QUANTUM $\mathcal{D}$-MODULES FOR TORIC NEF COMPLETE INTERSECTIONS

by

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Abstract. — Let X be a smooth projective toric variety with k ample line bundles. Let Z be the zero locus of k generic sections. It is well-known that the ambient quantum $\mathcal{D}$-module of Z is cyclic *i.e.*, is defined by an ideal of differential operators. In this paper, we give an explicit construction of this ideal as a quotient ideal of a GKZ system associated to the toric data of X and the line bundles. This description can be seen as a "left cancellation procedure". We consider some examples where this description enables us to compute generators of this ideal, and thus to give a presentation of the ambient quantum $\mathcal{D}$-module.

Contents

1. Introduction

Mirror symmetry has many different formulations in mathematics: equivalence of derived categories (known as Homological Mirror Symmetry by Kontsevich [**Kon95**]), isomorphism of Frobenius manifolds (see [**Bar00**]), comparison of Hodge numbers for Calabi-Yau varieties (see for example [**Bat94**]), isomorphism of Givental's conesb (see [**Giv98**]), isomorphism of pure polarized TERP structures (see [**Her06**]) or variation of non-commutative Hodge structures (see [**KKP08**]).

Inspired by the works of Givental (see for examples [**Giv96**] and [**Giv98**]), many authors have considered quantum cohomology with a differential module approach : see Kim [**Kim99**] and Rietsch (with Marsh and Pech-Williams) [**Rie12**] [**MR13**] [**PRW16**] for homogeneous spaces, see Coates-Corti-Lee-Tseng [**CLCT09**] and Guest-Sakai [**GS14**] for weighted projective spaces, see also the works of Iritani [**Iri06**], [**Iri07**], [**Iri08**] and [**Iri09**], the book of Cox-Katz [**CK99**] and the one of Guest [**Gue10**].

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From the small quantum product on a smooth projective variety Z, one can define a trivial vector bundle over $\mathbf{D}\times\mathbb{C}$ where $\mathbf{D}$ is an open subset of $H^2(Z,\mathbb{C})$ whose fibers are $H^{2*}(Z,\mathbb{C})$. This holomorphic bundle is endowed with a flat meromorphic connection and a non-degenerate pairing. These data collectively define the quantum $\mathcal{D}$-module of Z, which is denoted by $\mathrm{QDM}(Z)$. When Z is a smooth **toric** Fano variety, Givental (see also Iritani [**Iri09**] for toric weak Fano orbifolds) gives an explicit presentation of this $\mathcal{D}$-module using GKZ systems (Gelfand-Kapranov-Zelevinsky) in other words $\mathrm{QDM}(Z)$ is isomorphic to $\mathcal{D}/\mathcal{G}_Z$ where $\mathcal{G}_Z$ is the GKZ ideal associated to the toric data of Z. When Z is Fano, restricting this isomorphism to $\mathbf{D}\times\{0\}$ gives an isomorphism between the quantum cohomology ring of Z and a commutative algebra constructed by Batyrev in [**Bat93**].

In this paper, we investigate the non toric case where Z is a nef complete intersection subvariety in a smooth toric variety X. To be more precise, let $\mathcal{L}_1,\dots,\mathcal{L}_k$ be ample line bundles on X. Let Z be the zero locus of a generic section of $\mathcal{E}:=\oplus_{i=1}^k\mathcal{L}_i$. Denote by $\iota: Z\hookrightarrow X$ the closed embedding. By Lefschetz theorem, we have $H^*(Z,\mathbb{C})=\mathrm{Im}\,\iota^*\oplus\ker\iota_*$. The sub-vector space $\mathrm{Im}\,\iota^*$ is called the ambient part of the cohomology of Z, denoted by $H^*_{\mathrm{amb}}(Z)$. As $H^*_{\mathrm{amb}}(Z)$ is stable by the small quantum product of Z, we can define a sub $\mathcal{D}$-module, denoted by $\mathrm{QDM}_{\mathrm{amb}}(Z)$, whose fibers are $H^*_{\mathrm{amb}}(Z)$. A natural question is to find an explicit presentation of $\mathrm{QDM}_{\mathrm{amb}}(Z)$. It is well known that the GKZ ideal associated to $\mathcal{E}$, denoted by $\mathcal{G}_{(X,\mathcal{E})}$, is part of the equations. Cox and Katz addressed in the book [**CK99**, p.94-95 and p.101] the following question: what differential equations shall we add to $\mathcal{G}_{(X,\mathcal{E})}$ to get an isomorphism with $\mathrm{QDM}_{\mathrm{amb}}(Z)$?

