then use results from [5]. In Sec. 4, we prove Theorem 4.1.

2. Background

The weighted projective space $\mathbb{P}(1,3,4,4)$ is the quotient stack $\left[\mathbb{C}^4 - \{0\}/\mathbb{C}^\star\right]$, where $\mathbb{C}^\star$ acts diagonally with weights $w_0 = 1, w_1 = 3, w_2 = 4$ and $w_3 = 4$, it will be denoted by $\mathcal{X}$. The coarse moduli space $X := |\mathbb{P}(1,3,4,4)|$ is a projective variety whose singular locus is the disjoint union of the curve $C := \{[0:0:x_2:x_3]\} \subset X$ and the isolated point $\mathrm{P} := [0:1:0:0] \in X$. Along C, X has transverse A_3 singularities (see [13]), the point P is a singularity of type $\frac{1}{3}(1,1,1)$, according to Reid's notation [14].

2.1. *The Chen–Ruan cohomology*

To compute the Chen–Ruan cohomology ring $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ we follow [2]. The twisted sectors are indexed by the set $\mathrm{T} := \{\exp(2\pi \mathrm{i}\gamma) \mid \gamma \in \{0, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}\}$. For any $g \in \mathrm{T}$, $\mathcal{X}_{(g)}$ is a weighted projective space: set $\mathrm{I}(g) := \{i \in \{0,1,2,3\} \mid g^{w_i} = 1\}$, then $\mathcal{X}_{(g)} = \mathbb{P}(w_{\mathrm{I}(g)})$, where $(w_{\mathrm{I}(g)}) = (w_i)_{i\in \mathrm{I}(g)}$. The inertia stack is the disjoint union of the twisted sectors:

$$\mathrm{I}\mathcal{X} = \sqcup_{g\in\mathrm{T}}\mathbb{P}(w_{\mathrm{I}(g)}).$$

As a vector space, the Chen–Ruan cohomology is the cohomology of the inertia stack; the graded structure is obtained by shifting the degree of the cohomology of any twisted sector by twice the corresponding age, $\mathrm{age}(g)$. We have

$$\begin{aligned} H^p_{\mathrm{CR}}(\mathcal{X};\mathbb{C}) &= \oplus_{g\in\mathrm{T}} H^{p-2\mathrm{age}(g)}(\mathbb{P}(w_{\mathrm{I}(g)});\mathbb{C}) \\ &= H^p(\mathbb{P}(1,3,4,4);\mathbb{C}) \oplus H^{p-2}(\mathbb{P}(3);\mathbb{C}) \oplus H^{p-4}(\mathbb{P}(3);\mathbb{C}) \\ &\quad \oplus H^{p-2}(\mathbb{P}(4,4);\mathbb{C}) \oplus H^{p-2}(\mathbb{P}(4,4);\mathbb{C}) \oplus H^{p-2}(\mathbb{P}(4,4);\mathbb{C}). \end{aligned} \tag{2.1}$$

A basis of $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ is easily obtained in the following way: set

$$H, E_1, E_2, E_3, E_4 \in H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$$

be the image of $c_1(\mathcal{O}_\mathcal{X}(1)) \in H^2(\mathcal{X};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}/2))};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}))};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}3/2))};\mathbb{C})$ and $1 \in H^0(\mathcal{X}_{(\exp(2\pi\mathrm{i}/3))};\mathbb{C})$, respectively, under the inclusion $H^\star(\mathbb{P}(w_{\mathrm{I}(g)})) \to H^\star_{\mathrm{CR}}(\mathcal{X})$ determined by the decomposition (2.1). As a commutative $\mathbb{C}$-algebra, the Chen–Ruan cohomology ring is generated by H, E_1, E_2, E_3, E_4 with relations (see [2]):

$$HE_4,\ E_1E_1 - 3HE_2,\ E_1E_2 - 3HE_3,\ E_1E_3 - 3H^2,$$
$$E_2E_2 - 3H^2,\ E_2E_3 - HE_1,\ E_3E_3 - HE_2,\ 16H^3 - E_4^3,$$
$$H^2E_1,\ H^2E_2,\ H^2E_3,\ E_1E_4,\ E_2E_4,\ E_3E_4.$$

We see that the following elements form a basis of $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ which we fix for the rest of the paper:

$$1,\ H,\ E_1,\ E_2,\ E_3,\ E_4,\ H^2,\ HE_1,\ HE_2,\ HE_3,\ E_4^2,\ H^3. \tag{2.2}$$

Remark 2.1. Note that the elements of our basis are different from those used in [2] by a combinatorial factor.

