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A MODEL FOR THE ORBIFOLD CHOW RING OF WEIGHTED PROJECTIVE SPACES

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We construct an isomorphism of graded Frobenius algebras between the orbifold Chow ring of weighted projective spaces and graded algebras of groups of roots of the unity.

Key Words: Frobenius algebras; Orbifold cohomology; Weighted projective spaces.

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1. INTRODUCTION

Recently, particular attention has been given to the study of the orbifold cohomology ring of weighted projective spaces. This cohomology originates in physics and has been defined mathematically by Chen and Ruan (2004). It has been further developed and adapted to the language of stacks by Abramovich et al. (2002). Jiang (2007) studies the orbifold cohomology ring of weighted projective spaces by means of their simplicial toric fan with an explicit computation for $\mathbb{P}(1, 2, 2, 3, 3, 3)$. Borisov et al. (2005) prove a formula for the orbifold Chow ring of toric Deligne–Mumford stacks in terms of their stacky fan, that can be applied to weighted projective spaces. Chen and Hu (2006), obtain a general formula for the computation of the orbifold product of abelian orbifolds, and apply it to weighted projective spaces. Holm (2007) uses symplectic geometry to compute a presentation of their integral orbifold cohomology ring with generators and relations. Coates et al. (2007) calculate the small quantum orbifold cohomology of weighted projective spaces by proving an explicit formula for their small J-function. In this note, we give an alternative description: Starting from the computation of their orbifold cohomology as given by the second author in Mann (2008), we exhibit

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a short and comprehensible model as a graded group algebra over some specific group of roots of the unity.

More precisely, for a given sequence of weights $w := (w_0, \dots, w_n)$, we construct (see Theorem 4.9) an isomorphism of graded Frobenius algebras between the orbifold Chow ring $A^\star_{\mathrm{orb}}(\mathbb{P}(w))$ and a suitably graded algebra $\mathrm{gr}^\star_F \mathbb{C}[\mathcal{U}_{|w|}]$ where $|w| := w_0 + \cdots + w_n$ and $\mathcal{U}_{|w|}$ denotes the group of $|w|$-roots of the unity (see §4.1 for the construction of the grading). This construction uses some combinatorics associated to the weights, partially present in Douai and Sabbah (2004), and the second author in Mann (2008) (see §2).

This new description of the orbifold Chow ring of weighted projective spaces is interesting mainly for three reasons. First, this model is not a description with generators of the ring and a list of relations: Instead, it gives a presentation such that the generators of the ring are a basis of the underlying vector space, and the ring structure is natural. Second, this description is analogous to the one of a global symplectic quotient stack $[V/G]$ where V is a symplectic vector space and $G \subset \mathrm{Sp}(V)$ a finite group: Ginzburg and Kaledin (2004) observed that the ring $A^\star_{\mathrm{orb}}([V/G])$ is isomorphic to $\mathrm{gr}^\star_F \mathbb{C}[G]^G$ (for an appropriate grading). Third, the computation of this orbifold Chow ring is an essential step for studying the Cohomological Crepant Resolution Conjecture for weighted projective spaces (see Boissière et al., 2006 for such computations).

2. SOME COMBINATORICS

Let $n \geq 1$ be an integer and $w := (w_0, \dots, w_n)$ a sequence of positive integers (or *weights*). Set $|w| := w_0 + \cdots + w_n$. For $\ell \in \mathbb{N}^*$, denote by $\mathcal{U}_\ell$ the group of ℓth roots of the unity and set $\mathcal{U} := \bigcup_{\ell \in \mathbb{N}^*} \mathcal{U}_\ell$. We define an order on the group $\mathcal{U}$ by taking the principal determination of the argument, inducing a bijection

$$\begin{aligned} \gamma : \mathcal{U} &\longrightarrow [0, 1[\cap \mathbb{Q} \\ g &\longmapsto \gamma(g) \qquad \text{where } \exp(2i\pi\gamma(g)) = g. \end{aligned}$$

For $g \in \mathcal{U}$, put:

$$\begin{aligned} I(g) &:= \{ i \in [\![0, n]\!] \mid g^{w_i} = 1 \}, \\ a(g) &:= \sum_{i=0}^{n} \{ \gamma(g) w_i \}, \end{aligned}$$

where $\{\cdot\}$ is the fractional part. Note that $I(g) = I(g^{-1})$. One has

$$\{\gamma(g) w_i\} = \begin{cases} 0 & \text{if } i \in I(g) \\ 1 - \{\gamma(g^{-1}) w_i\} & \text{otherwise} \end{cases} \tag{2.1}$$

hence:

$$a(g) + a(g^{-1}) = n + 1 - \#I(g). \tag{2.2}$$

We order the disjoint union $\bigsqcup_{i=0}^{n} \mathcal{U}_{w_i}$ by the injection:

$$\begin{aligned} \bigsqcup_{i=0}^{n} \mathcal{U}_{w_i} &\longrightarrow [0,1[\times [\![0,n]\!] \\ g &\longmapsto (\gamma(g), i), \end{aligned}$$

where $[0,1[\times [\![0,n]\!]$ is given the lexicographic order. This induces an increasing bijection $s : [\![0, |w|-1]\!] \to \bigsqcup_{i=0}^{n} \mathcal{U}_{w_i}$.