Before giving an answer to this question in Theorem 1.1, we need to introduce some notations. Denote by $c_{\mathrm{top}}(\mathcal{E})$ the top Chern class of $\mathcal{E}$ and by $\widehat{c}_{\mathrm{top}}\in\mathcal{D}$ its associated operator (see Notation 4.3). Denote by $(\mathcal{G}_{(X,\mathcal{E})}:\widehat{c}_{\mathrm{top}})$ the left quotient ideal that is the left ideal of $\mathcal{D}$ defined by

$$(\mathcal{G}_{(X,\mathcal{E})}:\widehat{c}_{\mathrm{top}}):=\langle P\in\mathcal{D}\mid \widehat{c}_{\mathrm{top}}P\in\mathcal{G}_{(X,\mathcal{E})}\rangle.$$

Theorem 1.1 **(See Theorem 5.9)**. — *Let $\mathcal{L}_1,\dots,\mathcal{L}_k$ be ample line bundles on X, and assume that $\dim_{\mathbb{C}}X\geq k+3$. Let Z be the zero of a generic section of $\mathcal{E}:=\oplus_{i=1}^k\mathcal{L}_i$. Denote by $\iota: Z\hookrightarrow X$ the closed embedding. The ambient $\mathcal{D}$-module $\mathrm{QDM}_{\mathrm{amb}}(Z)$ is isomorphic to $\mathcal{D}/(\mathcal{G}_{(X,\mathcal{E})}:\widehat{c}_{\mathrm{top}})$.*

The quotient ideal $(\mathcal{G}_{(X,\mathcal{E})}:\widehat{c}_{\mathrm{top}})$ can be seen as a precise statement for the "left cancellation procedure" that appears in the works of Golyshev [**Gol07**, §2.9 and 2.10] and Guest-Sakai [**GS14**, p.287].

Reichelt-Sevenheck used this presentation of $\mathrm{QDM}_{\mathrm{amb}}(Z)$ to prove a mirror theorem for non affine Landau-Ginzburg model (see [**RS12**, Theorem 6.11]).

To prove our main theorem, we proceed in several steps.

1. In the first section, we review some standard facts on twisted quantum $\mathcal{D}$-module $\mathrm{QDM}(X,\mathcal{E})$ which is of rank $\dim_{\mathbb{C}}H^*(X)$ and is defined via the Gromov-Witten invariants twisted by $\mathcal{E}$. We have a surjective morphism $\varphi:\mathrm{QDM}(X,\mathcal{E})\to\mathrm{QDM}_{\mathrm{amb}}(Z)$ and we construct an explicit quotient of $\mathrm{QDM}(X,\mathcal{E})$ which gives an isomorphism with $\mathrm{QDM}_{\mathrm{amb}}(Z)$ (see Proposition 2.19).
2. Then we prove that we have an isomorphism of $\mathcal{D}$-modules $\varphi:\mathcal{D}/\mathcal{G}_{(X,\mathcal{E})}\to\mathrm{QDM}(X,\mathcal{E})$. To show this statement, we first define a surjective morphism. Then we prove that $\mathcal{D}/\mathcal{G}_{(X,\mathcal{E})}$ is locally free of rank $\dim_{\mathbb{C}}H^*(X)$. The freeness is proved in Section 4. To compute the rank, we restrict $\mathcal{D}/\mathcal{G}_{(X,\mathcal{E})}$ to $\mathbf{D}\times\{0\}$ and we get a commutative algebra. This algebra is a twisted version of the standard Batyrev algebras in [**Bat93**]. In Section 3, we prove that the spectrum of this algebra is locally free of rank $H^*(X)$ over some explicit open subset of $\mathbf{D}$ (see Theorem 3.18).
3. Using the isomorphism φ constructed above, we define a morphism $\overline{\varphi}:\mathcal{D}/(\mathcal{G}_{(X,\mathcal{E})}:\widehat{c}_{\mathrm{top}})\to\mathrm{QDM}_{\mathrm{amb}}(Z)$ which is surjective. To prove that $\overline{\varphi}$ is an isomorphism, we prove that $\mathcal{D}/(\mathcal{G}_{(X,\mathcal{E})}:\widehat{c}_{\mathrm{top}})$ is locally free of rank $\dim_{\mathbb{C}}H^*_{\mathrm{amb}}(Z)$. The freeness is proved in Section 4. To compute the rank, we restrict $\mathcal{D}/(\mathcal{G}_{(X,\mathcal{E})}:\widehat{c}_{\mathrm{top}})$ to $\mathbf{D}\times\{0\}$ and we get a commutative

algebra. In Section 3, we prove that the spectrum of this algebra is locally free of rank $H^*_{\mathrm{amb}}(Z)$ over some explicit open subset of $\mathbf{D}$ (see Theorem 3.18).