Other methods are suitable in order to compute the Chen–Ruan cup product of weighted projective spaces, here are a few: the results in [3] provide a presentation of the Chen–Ruan cohomology ring for a general toric Deligne–Mumford stack; results from [12] and from [7].

2.2. *The crepant resolution*

We study some properties of the crepant resolution of $X := |\mathbb{P}(1,3,4,4)|$. We begin with the following

Proposition 2.1. *The variety $X = |\mathbb{P}(1,3,4,4)|$ has a unique crepant resolution $\rho : Z \to X$, up to isomorphism.*

Proof. This is a direct consequence of the following facts: the 3-fold singularity $\frac{1}{3}(1,1,1)$ has a unique crepant resolution (see e.g. [10]) and any variety with transverse ADE singularities has a unique crepant resolution ([13, Proposition 4.2]), up to isomorphism. □

An explicit model for the crepant resolution $\rho : Z \to X$ can be constructed using methods from toric geometry, we follow [11]. The toric variety X is associated to the lattice $\mathbb{Z}^3$ and the fan Σ, where Σ is the fan whose cones are generated by proper subsets of $\{v_0 := (-3,-4,-4), v_1 := (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)\} \subset \mathbb{Z}^3$. The 3-dimensional cones which correspond to singular open affine subvarieties of X are: $\sigma_1 := \langle v_0, v_2, v_3 \rangle$, $\sigma_2 := \langle v_0, v_1, v_3 \rangle$, $\sigma_3 := \langle v_0, v_1, v_2 \rangle$. More precisely, σ_1 gives rise to the isolated singularity, while σ_2 and σ_3 to the transverse one. To resolve the isolated singularity, we subdivide σ_1 by inserting the ray generated by

$$Q_4 := (-1,-1,-1) = \frac{1}{3}v_0 + \frac{1}{3}v_2 + \frac{1}{3}v_3.$$

To resolve the transverse singularity, we subdivide σ_2 and σ_3 by inserting the rays generated by

$$Q_1 := (0,-1,-1) = \frac{1}{4}v_0 + \frac{3}{4}v_1,$$
$$Q_2 := (-1,-2,-2) = \frac{1}{2}v_0 + \frac{1}{2}v_1,$$

and

$$Q_3 := (-2,-3,-3) = \frac{3}{4}v_0 + \frac{1}{4}v_1.$$

The subdivision Σ' is shown in Fig. 1.

Then we set Z to be the toric variety associated to $\mathbb{Z}^3$ and Σ', and $\rho : Z \to X$ to be the morphism associated to the identity $\mathbb{Z}^3 \to \mathbb{Z}^3$.

Lemma 2.1. *The morphism $\rho : Z \to X$ defined above is a crepant resolution.*

Proof. Since any cone of Σ' is generated by a part of a basis of the lattice, it follows that Z is smooth, [11, Sec. 2.1]. The crepancy of ρ follows from the existence of a continuous Σ-piecewise linear function $h' : \mathbb{R}^3 \to \mathbb{R}$ such that $h'(Q_i) = h'(v_j) = -1$, for any $i \in \{1,2,3,4\}$ and $j \in \{0,1,2,3\}$, [11, Sec. 3.4]. □

Observe that Σ' is the maximal projective subdivision of the polytope of $|\mathbb{P}(1,3,4,4)|$ (see [9] or [16]).

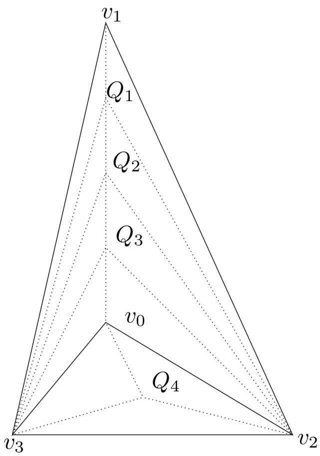

Fig. 1. Polar polytopes of $|\mathbb{P}(1,3,4,4)|$ and crepant resolution.