Example 2.3. Take $w = (1, 2, 3)$. The enumeration is

$$\begin{aligned} &\gamma : \mathcal{U}_1 \sqcup \mathcal{U}_2 \sqcup \mathcal{U}_3 \to \{0\} \sqcup \{0, 1/2\} \sqcup \{0, 1/3, 2/3\}, \\ &\gamma s : [\![0,5]\!] \to [0, 0, 0, 1/3, 1/2, 2/3]. \end{aligned}$$

The understanding of the growth of the map $\gamma s : [\![0, |w|-1]\!] \to [0, 1[$ is central in the sequel. First note that for $g \in \mathcal{U}$, the elements in the image of γs less or equal to $\gamma(g)$ are

$$0, \frac{1}{w_0}, \ldots, \frac{[\gamma(g)w_0]}{w_0}, 0, \frac{1}{w_1}, \ldots, \frac{[\gamma(g)w_1]}{w_1}, \ldots, 0, \frac{1}{w_n}, \ldots, \frac{[\gamma(g)w_n]}{w_n}, \tag{2.4}$$

where $[\cdot]$ is the integer part. In particular, $\#(\gamma s)^{-1}(\gamma(g)) = \#I(g)$. The growth is then controlled by the values, for $g \in \bigcup_{i=0}^{n} \mathcal{U}_{w_i}$:

$$\begin{aligned} k_{\min}(g) &:= \min\{k \in [\![0, |w|-1]\!] \mid \gamma s(k) = \gamma(g)\}, \\ k_{\max}(g) &:= \max\{k \in [\![0, |w|-1]\!] \mid \gamma s(k) = \gamma(g)\}. \end{aligned}$$

One has the relation $k_{\max}(g) = k_{\min}(g) + (\#I(g) - 1)$. Otherwise stated,

$$\gamma s(k_{\min}(g) + d) = \gamma(g) \qquad \forall d = 0, \ldots, \#I(g) - 1. \tag{2.5}$$

Another consequence of (2.4) is

$$\#\{k \in [\![0, |w|-1]\!] \mid \gamma s(k) \le \gamma(g)\} = n + 1 + \sum_{i=0}^{n} [\gamma(g) w_i].$$

One deduces

$$\begin{aligned} k_{\min}(g) &= (n + 1 - \#I(g)) + \sum_{i=0}^{n} [\gamma(g) w_i], \\ k_{\max}(g) &= n + \sum_{i=0}^{n} [\gamma(g) w_i]. \end{aligned}$$

Using that $\sum_{i=0}^{n} [\gamma(g) w_i] = |w|\gamma(g) - a(g)$ and Formula (2.2), one gets

$$k_{\min}(g) = a(g^{-1}) + |w|\gamma(g), \tag{2.6}$$

$$k_{\max}(g) = n + |w|\gamma(g) - a(g). \tag{2.7}$$

If $g \in \mathcal{U}$ but $g \notin \bigcup_{i=0}^{n} \mathcal{U}_{w_i}$, then $I(g) = \emptyset$ and

$$\begin{aligned}\#\{k \in [\![0, |w|-1]\!] \mid \gamma s(k) < \gamma(g)\} &= n+1+\sum_{i=0}^{n}[\gamma(g)w_i] \\ &= n+1+|w|\gamma(g) - a(g) \\ &= |w|\gamma(g) + a(g^{-1}) \qquad \text{by Formula (2.2)}\end{aligned}$$

so we can extend the definition of $k_{\min}(g)$ by setting

$$k_{\min}(g) := a(g^{-1}) + |w|\gamma(g) \qquad \forall g \in \mathcal{U}, \tag{2.8}$$

with the property that $k \geq k_{\min}(g)$ if and only if $\gamma s(k) \geq \gamma(g)$ (resp., $\gamma s(k) > \gamma(g)$, if $g \notin \bigcup_{i=0}^{n} \mathcal{U}_{w_i}$).