The plan of this article is the following.

Section 2 contains a brief discussion of the twisted quantum $\mathcal{D}$-module $\mathrm{QDM}(X, \mathcal{E})$.

In Section 3, we define and study twisted Batyrev algebras for a quasi-projective toric variety. The main result of this section is Theorem 3.18. Notice that this section can be read independently of the rest of the paper.

In Section 4, we prove that the GKZ modules of $\mathcal{E}^\vee$ and its residual are locally free sheaves. Using Section 3, we compute their ranks. The main result of this section is Theorem 4.15.

In Section 5, we state and prove Theorem 1.1 in Subsection 5.2.

In Section 6, we give two explicit computations of the generators of the quotient ideal for $\mathbb{P}^n$ with the line bundle $\mathcal{O}(a)$ with $a \in \{1, \dots, n+1\}$ and the blow-up of $\mathbb{P}^n$ at a point with the line bundle $\mathcal{O}(aH + bE)$ with $b \in \{-1, \dots, 1-n\}$ and $a + b \in \{1, 2\}$.

For the sake of completeness, lacking references in the literature, we review the axioms for twisted Gromov–Witten theory in Appendix A.

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Notation 1.2. — We use calligraphic letters for the sheaves such as $\mathcal{D}, \mathcal{G}, \mathcal{M}, \mathcal{M}^{\mathrm{res}}$. We use bold letters for modules or ideals on non commutative rings such as $\mathbb{D}, \mathbb{G}, \mathbb{M}, \mathbb{M}^{\mathrm{res}}$.

2. Twisted and reduced quantum $\mathcal{D}$-modules with geometric interpretation

Let X be a smooth projective complex variety of dimension n and $\mathcal{L}_1, \dots, \mathcal{L}_k$ be globally generated line bundles. Denote by $\mathcal{E}$ the sum $\mathcal{E} := \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$.

Notation 2.1. — For $0 \leq i \leq 2n$, denote by $H^i(X) := H^i(X, \mathbb{C})$ the complex cohomology group of classes of degree i. Also denote by $H^*(X)$ the complex cohomology ring $\oplus_{i=0}^{2n} H^i(X)$; the even part of this ring will be written $H^{2*}(X)$. Put $s = \dim_{\mathbb{C}} H^{2*}(X)$ and $r = \dim_{\mathbb{C}} H^2(X)$.

We fix, once and for all, a homogeneous basis $(T_0, \dots, T_{s-1})$ of $H^{2*}(X)$ such that $T_0 = \mathbf{1}$ is the unit for the cup product and that the classes $T_1, \dots, T_r$ form a basis of $H^2(X, \mathbb{Z})$ modulo torsion. Also denote by $(T^0, \dots, T^{s-1})$ the Poincaré dual in $H^{2*}(X)$ of $(T_0, \dots, T_{s-1})$.

As a convention, we will write $H_2(X, \mathbb{Z})$ for the degree 2 integer homology modulo torsion. Denote by $(B_1, \dots, B_r)$ the dual basis of $(T_1, \dots, T_r)$ in $H_2(X, \mathbb{Z})$. The associated coordinates will be denoted by $(d_1, \dots, d_r)$.

We denote by $\mathcal{T}_X$ the tangent bundle of X and by ω_X its canonical sheaf.

As a convention, we will make no notational distinction between vector bundles and locally free sheaves, writing –for example– $\mathcal{E}$ for both.