2.3. *Quantum corrected cohomology ring of Z*

Let us denote by b_j (respectively e_i) the first Chern class of the holomorphic line bundle associated to the torus invariant divisor in Z corresponding to the ray generated by v_j (respectively Q_i), for any $j \in \{0,1,2,3\}$ ($i \in \{1,2,3,4\}$ respectively). Set $h = \frac{1}{12}(\sum_{i=0}^{3} b_i + \sum_{j=1}^{4} e_j)$, then $H^\star(Z;\mathbb{C}) \cong \mathbb{C}[h, e_1, e_2, e_3, e_4]/I$ where I is the ideal generated by (see [1] for more details)

$$3he_4,\ e_1e_3,\ e_1e_4,\ e_2e_4,\ e_3e_4,$$

$$e_1^2 - 10he_1 - 4he_2 - 2he_3 + 24h^2,$$

$$e_1e_2 + 3he_1 + 2he_2 + he_3 - 12h^2,$$

$$e_2^2 - 6he_1 - 12he_2 - 2he_3 + 24h^2,$$

$$e_2e_3 + 3he_1 + 6he_2 + he_3 - 12h^2,$$

$$e_3^2 - 6he_1 - 12he_2 - 14he_3 + 24h^2,$$

$$16h^2e_1,\ 16h^2e_2,\ 16h^2e_3, 16h^3 - \tfrac{1}{27}e_4^3.$$

Let us fix the following basis of $H^\star(Z;\mathbb{C})$:

$$\begin{gathered}\psi_0 = 1, \quad \psi_1 = h, \quad \psi_2 = e_1, \quad \psi_3 = e_2, \quad \psi_4 = e_3, \quad \psi_5 = e_4, \\ \psi_6 = h^2, \quad \psi_7 = he_1, \quad \psi_8 = he_2, \quad \psi_9 = he_3, \quad \psi_{10} = e_4^2, \quad \psi_{11} = h^3,\end{gathered} \tag{2.3}$$

with dual basis:

$$\psi^0 = 48h^3, \quad \psi^1 = 48h^2, \quad \psi^2 = -3he_1 - 2he_2 - he_3, \quad \psi^3 = -2he_1 - 4he_2 - 2he_3,$$

$$\psi^4 = -he_1 - 2he_2 - 3he_3, \quad \psi^5 = \frac{1}{9}e_4^2, \quad \psi^6 = 48h, \quad \psi^7 = -3e_1 - 2e_2 - e_3,$$

$$\psi^8 = -2e_1 - 4e_2 - 2e_3, \quad \psi^9 = -e_1 - 2e_2 - 3e_3, \quad \psi^{10} = \frac{1}{9}e_4, \quad \psi^{11} = 48 \cdot 1.$$

Let $\mathrm{M}_\rho(Z) \subset \mathrm{A}_1(Z;\mathbb{Z})$ be the cone of effective 1-cycles in Z which are contracted by ρ. It is freely generated by $\Gamma_i := \mathrm{PD}(4he_i)$ for $i \in \{1,2,3\}$ and $\Gamma_4 := \mathrm{PD}(-\frac{1}{3}e_4^2)$, where PD means Poincaré dual (see e.g. [1]). Let q_1, q_2, q_3, q_4 be formal variables and let $\Lambda := \mathbb{C}[\![q_1,\dots,q_4]\!]$ be the ring of formal power series in $q_1,\dots,q_4$. We have an associative product on $H^\star(Z;\Lambda)$ defined as:

$$\psi_i \star_\rho \psi_j := \sum_{\Gamma \in \mathrm{M}_\rho(Z)} \sum_{\ell=0}^{11} \langle \psi_i, \psi_j, \psi^\ell \rangle_{0,3,\Gamma}^Z \psi_\ell q^\Gamma, \tag{2.4}$$

where $\langle \cdots \rangle_{0,3,\Gamma}^Z$ is the Gromov–Witten invariant of Z of genus zero, three marked points, homology class Γ and $q^\Gamma := q_1^{d_1} \cdots q_4^{d_4}$ for $\Gamma = d_1\Gamma_1 + \cdots + d_4\Gamma_4 \in \mathrm{M}_\rho(Z)$.