For $g, h \in \bigcup_{i=0}^{n} \mathcal{U}_{w_i}$, set

$$J(g, h) := \{i \in [\![0, n]\!] \mid \{\gamma(g)w_i\} + \{\gamma(h)w_i\} + \{\gamma(gh)^{-1}w_i\} = 2\}.$$

Using Formula (2.1) and noting that

$$\{\gamma(gh)w_i\} \equiv \{\gamma(g)w_i\} + \{\gamma(h)w_i\} \qquad \bmod 1, \tag{2.9}$$

one gets the following decomposition in disjoint union:

$$[\![0, n]\!] = (I(g) \cup I(h)) \sqcup (I(gh)\backslash(I(g) \cap I(h))) \sqcup J(g, h) \sqcup J(g^{-1}, h^{-1})$$

or more precisely,

$$\{\gamma(g)w_i\} + \{\gamma(h)w_i\} - \{\gamma(gh)w_i\} = \begin{cases} 0 & \text{if } i \in I(g) \cup I(h) \\ 1 & \text{if } i \in I(gh)\backslash(I(g) \cap I(h)) \\ 0 & \text{if } i \in J(g^{-1}, h^{-1}) \\ 1 & \text{if } i \in J(g, h). \end{cases} \tag{2.10}$$

This implies

$$a(g) + a(h) - a(gh) = \#(I(gh)\backslash(I(g) \cap I(h))) + \#J(g, h). \tag{2.11}$$

3. ORBIFOLD CHOW RING OF WEIGHTED PROJECTIVE SPACES

3.1. Weighted Projective Spaces

Let $w := (w_0, \ldots, w_n)$ be a sequence of weights. The group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}\backslash\{0\}$ by

$$\lambda \cdot (x_0, \ldots, x_n) := (\lambda^{w_0}x_0, \ldots, \lambda^{w_n}x_n).$$

The *weighted projective stack* $\mathbb{P}(w)$ is defined as the quotient stack $[(\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^*]$. It is a smooth proper Deligne–Mumford stack whose coarse moduli space, denoted $|\mathbb{P}(w)|$, is a projective variety of dimension n.

For any subset $I := \{i_1, \dots, i_k\} \subset \{0, \dots, n\}$, set $w_I := (w_{i_1}, \dots, w_{i_k})$. There is a natural closed embedding $\iota_I : \mathbb{P}(w_I) \hookrightarrow \mathbb{P}(w)$. The weighted projective stack $\mathbb{P}(w)$ comes with a natural invertible sheaf $\mathscr{O}_{\mathbb{P}(w)}(1)$ defined as follows: For any scheme X and any stack morphism $X \to \mathbb{P}(w)$ given by a principal $\mathbb{C}^*$-bundle $P \to X$ and a $\mathbb{C}^*$-equivariant morphism $P \to \mathbb{C}^{n+1}\backslash\{0\}$, one defines $\mathscr{O}_{\mathbb{P}(w)}(1)_X$ as the sheaf of sections of the associated line bundle of P. This sheaf is compatible with the embedding ι_I in the sense that $\iota_I^*\mathscr{O}_{\mathbb{P}(w)}(1) = \mathscr{O}_{\mathbb{P}(w_I)}(1)$.

3.2. Computation of the Orbifold Chow Ring

We denote by $A^\star_{\mathrm{orb}}(\mathscr{X})$ the orbifold Chow ring with *complex* coefficients of a Deligne–Mumford stack (or *orbifold*) $\mathscr{X}$. For toric stacks, such as weighted projective spaces (see Boissière et al., 2006), it is isomorphic to the even orbifold cohomology. As a vector space, $A^\star_{\mathrm{orb}}(\mathscr{X}) = A^\star(\mathscr{I}\mathscr{X})$ where $\mathscr{I}\mathscr{X}$ is the *inertia stack* of $\mathscr{X}$.

We recall the results of the second author in Mann (2008). The inertia stack of $\mathbb{P}(w)$ decomposes as

$$\mathscr{I}\,\mathbb{P}(w) = \coprod_{g\in\bigcup_{i=0}^n \mathscr{U}_{w_i}} \mathbb{P}(w_{I(g)}).$$

Note that $\dim \mathbb{P}(w_{I(g)}) = \#I(g) - 1$.

Example 3.1. Take again $w = (1, 2, 3)$. The components of the inertia stack of $\mathbb{P}(1, 2, 3)$ are indexed by the roots $1, j, -1, j^2$, where j is the primitive third root of the unity, so that

$$\mathscr{I}\,\mathbb{P}(1, 2, 3) = \mathbb{P}(1, 2, 3) \amalg \mathbb{P}(3) \amalg \mathbb{P}(2) \amalg \mathbb{P}(3).$$

For $g \in \bigcup_{i=0}^n \mathscr{U}_{w_i}$ and $d \in \{0, \dots, \dim \mathbb{P}(w_{I(g)})\}$, define the classes[1]

$$\eta_g^d := \left(\prod_{i=0}^n w_i^{-\{\gamma(g)w_i\}}\right) \cdot c_1(\mathscr{O}_{\mathbb{P}(w_{I(g)})}(1))^d \in A^d(|\mathbb{P}(w_{I(g)})|).$$

The first result concerns the vector space decomposition of $A^\star_{\mathrm{orb}}(\mathbb{P}(w))$.