2.1. Twisted quantum $\mathcal{D}$-module. —

2.1.a. Twisted Gromov-Witten invariants. — Let ℓ be in $\mathbb{N}$ and d be in $H_2(X,\mathbb{Z})$. Denote by $X_{0,\ell,d}$ the moduli space of stable maps of degree d from rational curves with ℓ marked points to X. The universal curve over $X_{0,\ell,d}$ is $X_{0,\ell+1,d}$

$$\begin{array}{ccc} X_{0,\ell+1,d} & \xrightarrow{\mathrm{e}_{\ell+1}} & X \\ \downarrow{\scriptstyle \pi} & & \\ X_{0,\ell,d} & & \end{array}$$

where π is the map that forgets the $(\ell+1)$-th point and stabilises, and $\mathrm{e}_{\ell+1}$ is the evaluation at the $(\ell+1)$-th marked point. By Lemma 10 in [**FP97**]) the sheaf $\mathcal{E}_{0,\ell,d} := R^0\pi_*\,\mathrm{e}_{\ell+1}^*\,\mathcal{E}$ is locally free of rank $\int_d c_1(\mathcal{E}) + k$.

For j in $\{1,\dots,\ell\}$, we define the surjective morphism $\mathcal{E}_{0,\ell,d} \to \mathrm{e}_j^*\,\mathcal{E}$ by evaluating the section at the j-th marked point. We define $\mathcal{E}_{0,\ell,d}(j)$ to be the kernel of this map ; that is, we have the following exact sequence

$$0 \longrightarrow \mathcal{E}_{0,\ell,d}(j) \longrightarrow \mathcal{E}_{0,\ell,d} \longrightarrow \mathrm{e}_j^*\,\mathcal{E} \longrightarrow 0 \tag{2.2}$$

For any $j \in \{1,\dots,\ell\}$ the bundle $\mathcal{E}_{0,\ell,d}(j)$ has rank $\int_d c_1(\mathcal{E})$. For $i \in \{1,\dots,\ell\}$, denote by ψ_i the first Chern class of the line bundle on $X_{0,\ell,d}$ whose fiber at a point $(C, x_1, \dots, x_\ell,\ f:\ C \to X)$ is the cotangent space $\mathcal{T}^*C_{x_i}$.

Definition 2.3. — Let ℓ be in $\mathbb{N}$, $\gamma_1,\dots,\gamma_\ell$ be classes in $H^{2*}(X)$, d be in $H_2(X,\mathbb{Z})$ and $(m_1,\dots,m_\ell)$ be in $\mathbb{N}^\ell$. For j in $\{1,\dots,\ell\}$, the (j-th) *twisted Gromov-Witten invariant with descendants* is defined by

$$\left\langle \tau_{m_1}(\gamma_1),\dots,\widetilde{\tau_{m_j}(\gamma_j)},\dots,\tau_{m_\ell}(\gamma_\ell)\right\rangle_{0,\ell,d} := \int_{[X_{0,\ell,d}]^{\mathrm{vir}}} c_{\mathrm{top}}(\mathcal{E}_{0,\ell,d}(j)) \prod_{i=1}^{\ell} \psi_i^{m_i}\,\mathrm{e}_i^*\,\gamma_i$$

where $\mathrm{e}_i : X_{0,\ell,d} \to X$ $(1 \le i \le \ell)$ is the evaluation morphism to the i-th marked point and $[X_{0,\ell,d}]^{\mathrm{vir}}$ is the virtual class of $X_{0,\ell,d}$ (see [**BF97**]).

2.1.b. Twisted quantum product. —

Notation 2.4. — Denote by $\mathrm{NE}(X) \subset H_2(X,\mathbb{Z})$ the Mori cone of X, generated as a semi-group by numerical classes of irreducible curves in X

$$\mathrm{NE}(X) = \left\{ \sum_{\substack{C \text{ irreducible curve,}\\ \text{finite sum}}} n_C[C],\quad n_C \in \mathbb{N}, [C] \text{ numeric class of } C \right\}.$$

The semigroup algebras of $\mathrm{NE}(X)$ and $H_2(X,\mathbb{Z})$ will be respectively denoted by Λ and Π :

$$\Lambda = \mathbb{C}[\mathrm{NE}(X)] = \mathbb{C}[Q^d, d \in \mathrm{NE}(X)], \qquad \Pi = \mathbb{C}[H_2(X,\mathbb{Z})] = \mathbb{C}[Q^d, d \in H_2(X,\mathbb{Z})],$$

where Q^d are indeterminates satisfying relations : $Q^d.Q^{d'} = Q^{d+d'}$. Associated schemes to Λ and Π are :

$$\mathbf{S} := \mathrm{Spec}\,\Lambda, \qquad \mathbf{T} := \mathrm{Spec}\,\Pi.$$