In [1], we computed explicitly the product (2.4) whenever $\psi_i \neq e_4$ or $\psi_j \neq e_4$; as a result, it follows that, in these cases, the power series involved in (2.4) converge in a neighborhood of the origin. We will see in Sec. 3 that also $e_4 \star_\rho e_4$ is convergent in a neighborhood of the origin. Therefore, the expression (2.4) defines a family of $\mathbb{C}$-algebras over the vector space $H^\star(Z;\mathbb{C})$ whose structure constants are analytic functions defined in some region of the complex space $\mathbb{C}^4$. For any $c_1,\dots,c_4$ in this region, the algebra obtained by setting $q_i = c_i$ is the *quantum corrected cohomology ring* of Z with quantum parameters specialized to $c_1,\dots,c_4$ and is denoted by $H^\star(Z;\mathbb{C})(c_1,\dots,c_4)$ [15].

In particular, we have the following presentation for $H^\star_\rho(Z)(\mathrm{i},\mathrm{i},\mathrm{i},c_4)$ (see [1] for more details):

$$\begin{aligned}
\alpha_1 \star_\rho \alpha_2 &= \alpha_1 \cup \alpha_2, \quad \text{if } \deg(\alpha_1) \neq 2,\\
h \star_\rho \alpha &= h \cup \alpha, \quad \text{for any } \alpha,\\
e_1 \star_\rho e_1 &= -24h^2 + (-2+6\mathrm{i})he_1 - 4he_2 + (-2-2\mathrm{i})he_3,\\
e_1 \star_\rho e_2 &= 12h^2 + (-1-4\mathrm{i})he_1 + (2-4\mathrm{i})he_2 + he_3,\\
e_1 \star_\rho e_3 &= -2\mathrm{i}he_1 - 2\mathrm{i}he_3,\\
e_2 \star_\rho e_2 &= -24h^2 + (2+2\mathrm{i})he_1 + 8\mathrm{i}he_2 + (-2+2\mathrm{i})he_3,\\
e_2 \star_\rho e_3 &= 12h^2 - he_1 + (-2-4\mathrm{i})he_2 + (1-4\mathrm{i})he_3,\\
e_3 \star_\rho e_3 &= -24h^2 + (2-2\mathrm{i})he_1 + 4he_2 + (2+6\mathrm{i})he_3,\\
e_4 \star_\rho e_4 &= \epsilon(c_4)e_4^2;
\end{aligned} \tag{2.5}$$

where $\cup$ is the usual cup product, and

$$\begin{aligned}
\epsilon(q) &:= 1 + \frac{1}{9}\left(\int_{\Gamma_4} e_4\right)^3 \sum_{a=1}^{\infty} a^3 \deg[\overline{\mathcal{M}}_{0,0}(Z,a\Gamma_4)]^{\mathrm{vir}} q^a\\
&= 1 - 3\sum_{a=1}^{\infty} a^3 \deg[\overline{\mathcal{M}}_{0,0}(Z,a\Gamma_4)]^{\mathrm{vir}} q^a.
\end{aligned} \tag{2.6}$$

A similar presentation holds for $H^\star(Z)(-\mathrm{i},-\mathrm{i},-\mathrm{i},c_4)$, also in this case, $e_4 \star_\rho e_4 = \epsilon(c_4)e_4^2$ with $\epsilon(q)$ defined in (2.6).

3. Relations with the Quantum Cohomology of $\mathbb{F}_3$

To state our main result (Theorem 4.1), we need to know that the series (2.6) converges in a neighborhood of the origin to an analytic function that admits analytic continuation at the point $1 \in \mathbb{C}$. To this aim, we show in this section that (2.6) appears as a structure constant of the small quantum cohomology of $\mathbb{F}_3$, the crepant resolution of $|\mathbb{P}(1,1,1,3)|$, so that we can use the results of [5, 8]. For reader's convenience, we stick to the notation of [5].