Proposition 3.2 (Mann, 2008, Proposition 3.9 & Corollary 3.11)**.**

(1) *The structure of graded vector space of* $A^\star_{\mathrm{orb}}(\mathbb{P}(w))$ *is*:

$$A^\star_{\mathrm{orb}}(\mathbb{P}(w)) = \bigoplus_{g\in\bigcup_{i=0}^n \mathscr{U}_{w_i}} A^{\star-a(g)}(|\mathbb{P}(w_{I(g)})|).$$

(2) *The dimension of the vector space* $A^\star_{\mathrm{orb}}(\mathbb{P}(w))$ *is* $|w| = w_0 + \cdots + w_n$.

[1]The normalization factor differs from Mann (2008).

(3) *The set* $\boldsymbol{\eta} := \{\eta_g^d \mid g \in \bigcup_{i=0}^n \mathcal{U}_{w_i}, d \in [\![0, \#I(g)-1]\!]\}$ *is a basis of* $A^\star_{\text{orb}}(\mathbb{P}(w))$. *The orbifold degree of* η_g^d *is* $\deg(\eta_g^d) = d + a(g)$.

The second result expresses the orbifold Poincaré duality, denoted by $\langle -, - \rangle$, in the basis $\boldsymbol{\eta}$. We set $\langle w \rangle := \prod_{i=0}^n w_i$.

Proposition 3.3. *Let* $\eta_{g_0}^{d_0}$ *and* $\eta_{g_1}^{d_1}$ *be two elements of the basis* $\boldsymbol{\eta}$.

(1) *If* $g_0 g_1 \neq 1$, *then* $\langle \eta_{g_0}^{d_0}, \eta_{g_1}^{d_1} \rangle = 0$.
(2) *If* $g_0 g_1 = 1$, *then*:

$$\langle \eta_{g_0}^{d_0}, \eta_{g_1}^{d_1} \rangle = \begin{cases} \dfrac{1}{\langle w \rangle} & \textit{if } \deg(\eta_{g_0}^{d_0}) + \deg(\eta_{g_1}^{d_1}) = n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The vanishings come from the definition of the orbifold Poincaré duality. Assume that $\deg(\eta_{g_0}^{d_0}) + \deg(\eta_{g_0^{-1}}^{d_1}) = n$. According to Mann (2008, Proposition 3.13), one has

$$\langle \eta_{g_0}^{d_0}, \eta_{g_0^{-1}}^{d_1} \rangle = \left(\prod_{i=0}^n w_i^{-\{\gamma(g_0) w_i\} - \{\gamma(g_0^{-1}) w_i\}} \right) \prod_{i \in I(g_0)} w_i^{-1},$$

so Formula (2.1) gives $\langle \eta_{g_0}^{d_0}, \eta_{g_0^{-1}}^{d_1} \rangle = \frac{1}{\langle w \rangle}$. □

The third result computes the orbifold cup product, denoted $\cup$, in the basis $\boldsymbol{\eta}$.

Proposition 3.4. *Let* $\eta_{g_0}^{d_0}$ *and* $\eta_{g_1}^{d_1}$ *be two elements of the basis* $\boldsymbol{\eta}$. *It is*

$$\eta_{g_0}^{d_0} \cup \eta_{g_1}^{d_1} = \eta_{g_0 g_1}^d.$$

with $d := \deg(\eta_{g_0}^{d_0}) + \deg(\eta_{g_1}^{d_1}) - a(g_0 g_1)$.

Remark 3.5. By Proposition 3.2 and Formula (2.11), one has

$$d = a(g_0) + a(g_1) - a(g_0 g_1) + d_0 + d_1 \geq 0.$$

The formula for the cup product makes sense only with the following conventions:

(i) If $g_0 g_1 \notin \bigcup_{i=0}^n \mathcal{U}_{w_i}$, then $\eta_{g_0 g_1}^d = 0$. The reason is that the component of the inertia stack corresponding to $g_0 g_1 \in \mathcal{U}$ is empty.
(ii) If $d > \dim \mathbb{P}(w_{I(g_0 g_1)})$, then $\eta_{g_0 g_1}^d = 0$.