The scheme $\mathbf{S}$ is an irreducible, possibly singular, affine variety of dimension r. Points of $\mathbf{S}$ are characters of $\mathrm{NE}(X)$. If q is such a character, denote by q^d its evaluation on d in $\mathrm{NE}(X)$. Since X is projective, the Mori cone is strictly convex and there exists a unique character sending any d in $\mathrm{NE}(X) \setminus \{0\}$ to 0 ; it corresponds to the maximal ideal $\langle Q^d, d \in \mathrm{NE}(X) \setminus \{0\}\rangle$. We will denote this point by $\mathbf{0}$.

The scheme $\mathbf{T} \simeq (\mathbb{C}^*)^r$ is an algebraic torus of rank r. In [**CK99**], the point $\mathbf{0} \in \mathbf{S} \setminus \mathbf{T}$ is called the *large radius limit* of $\mathbf{T}$.

The *small twisted quantum product* can now be defined. Let q be in $\mathbf{S}$ and γ_1, γ_2 be in $H^{2*}(X)$. The twisted small quantum product is defined by

$$\gamma_1 \bullet_q^{\mathrm{tw}} \gamma_2 := \sum_{a=0}^{s-1} \sum_{d \in H_2(X,\mathbb{Z})} q^d \left\langle \gamma_1, \gamma_2, \widetilde{T}_a \right\rangle_{0,3,d} T^a \tag{2.5}$$

whenever this sum is convergent. Notice that this twisted quantum product is the non-equivariant limit of $\bullet_\tau^{\mathbf{e}_\lambda}$ in [**Iri11**, p.5]. Remark 2.2 in [**Iri11**] implies that the twisted quantum product $\bullet_q^{\mathrm{tw}}$ is associative, commutative, with unity $T_0 := \mathbf{1}$.

Assumption 2.6. — We assume that $(\omega_X \otimes \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_k)^\vee$ is nef.

Iritani proves in [**Iri07**], that under this assumption, there exists an open subset $\overline{\mathbf{D}}$ of $\mathbf{S}$ containing $\mathbf{0}$ such that :

$$\forall q \in \overline{\mathbf{D}}, \forall \gamma_1, \gamma_2 \in H^{2*}(X), \gamma_1 \bullet_q^{\mathrm{tw}} \gamma_2 \text{ is convergent.}$$

Notation 2.7. — We denote by $\mathbf{D}$ the complex nonsingular variety $\mathbf{D} := \overline{\mathbf{D}} \cap \mathbf{T}$.

2.1.c. Twisted quantum D-module. — Let $(B_1, \ldots, B_r)$ be the basis of $H_2(X, \mathbb{Z})$ fixed in Notation 2.1. For $a \in \{1, \ldots, r\}$, put $q_a = Q^{B_a}$. We have:

$$\Pi = \mathbb{C}[H_2(X, \mathbb{Z})] \xrightarrow{\sim} \mathbb{C}[q_1^\pm, \ldots, q_r^\pm];$$

If $d = \sum_{a=1}^r d_a B_a$ we get $Q^d = \prod_{a=1}^r q_a^{d_a}$. Viewing the q_a's as coordinates of $\mathbf{T}$ we get, for any $q \in \mathbf{T}$, $q^d = \prod_{a=1}^r q_a^{d_a}$.

Let z be another variable ; we write $\mathbb{C}$ for $\operatorname{Spec} \mathbb{C}[z]$. We define $r+1$ differential operators on $\mathbf{T} \times \mathbb{C}$ by :

$$\delta_a := q_a \partial_{q_a}, a \in \{1, \ldots, r\}, \text{ and } \delta_z := z \partial_z.$$

We denote by F the trivial holomorphic vector bundle of fiber $H^{2*}(X)$ over $\mathbf{D} \times \mathbb{C}$ together with the following meromorphic connection:

$$\nabla_{\delta a} := \delta_a + \frac{1}{z} T_a \bullet_q^{\mathrm{tw}}, \quad \nabla_{\delta_z} := \delta_z - \frac{1}{z} \mathfrak{E} \bullet_q^{\mathrm{tw}} + \mu \tag{2.8}$$

where μ is the diagonal morphism defined by $\mu(T_a) := \frac{1}{2} (\deg(T_a) - (\dim_{\mathbb{C}} X - \operatorname{rk} \mathcal{E})) T_a$ and $\mathfrak{E} := c_1(\mathcal{T}_X) - c_1(\mathcal{E})$. The couple (F, ∇) is called the *twisted Quantum D-module* of $(X, \mathcal{E})$ and denoted by $\mathrm{QDM}(X, \mathcal{E})$.