Let us consider the coarse moduli space of the weighted projective space $\mathbb{P}(1,1,1,3)$, it has an isolated singularity of type $\frac{1}{3}(1,1,1)$. Let $\mathbb{F}_3 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2})$, then there exists a morphism $\chi : \mathbb{F}_3 \to |\mathbb{P}(1,1,1,3)|$ which is a crepant resolution. Following the notation of [5], let $p_1 \in H^2(\mathbb{F}_3;\mathbb{C})$ be the class Poincaré dual to the preimage in $\mathbb{F}_3$ of a hyperplane in $\mathbb{P}^2$; let $p_2 \in H^2(\mathbb{F}_3;\mathbb{C})$ the class Poincaré dual to the infinity section. The cohomology ring of $\mathbb{F}_3$ has the following presentation

$$H^\star(\mathbb{F}_3;\mathbb{C}) = \mathbb{C}[p_1,p_2]/\langle p_1^3, p_2^2 - 3p_1p_2\rangle.$$

Let us fix the following basis of $H^\star(\mathbb{F}_3;\mathbb{C})$ (as in [5])

$$\phi_0 = 1, \quad \phi_1 = \frac{p_2}{3}, \quad \phi_2 = \frac{p_1p_2}{3}, \quad \phi_3 = \frac{p_2 - 3p_1}{3},$$
$$\phi_4 = -\frac{p_1(p_2-3p_1)}{3}, \quad \phi_5 = \frac{p_1^2p_2}{3},$$

then its dual is

$$\phi^0 = p_1^2p_2, \quad \phi^1 = p_1p_2, \quad \phi^2 = p_2, \quad \phi^3 = -p_1(p_2-3p_1),$$
$$\phi^4 = p_2 - 3p_1, \quad \phi^5 = 3.$$

The Poincaré dual to the exceptional divisor is $\mathfrak{p} := p_2 - 3p_1$ (in [5] $\mathfrak{p}$ is denoted by $\mathfrak{p}_1$). We want to compute the product $\mathfrak{p} \circ_{\hat{q}} \mathfrak{p}$ in the small quantum cohomology of $\mathbb{F}_3$ with $(\hat{q}_1,\hat{q}_2) = (1,0)$ (see (3.2)).

By definition, we have (see [5, Sec. 2.4])

$$\mathfrak{p} \circ_{\hat{q}} \mathfrak{p} = \sum_{\gamma \in \mathrm{M}(\mathbb{F}_3)} \sum_{\ell=0}^{5} \langle \mathfrak{p}, \mathfrak{p}, \phi_\ell \rangle_{0,3,\gamma}^{\mathbb{F}_3} \hat{q}^\gamma \phi^\ell, \tag{3.1}$$

where $\mathrm{M}(\mathbb{F}_3) \subset H_2(\mathbb{F}_3;\mathbb{Z})$ is the cone of the classes of effective curves, for $\gamma \in \mathrm{M}(\mathbb{F}_3)$ $\hat{q}^\gamma : \frac{H^2(\mathbb{F}_3;\mathbb{C})}{2\pi \mathrm{i} H^2(\mathbb{F}_3;\mathbb{Z})} \to \mathbb{C}^\star$ is the function $[\tau] \mapsto \exp(\int_\gamma \tau)$, $\langle \cdots \rangle_{0,3,\gamma}^{\mathbb{F}_3}$ is the Gromov–Witten invariant of $\mathbb{F}_3$ of genus zero, three marked points and class γ.

Since p_1, p_2 form a basis of $H^2(\mathbb{F}_3;\mathbb{C})$, any $[\tau]$ can be written as $[\tau] = [\tau^1 p_1 + \tau^2 p_2]$, therefore

$$\hat{q}^{\gamma}([\tau]) = \hat{q}_1^{\int_\gamma p_1} \hat{q}_2^{\int_\gamma p_2}, \tag{3.2}$$

where $\hat{q}_i := \exp(\tau^i)$, $i \in \{1,2\}$. Let us consider the classes $\gamma_1 := \mathrm{PD}(-\frac{1}{3}\mathfrak{p}^2 = p_1(p_2 - 3p_1))$ and $\gamma_2 := \mathrm{PD}(p_1^2)$, they form a basis for $\mathrm{M}(\mathbb{F}_3)$ such that $\int_{\gamma_i} p_j = \delta_{ij}$, $i,j \in \{1,2\}$. It follows that the product $\mathfrak{p} \circ_{\hat{q}} \mathfrak{p}$ restricted to $\hat{q} = (\hat{q}_1, 0)$ is given by:

$$\mathfrak{p} \circ_{(\hat{q}_1,0)} \mathfrak{p} = \sum_{a \geq 0} \sum_{\ell=0}^{5} \langle \mathfrak{p}, \mathfrak{p}, \phi_\ell \rangle_{0,3,a\gamma_1}^{\mathbb{F}_3} \hat{q}_1^a \phi^\ell.$$