Proof of Proposition 3.4. Set $K(g_0, g_1) := J(g_0, g_1) \bigsqcup (I(g_0 g_1) \backslash (I(g_0) \cap I(g_1)))$. According to Mann (2008, Corollary 3.18), we have

$$\eta_{g_0}^{d_0} \cup \eta_{g_1}^{d_1} = \left(\prod_{i=0}^n w_i^{-\{\gamma(g_0 w_i)\} - \{\gamma(g_1) w_i\} + \{\gamma(g_0 g_1) w_i\}} \cdot \prod_{i \in K(g_0, g_1)} w_i \right) \cdot \eta_{g_0 g_1}^d.$$

Formula (2.10) gives the result. □

4. THE MODEL

4.1. Construction of the Model

Let $w := (w_0, \dots, w_n)$ be a sequence of weights. Consider the group $\mathcal{U}_{|w|}$ of $|w|$th roots of the unity, and take $\xi := \exp(2i\pi/|w|)$ as primitive $|w|$th root. The set $\boldsymbol{\xi} := \{1, \xi, \dots, \xi^{|w|-1}\}$ is a basis of the group algebra $\mathbb{C}[\mathcal{U}_{|w|}]$, and we define

$$\deg(\xi^j) := j - |w|\gamma s(j) \qquad \forall j = 0, \dots, |w| - 1.$$

Example 4.1. Take again $w = (1, 2, 3)$. The degrees in the group $\mathcal{U}_6$ are

$\mathcal{U}_6$	1	ξ	ξ^2	ξ^3	ξ^4	ξ^5
deg	0	1	2	1	1	1

Lemma 4.2.

(1) *For all* j, $0 \le \deg(\xi^j) \le n$.
(2) *For all* j, k, $\deg(\xi^{j+k}) \le \deg(\xi^j) + \deg(\xi^k)$.

Proof. (1) By definition, $k_{\min}(s(j)) \le j \le k_{\max}(s(j))$. Formulas (2.6) and (2.7) give

$$0 \le a(s(j)^{-1}) \le \deg(\xi^j) \le n - a(s(j)) \le n.$$

(2) Set $g_0 := s(j)$ and $g_1 := s(k)$. Then

$$\begin{aligned} j &= k_{\min}(g_0) + d_0 \qquad \text{with } d_0 \le \#I(g_0) - 1,\\ k &= k_{\min}(g_1) + d_1 \qquad \text{with } d_1 \le \#I(g_1) - 1. \end{aligned}$$

Using Formulas (2.5) and (2.6) one gets

$$\begin{aligned} \deg(\xi^j) &= j - |w|\gamma(g_0) = k_{\min}(g_0) + d_0 - |w|\gamma(g_0) = a(g_0^{-1}) + d_0,\\ \deg(\xi^k) &= k - |w|\gamma(g_1) = k_{\min}(g_1) + d_1 - |w|\gamma(g_1) = a(g_1^{-1}) + d_1. \end{aligned}$$

One computes with Formula (2.6)

$$\begin{aligned} j + k &= k_{\min}(g_0) + d_0 + k_{\min}(g_1) + d_1\\ &= a(g_0^{-1}) + |w|\gamma(g_0) + a(g_1^{-1}) + |w|\gamma(g_1) + d_0 + d_1\\ &= \deg(\xi^j) + \deg(\xi^k) + |w|(\gamma(g_0) + \gamma(g_1)). \end{aligned}$$

Setting $d := \deg(\xi^j) + \deg(\xi^k) - a((g_0 g_1)^{-1})$ and using that

$$\gamma(g_0) + \gamma(g_1) \equiv \gamma(g_0 g_1) \qquad \mod 1,$$

one gets:

$$\xi^j \cdot \xi^k = \xi^{j+k} = \xi^{k_{\min}(g_0 g_1) + d}. \tag{4.3}$$

(a) If $k_{\min}(g_0g_1) + d \leq |w| - 1$, then

$$\begin{aligned} \deg(\xi^{j+k}) &= k_{\min}(g_0g_1) + d - |w|\gamma s(k_{\min}(g_0g_1) + d) \\ &= |w|(\gamma(g_0g_1) - \gamma s(k_{\min}(g_0g_1) + d)) + \deg(\xi^j) + \deg(\xi^k). \end{aligned}$$

By Formula (2.8) one has $\gamma(g_0g_1) \leq \gamma(k_{\min}(g_0g_1) + d)$, hence the result.