We define a multi-valued meromorphic section L^{tw} of $\operatorname{Hom}(F, F)$ by :

$$L^{\mathrm{tw}}(q, z)\gamma = q^{-T/z}\gamma - \sum_{a=0}^{s-1} \sum_{\substack{H_2(X,\mathbb{Z}) \\ d \neq 0}} q^d \left\langle \frac{q^{-T/z}\gamma}{z + \psi}, \widetilde{T}_a \right\rangle_{0,2,d} T^a \tag{2.9}$$

where $\frac{1}{z+\psi} := \sum_{k \in \mathbb{N}} (-1)^k \psi^k z^{-k-1}$ and $q^{-T/z} := q^{-T_1/z} \ldots q^{-T_r/z} := e^{-z^{-1} \sum_{a=1}^r T_a \log(q_a)}$ and $\log(q_a)$ is the multi-valued function, or any determination of the logarithm on a simply connected open subset of $\mathbf{D}$.

Define a pairing by: $(\gamma_1, \gamma_2)^{\mathrm{tw}} := \int_X \gamma_1 \cup \gamma_2 \cup c_{\mathrm{top}}(\mathcal{E})$. This pairing is degenerated and its kernel is $\ker m_{c_{top}}$ where $m_{c_{top}} : H^{2*}(X) \to H^{2*}(X)$ sends α to $c_{\mathrm{top}}(\mathcal{E}) \cup \alpha$.

Proposition 2.10. — 1. *The connection ∇ is flat.*

2. *For a in $\{1, \ldots, r\}$ and $\gamma \in H^{2*}(X)$ we have*

$$\nabla_{\delta_a} L^{\mathrm{tw}}(q, z)\gamma = 0, \qquad \nabla_{\delta_z} L^{\mathrm{tw}}(q, z)\gamma = L^{\mathrm{tw}}(q, z) \left(\mu - \frac{c_1(\mathcal{T}_X) - c_1(\mathcal{E})}{z} \right) \gamma$$

3. *For any endomorphism u of $H^{2*}(X)$, we put $z^u := \exp(u \log z)$. The multi-valued cohomological function $L^{\mathrm{tw}}(q, z) z^{-\mu} z^{c_1(\mathcal{T}_X) - c_1(\mathcal{E})}$ is a fundamental solution of ∇.*

4. *For any* $\gamma_1, \gamma_2 \in H^{2*}(X)$, *we have*

$$(L^{\mathrm{tw}}(q,-z)\gamma_1, L^{\mathrm{tw}}(q,z)\gamma_2)^{\mathrm{tw}} = (\gamma_1,\gamma_2)^{\mathrm{tw}}.$$

Proof. — This proof is completely parallel to the one of Proposition 2.4 in [**Iri09**], using the twisted axioms (see Appendix A). □

2.2. Quantum $\mathcal{D}$-module for complete intersection subvarieties. —

Assumption 2.11. — In this section, we assume that $\dim_{\mathbb{C}} X \geq k+3$ and that the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are ample. This makes it possible to use Hyperplane and Hard Lefschetz Theorems.

Notation 2.12. — Fix a generic section of $\mathcal{E}$, and denote by Z the projective subvariety defined by this section. By Bertini's theorem, Z is a smooth complete intersection subvariety of X. Denote by $\iota : Z \hookrightarrow X$ the corresponding closed embedding.

By Lefschetz theorem we have

$$H^{2*}(Z) = \operatorname{Im}\iota^* \oplus \ker \iota_* \tag{2.13}$$

and $\ker \iota_* \subset H^{\dim_{\mathbb{C}} Z}(Z)$. We put $H^{2*}_{\mathrm{amb}}(Z) := \operatorname{Im}\iota^*$, this is the part of the cohomology of Z coming from the ambient space X. We have an isomorphism $H^2(X) \simeq H^2(Z)$.