Using the degree axiom, the divisor axiom and $\int_{\gamma_1} p_2 = 0$, we deduce that

$$\mathfrak{p} \circ_{(\hat{q}_1,0)} \mathfrak{p} = \sum_{a=0}^{\infty} \langle \mathfrak{p}, \mathfrak{p}, \frac{1}{3}\mathfrak{p} \rangle_{0,3,a\gamma_1}^{\mathbb{F}_3} \cdot (-p_1(p_2 - 3p_1))\hat{q}_1^a.$$

Finally, since $\mathfrak{p}^2 := \mathfrak{p} \cup \mathfrak{p} = -3p_1(p_2 - 3p_1)$, using again the divisor axiom, we rewrite the previous expression as

$$\mathfrak{p} \circ_{(\hat{q}_1,0)} \mathfrak{p} = \left(1 - 3\sum_{a=1}^{\infty} a^3 \deg\left[\overline{\mathcal{M}}_{0,0}(\mathbb{F}_3, a\gamma_1)\right]^{\mathrm{vir}} \hat{q}_1^a\right) \mathfrak{p}^2. \tag{3.3}$$

Then we have:

Lemma 3.1. *The power series in* (3.3) *is equal to the power series* ϵ (*see Formula* (2.6)).

Proof. From the uniqueness of the crepant resolution of the singularity $\frac{1}{3}(1,1,1)$, there exists an isomorphism between a neighborhood in Z of the component of the exceptional divisor over $[0:1:0:0] \in |\mathbb{P}(1,3,4,4)|$ and a neighborhood in $\mathbb{F}_3$ of the exceptional divisor of χ. The isomorphism between these neighborhoods induces isomorphisms of stacks (see e.g. [13, Lemma 7.1]):

$$\overline{\mathcal{M}}_{0,0}(Z, a\Gamma_4) \cong \overline{\mathcal{M}}_{0,0}(\mathbb{F}_3, a\gamma_1) \quad \forall a \in \mathbb{N} - \{0\}. \tag{3.4}$$

Since the virtual fundamental classes of the above stacks depend only on neighborhoods of the exceptional divisors, it follows that

$$\deg[\overline{\mathcal{M}}_{0,0}(Z, a\Gamma_4)]^{\mathrm{vir}} = \deg[\overline{\mathcal{M}}_{0,0}(\mathbb{F}_3, a\gamma_1)]^{\mathrm{vir}} \quad \forall a \in \mathbb{N} - \{0\}.$$

Hence, the result follows. □

As a consequence, we have:

Corollary 3.1. *The series* (2.6) *converges in a neighborhood of the origin to an analytic function* $f(q)$ *that has analytic continuation in* $1 \in \mathbb{C}$. *Moreover, the value of* $f(q)$ *at* $q = 1$ *is*

$$f(1) = \frac{2\pi\beta_1}{9\beta_2^2} = \frac{(2\pi)^6}{27\Gamma\left(\frac{1}{3}\right)^9},$$

where $\beta_i = \frac{2\pi}{9\Gamma(\frac{i}{3})^3}$, $i \in \{1,2\}$.

Proof. It follows from [5] (see also [8, Theorem 7.2]) that the statement is true for (3.3) and Lemma 3.1 implies that it is also true for (2.6).

To compute $f(1)$, consider the map $\Theta(q) : H^\star_{\mathrm{CR}}(\mathbb{P}(1,1,1,3)) \to H^\star(\mathbb{F}_3)$ defined in [5, p. 56] and set $q = 0$. We consider $H^\star(\mathbb{F}_3)$ with the algebra structure coming from the quantum cohomology when we set the quantum parameter $\hat{q}$ at $(1,0)$ (see (3.2)). $\Theta(0)$ is a morphism of algebras, so we have [5][a]:

$$\frac{2\pi}{3}\beta_1\mathfrak{p}^2 = \Theta(0)(\mathbf{1}_{\frac{1}{3}}) = \Theta(0)(\mathbf{1}_{\frac{2}{3}}) \circ_{(1,0)} \Theta(0)(\mathbf{1}_{\frac{2}{3}}) = 3\beta_2^2\mathfrak{p} \circ_{(1,0)} \mathfrak{p},$$

therefore

$$\mathfrak{p} \circ_{(1,0)} \mathfrak{p} = f(1)\mathfrak{p}^2 = \frac{2\pi\beta_1}{9\beta_2^2}\mathfrak{p}^2 \quad \text{and} \quad f(1) = \frac{2\pi\beta_1}{9\beta_2^2}.$$