(b) If $k_{\min}(g_0g_1) + d \geq |w|$, then

$$\begin{aligned} \deg(\xi^{j+k}) &= k_{\min}(g_0g_1) + d - |w| - |w|\gamma s(k_{\min}(g_0g_1) + d - |w|) \\ &= |w|(\gamma(g_0g_1) - 1 - \gamma s(k_{\min}(g_0g_1) + d - |w|)) + \deg(\xi^j) + \deg(\xi^k) \end{aligned}$$

and $\gamma(g_0g_1) \leq 1$, hence the result. □

Remark 4.4. Looking at (2.4), one observes that if w_i divides $|w|$ for all i, then $|w|\gamma s(j) \in \mathbb{N}$ for all j so $\deg(\xi^j)$ is an integer for all j. In this case, the orbifold $\mathbb{P}(w)$ is Gorenstein.

For any element $z := \sum_{\sigma \in \mathcal{U}_{|w|}} z_\sigma \cdot \sigma \in \mathbb{C}[\mathcal{U}_{|w|}]$ we set $\deg(z) := \max\{\deg \sigma \mid z_\sigma \neq 0\}$. Introduce the increasing filtration

$$F^u\mathbb{C}[\mathcal{U}_{|w|}] := \{z \in \mathbb{C}[\mathcal{U}_{|w|}] \mid \deg(z) \leq u\} \qquad \text{for } u \in [0, n] \cap \mathbb{Q}.$$

By Lemma 4.2, the natural ring structure on $\mathbb{C}[\mathcal{U}_{|w|}]$ is compatible with this filtration

$$F^u\mathbb{C}[\mathcal{U}_{|w|}] \cdot F^v\mathbb{C}[\mathcal{U}_{|w|}] \subset F^{u+v}\mathbb{C}[\mathcal{U}_{|w|}].$$

Set $F^{<u}\mathbb{C}[\mathcal{U}_{|w|}] := \{z \in \mathbb{C}[\mathcal{U}_{|w|}] \mid \deg(z) < u\}$. The induced product on the graded space $\mathrm{gr}_F^\star\mathbb{C}[\mathcal{U}_{|w|}] := \bigoplus_{u\in[0,n]\cap\mathbb{Q}} F^u\mathbb{C}[\mathcal{U}_{|w|}]/F^{<u}\mathbb{C}[\mathcal{U}_{|w|}]$ defines a structure of graded ring denoted $\cup$. For $\sigma_1, \sigma_2 \in \mathcal{U}_{|w|}$, it is

$$\sigma_1 \cup \sigma_2 = \begin{cases} \sigma_1\sigma_2 & \text{if } \deg(\sigma_1) + \deg(\sigma_2) = \deg(\sigma_1\sigma_2) \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.5. Take again $w = (1, 2, 3)$. The ring structure on $\mathrm{gr}_F^\star\mathbb{C}[\mathcal{U}_6]$ is given by the following table

$\mathrm{gr}_F^\star\mathbb{C}[\mathcal{U}_6]$	1	ξ	ξ^2	ξ^3	ξ^4	ξ^5
1	1	ξ	ξ^2	ξ^3	ξ^4	ξ^5
ξ	ξ	ξ^2	0	0	0	0
ξ^2	ξ^2	0	0	0	0	0
ξ^3	ξ^3	0	0	0	0	ξ^2
ξ^4	ξ^4	0	0	0	ξ^2	0
ξ^5	ξ^5	0	0	ξ^2	0	0

We define an integral $\int : \mathrm{gr}_F^\star \mathbb{C}[\mathcal{U}_{|w|}] \to \mathbb{C}$ by setting for $j \in [\![0, |w|-1]\!]$

$$\int \xi^j = \begin{cases} \dfrac{1}{\langle w \rangle} & \text{if } j = n \\ 0 & \text{otherwise} \end{cases}$$

and extending by linearity. The reason is that the only j such that $s(j) = 1$ (nontwisted sector) and $\deg(\xi^j) = n$ is $j = n$. We further define a pairing $\langle\langle -, - \rangle\rangle$ on $\mathrm{gr}_F^\star \mathbb{C}[\mathcal{U}_{|w|}]$ by setting for $\sigma_1, \sigma_2 \in \mathcal{U}_{|w|}$

$$\langle\langle \sigma_1, \sigma_2 \rangle\rangle := \int \sigma_1 \cup \sigma_2$$

and extending by bilinearity.