Remark 2.14. — It should be possible to improve Assumption 2.11, at least for toric varieties. For example, if X is a toric projective variety of dimension at least 3, $k=1$ and $\mathcal{L}_1$ is a nef (not necessary ample) line bundle on X, then Theorem 5.1 of [**Mav03**] ensures that Z is a smooth connected hypersurface satisfying : $H^{2*}(Z) = \operatorname{Im}\iota^* \oplus \ker \iota_*$.

Proposition 2.15 **(See Corollary 2.3 in [Iri11])**. — *Using Notation 2.12, and under Assumption 2.11, for any* $\gamma_1, \gamma_2 \in H^{2*}(X)$

$$\iota^*(\gamma_1 \bullet_q^{\mathrm{tw}} \gamma_2) = \iota^*(\gamma_1) \bullet_q^Z \iota^*(\gamma_2),$$

where $\bullet^Z$ *is the quantum product on* Z. □

We define the trivial vector bundle, denoted by F^Z_{amb}, of fiber $H^{2*}_{\mathrm{amb}}(X)$ over $\mathbf{D}_Z \times \mathbb{C}$ where $\mathbf{D}_Z$ is the subset of $H^2(Z,\mathbb{C})/\operatorname{Pic}(Z)$ where the quantum product on Z is convergent[(1)]. The connection ∇^Z is defined via the same formula as ∇ with the quantum product of Z and $\mathfrak{E} := c_1(\mathcal{T}_Z)$ and

$$\mu^Z(\psi_a) = \frac{1}{2}\left(\deg(\psi_a) - \dim_{\mathbb{C}} Z\right)\psi_a.$$

where (ψ_a) is a basis of $H^{2*}(Z)$. Proposition 2.15 implies that this bundle is stable by ∇^Z. We denote by $\mathrm{QDM}_{\mathrm{amb}}(Z) = (F^Z_{\mathrm{amb}}, \nabla^Z)$.

Corollary 2.16. — *Using Notation 2.12, and under Assumption 2.11. The morphism* ι^* *induces a surjective morphism* $\iota^* : \mathrm{QDM}(X,\mathcal{E}) \twoheadrightarrow \mathrm{QDM}_{\mathrm{amb}}(Z)$.

Proof. — It is clearly a surjective morphism of vector bundles. Proposition 2.15 implies that $\iota^*(\nabla_{\delta_a}\gamma) = \nabla^Z_{\delta_a}\iota^*\gamma$. The adjunction formula gives : $c_1(\mathcal{T}_Z) = \iota^*(c_1(\mathcal{T}_X) - c_1(\mathcal{E}))$. Since the dimension of Z is the dimension of X minus the rank of $\mathcal{E}$, we deduce that $\mu^Z(\iota^*\gamma) = \iota^*(\mu(\gamma))$. This implies $\iota^*(\nabla_{\delta_z}\gamma) = \nabla^Z_{\delta_z}\iota^*\gamma$. □

(1) We use the same parameter q because of the isomorphism $\iota^* : H^2(X) \simeq H^2(Z)$

2.3. Reduced quantum D-module. — Consider the quotient $\overline{H^{2*}(X)} := H^{2*}(X)/\ker m_{c_{top}}$ and call it the *reduced cohomology ring of* $(X, \mathcal{E})$. In this section, we define a "reduced" quantum product on $\overline{H^{2*}(X)}$, which enables us to define a "reduced" quantum $\mathcal{D}$-module.

This reduced quantum D-module turns out to be isomorphic to the ambient part of the quantum D-module of the subvariety Z defined in Subsection 2.2.

Since $m_{c_{top}}$ is a graded morphism, the reduced cohomology ring $\overline{H^{2*}(X)} = H^{2*}(X)/\ker m_{c_{top}}$ is naturally graded. For $\gamma \in H^{2*}(X)$, we denote by $\overline{\gamma}$ its class in $\overline{H^{2*}(X)}$. Denote by $\overline{F}$ the trivial bundle with fiber $\overline{H^{2*}(X)}$ over $\mathbf{D}\times\mathbb{C}$. For any $\gamma_1, \gamma_2 \in H^{2*}(X)$, define the reduced pairing $(\cdot,\cdot)^{\mathrm{red}}$ which is a bilinear form on $\overline{H^{2*}(X)}$ by

$$(\overline{\gamma}_1, \overline{\gamma}_2)^{\mathrm{red}} := (\gamma_1, \gamma_2)^{\mathrm{tw}}. \tag{2.17}$$