The second equality follows from standard identities of the Γ-function. □

4. The Main Result

From the results in [1] and Corollary 3.1, it follows that the power series in (2.4) converge in a neighborhood of the origin of $\mathbb{C}^4$ to analytic functions that admit analytic continuations in $(\mathrm{i},\mathrm{i},\mathrm{i},1)$ and in $(-\mathrm{i},-\mathrm{i},-\mathrm{i},1)$, moreover, the following result holds.

Theorem 4.1. *For* $(c_1,c_2,c_3,c_4) \in \{(\mathrm{i},\mathrm{i},\mathrm{i},1),(-\mathrm{i},-\mathrm{i},-\mathrm{i},1)\}$ *there is a ring isomorphism*

$$H^\star_\rho(Z;\mathbb{C})(c_1,c_2,c_3,c_4) \xrightarrow{\cong} H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4);\mathbb{C})$$

which is an isometry with respect to the Poincaré pairings.

Proof. Let us first consider the case where $(c_1,\ldots,c_4) = (\mathrm{i},\mathrm{i},\mathrm{i},1)$. Let

$$\Xi : H^\star_\rho(Z;\mathbb{C})(\mathrm{i},\mathrm{i},\mathrm{i},1) \to H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4);\mathbb{C}) \tag{4.1}$$

be the linear map defined by the following matrix with respect to the basis (2.2) and (2.3),

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\sqrt{2} & -2\mathrm{i} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -\mathrm{i}\sqrt{2} & 2\mathrm{i} & -\mathrm{i}\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \sqrt{2} & -2\mathrm{i} & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & -2\mathrm{i} & \sqrt{2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\sqrt{2} & 2\mathrm{i} & -\mathrm{i}\sqrt{2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & -2\mathrm{i} & -\sqrt{2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

[a]The careful reader will notice that, in [5], $\mathbb{P}(1,1,1,3)$ is defined as the quotient $[\mathbb{C}^4 - \{0\}/\mathbb{C}^\star]$ where the action has negative weights.

where α and β are complex numbers to be determined. Note that Ξ coincides with the map (4.3) in [1] except possibly for the image of e_4 and ${e_4}^2$.

Clearly, Ξ is an isomorphism of vector spaces if and only if $\alpha \cdot \beta \neq 0$. We have that $\Xi(e_i \star_\rho e_j) = \Xi(e_i) \cup_{\mathrm{CR}} \Xi(e_j)$ for $i, j \in \{1,2,3\}$ [1], therefore, it remains to find α and β such that $\alpha \cdot \beta \neq 0$, $\Xi(e_4 \star_\rho e_4) = \Xi(e_4) \cup_{\mathrm{CR}} \Xi(e_4)$ and $\Xi(e_4 \star_\rho e_4^2) = \Xi(e_4) \cup_{\mathrm{CR}} \Xi(e_4^2)$. This is equivalent to the equations:

$$\alpha^3 = 27 f(1) \quad \text{and} \quad \alpha \cdot \beta = 27. \tag{4.2}$$

The existence of α and β verifying (4.2) follows from Corollary 3.1. The fact that Ξ is an isometry is then a direct consequence of [1] and (4.2).

The case where $(c_1, \ldots, c_4) = (-\mathrm{i}, -\mathrm{i}, -\mathrm{i}, 1)$ is similar to the previous one, the linear map $H^\star_\rho(Z;\mathbb{C})(-\mathrm{i}, -\mathrm{i}, -\mathrm{i}, 1) \to H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4);\mathbb{C})$ coincides with (4.4) in [1] on $1, e_1, e_2, e_3, h^2, he_1, he_2, he_3, h^3$ and coincides with (4.1) on e_4 and e_4^2. Also, in this case, the fact that it is a ring isomorphism and that preserves the metrics follows from [1] and (4.2). This concludes the proof. □

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