Example 4.6. Take again $w = (1, 2, 3)$. The matrix of the pairing $\langle\langle -, - \rangle\rangle$ in the basis $\boldsymbol{\xi}$ is

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1/6 & 0 & 0 & 0 \\ 0 & 1/6 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1/6 & 0 & 0 \end{array}\right).$$

Lemma 4.7. *The pairing $\langle\langle -, - \rangle\rangle$ is perfect.*

Proof. As in the proof of Lemma 4.2, for $\xi^j = \xi^{k_{\min}(g_0)+d_0}$ with $g_0 = s(j)$ and $d_0 \leq \#I(g_0) - 1$, set $k := k_{\min}(g_0^{-1}) + d_1$ with $d_1 := \#I(g_0) - 1 - d_0$. Then by Formula (4.3), $\xi^j \cdot \xi^k = \xi^n$ with $\deg(\xi^n) = n$ and by Formula (2.2)

$$\deg(\xi^j) + \deg(\xi^k) = a(g_0) + a(g_0^{-1}) + \#I(g_0) - 1 = n$$

so $\langle\langle \xi^j, \xi^k \rangle\rangle = \frac{1}{\langle w \rangle}$. □

As a consequence, the structure $(\mathrm{gr}_F^\star \mathbb{C}[\mathcal{U}_{|w|}], \cup, \langle\langle -, - \rangle\rangle)$ is a graded Frobenius algebra, as $(A^\star_{\mathrm{orb}}(\mathbb{P}(w)), \cup, \langle -, - \rangle)$ is.

4.2. Isomorphism with the Orbifold Chow Ring

Define a linear map $\Xi : A^\star_{\mathrm{orb}}(\mathbb{P}(w)) \to \mathrm{gr}_F^\star \mathbb{C}[\mathcal{U}_{|w|}]$ by setting

$$\Xi(\eta_g^d) := \xi^{k_{\min}(g^{-1})+d}$$

and extending by linearity.

Example 4.8. Take again $w = (1, 2, 3)$. The map Ξ is given by

$$\Xi(\eta_1^0) = 1, \qquad \Xi(\eta_1^1) = \xi, \qquad \Xi(\eta_1^2) = \xi^2,$$
$$\Xi(\eta_{\mathrm{j}}^0) = \xi^5, \qquad \Xi(\eta_{-1}^0) = \xi^4, \qquad \Xi(\eta_{\mathrm{j}^2}^0) = \xi^3.$$

Theorem 4.9. *The map* $\Xi : A^\star_{\mathrm{orb}}(\mathbb{P}(w)) \to \mathrm{gr}^\star_F \mathbb{C}[\mathcal{U}_{|w|}]$ *is an isomorphism of graded Frobenius algebras.*

Proof. **Step 1.** Ξ *is an isomorphim.* By definition of $k_{\min}(g)$ and since $d \le \#I(g) - 1$, as g and d vary, the numbers $k_{\min}(g^{-1}) + d$ are all distinct and cover $[\![0, |w| - 1]\!]$, so the map Ξ maps the basis $\boldsymbol{\eta}$ onto the basis $\boldsymbol{\xi}$.

Step 2. Ξ *is graded.* It is

$$\begin{aligned}\deg(\xi^{k_{\min}(g^{-1})+d}) &= k_{\min}(g^{-1}) + d - |w|\gamma s(k_{\min}(g^{-1}) + d)\\ &= k_{\min}(g^{-1}) + d - |w|\gamma(g^{-1}) \qquad \text{by Formula (2.5)}\\ &= a(g) + d \qquad \text{by Formula (2.6)}\\ &= \deg(\eta_g^d).\end{aligned}$$

Step 3. Ξ *is a ring morphism.* We use notation of Proposition 3.4 and Step 2. The same computation as in Formula (4.3) gives

$$\xi^{k_{\min}(g_0^{-1})+d_0} \cdot \xi^{k_{\min}(g_1^{-1})+d_1} = \xi^{k_{\min}((g_0g_1)^{-1})+d}. \tag{4.10}$$

i. Assume that $(g_0g_1)^{-1} \in \bigcup_{i=0}^n \mathcal{U}_{w_i}$ and $d \le \dim \mathbb{P}(w_{I(g_0g_1)})$. Then Formula (4.10) means

$$\Xi(\eta_{g_0}^{d_0}) \cdot \Xi(\eta_{g_1}^{d_1}) = \Xi(\eta_{g_0g_1}^{d}) = \Xi(\eta_{g_0}^{d_0} \cup \eta_{g_1}^{d_1}).$$

Since Ξ is graded, this implies

$$\Xi(\eta_{g_0}^{d_0}) \cup \Xi(\eta_{g_1}^{d_1}) = \Xi(\eta_{g_0}^{d_0} \cup \eta_{g_1}^{d_1}).$$

ii. Assume that $(g_0g_1)^{-1} \notin \bigcup_{i=0}^n \mathcal{U}_{w_i}$ or $d > \dim \mathbb{P}(w_{I(g_0g_1)})$. Since

$$\deg(\xi^{k_{\min}(g_0^{-1})+d_0}) + \deg(\xi^{k_{\min}(g_1^{-1})+d_1}) = \deg(\eta_{g_0}^{d_0}) + \deg(\eta_{g_1}^{d_1}) = a(g_0g_1) + d,$$

we have to show that $\deg(\xi^{k_{\min}((g_0g_1)^{-1})+d}) < a(g_0g_1) + d$.

- If $k_{\min}((g_0g_1)^{-1}) + d \le |w| - 1$, then

$$\begin{aligned}\deg(\xi^{k_{\min}((g_0g_1)^{-1})+d}) &= k_{\min}((g_0g_1)^{-1}) + d - |w|\gamma s(k_{\min}((g_0g_1)^{-1}) + d)\\ &= a(g_0g_1) + d + |w|\gamma((g_0g_1)^{-1}) - |w|\gamma s(k_{\min}((g_0g_1)^{-1}) + d).\end{aligned}$$

In both cases $d > \#I(g_0g_1) - 1$ or $(g_0g_1)^{-1} \notin \bigcup_{i=0}^n \mathcal{U}_{w_i}$, one has by Formula (2.8) $\gamma s(k_{\min}((g_0g_1)^{-1}) + d) > \gamma((g_0g_1)^{-1})$, hence the result.

- If $k_{\min}((g_0g_1)^{-1}) + d \geq |w|$, then

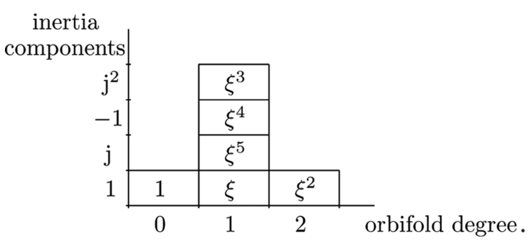
$$\begin{aligned}\deg(\xi^{k_{\min}((g_0g_1)^{-1})+d}) &= k_{\min}((g_0g_1)^{-1}) + d - |w| \\ &\quad - |w|\gamma s(k_{\min}((g_0g_1)^{-1}) + d - |w|) \\ &= a(g_0g_1) + d + |w|\gamma((g_0g_1)^{-1}) - |w| \\ &\quad - |w|\gamma s(k_{\min}((g_0g_1)^{-1}) + d - |w|).\end{aligned}$$

Since $\gamma((g_0g_1)^{-1}) < 1 + \gamma s(k_{\min}((g_0g_1)^{-1}) + d - |w|)$, one gets the result.

Step 4. Ξ *is compatible with the pairings.* We use notation of Proposition 3.3.

a) If $g_0g_1 \neq 1$, then $s(k_{\min}((g_0g_1)^{-1}) + d) \neq 1$ so using Formula (4.3), we see that $\langle\langle \Xi(\eta_{g_0}^{d_0}), \Xi(\eta_{g_1}^{d_1})\rangle\rangle = 0 = \langle \eta_{g_0}^{d_0}, \eta_{g_1}^{d_1}\rangle$.
b) If $g_0g_1 = 1$, then $d = \deg(\eta_{g_0}^{d_0}) + \deg(\eta_{g_1}^{d_1})$ and $\deg(\xi^{k_{\min}((g_0g_1)^{-1})+d}) = d$ so if $d < n$, then $\langle\langle \Xi(\eta_{g_0}^{d_0}), \Xi(\eta_{g_1}^{d_1})\rangle\rangle = 0 = \langle \eta_{g_0}^{d_0}, \eta_{g_1}^{d_1}\rangle$ and if $d = n$, then $\langle\langle \Xi(\eta_{g_0}^{d_0}), \Xi(\eta_{g_1}^{d_1})\rangle\rangle = \frac{1}{\langle w\rangle} = \langle \eta_{g_0}^{d_0}, \eta_{g_1}^{d_1}\rangle$. $\square$

Example 4.11. Take again $w = (1, 2, 3)$. The orbifold Chow ring of $\mathbb{P}(1, 2, 3)$ can be pictured as follows:



Example 4.12. Take $w = (1, \ldots, 1)$ ($n+1$ times). Then $\mathrm{gr}_F^{\star}\mathbb{C}[\mathcal{U}_{n+1}] \cong \mathbb{C}[h]/h^{n+1}$ where h has degree one. On the other hand, $\mathbb{P}(w) \cong \mathbb{P}_{\mathbb{C}}^n$ and $\mathcal{I}\mathbb{P}(w) = \mathbb{P}_{\mathbb{C}}^n$ so we recover the well-known fact

$$A_{\mathrm{orb}}^{\star}(\mathbb{P}(1, \ldots, 1)) \cong \mathrm{gr}_F^{\star}\mathbb{C}[\mathcal{U}_{n+1}] \cong \mathbb{C}[h]/h^{n+1} \cong A^{\star}(\mathbb{P}_{\mathbb{C}}^n).$$

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