The reduced pairing is a well defined and a non degenerate bilinear form. Put $s' = \dim_{\mathbb{C}} \overline{H^{2*}(X)}$. Let $(\phi_0, \dots, \phi_{s'-1})$ be a homogeneous basis of $\overline{H^{2*}(X)}$ and denote $(\phi^0, \dots, \phi^{s'-1})$ its dual basis with respect to $(\cdot,\cdot)^{\mathrm{red}}$. Let $\gamma_1, \dots, \gamma_\ell$ be classes in $H^{2*}(X)$. Let d be in $H_2(X,\mathbb{Z})$. Using Definition 2.3, we define the *reduced Gromov-Witten invariant* by

$$\langle \overline{\gamma}_1, \dots, \overline{\gamma}_\ell \rangle^{\mathrm{red}}_{0,\ell,d} := \langle \gamma_1, \dots, \widetilde{c_{\mathrm{top}}(\mathcal{E})\gamma_\ell} \rangle_{0,\ell,d} = \int_{[X_{0,\ell,d}]^{\mathrm{vir}}} c_{\mathrm{top}}(\mathcal{E}_{0,n,d}) \prod_{i=1}^{\ell} \mathrm{e}_i^* \, \gamma_i$$

By the twisted S_ℓ-symmetric axiom (*cf.* Axiom A.1), the reduced Gromov-Witten invariants are well defined on the class in $\overline{H^{2*}(X)}$. Notice that the reduced Gromov-Witten invariants are symmetric with respect to the ℓ entries.

The *reduced quantum product* is

$$\overline{\gamma}_1 \bullet_q^{\mathrm{red}} \overline{\gamma}_2 := \sum_{a=0}^{s'-1} \sum_{d\in H_2(X,\mathbb{Z})} q^d \, \langle \overline{\gamma}_1, \overline{\gamma}_2, \phi_a \rangle^{\mathrm{red}}_{0,3,d} \, \phi^a.$$

The convergence domain of $\bullet_q^{\mathrm{red}}$ contains $\mathbf{D}$. We will restrict ourselves to $\mathbf{D}$.

Define the following connection on the trivial bundle $\overline{F}$:

$$\begin{aligned} \forall a \in \{1,\dots,r\}, \quad \overline{\nabla}_{\delta a} &:= \delta_a + \frac{1}{z} \overline{T}_a \bullet_q^{\mathrm{red}} \\ \overline{\nabla}_{\delta_z} &:= \delta_z - \frac{1}{z} \overline{\mathfrak{E}} \bullet_q^{\mathrm{red}} + \overline{\mu} \end{aligned}$$

where $\overline{\mu}$ is the diagonal morphism defined by $\overline{\mu}(\phi_a) := \frac{1}{2}\left(\deg(\phi_a) - (\dim_{\mathbb{C}} X - \mathrm{rk}\,\mathcal{E})\right)\phi_a$ and $\overline{\mathfrak{E}} := \overline{c_1(\mathcal{T}_X)} - \overline{c_1(\mathcal{E})}$.

Definition 2.18. — The couple $(\overline{F}, \overline{\nabla})$ is called the *reduced quantum D-module of* $(X, \mathcal{E})$ and denoted by $\overline{\mathrm{QDM}}(X, \mathcal{E})$.

Proposition 2.19. — 1. *The connection $\overline{\nabla}$ is flat.*

2. *Under assumption 2.11, let Z be the subvariety defined by a generic section of $\mathcal{E}$. There exists an isomorphism of $\mathcal{D}$-modules $f : \overline{\mathrm{QDM}}(X, \mathcal{E}) \xrightarrow{\sim} \mathrm{QDM}_{\mathrm{amb}}(Z)$ making the following diagram commutative:*

$$\begin{array}{ccc} & \mathrm{QDM}(X,\mathcal{E}) & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \iota^*} \\ \overline{\mathrm{QDM}}(X,\mathcal{E}) & \xrightarrow[f]{\sim} & \mathrm{QDM}_{\mathrm{amb}}(Z) \end{array}$$

where p is the natural projection on the quotient.

Proof. — For any $\gamma_1, \gamma_2 \in H^{2*}(X)$ and any $a \in \{0, \dots, s-1\}$ we have :

$$\overline{\gamma_1 \bullet_q^{\mathrm{tw}} \gamma_2} = \overline{\gamma}_1 \bullet_q^{\mathrm{red}} \overline{\gamma}_2 \quad \text{and} \quad \overline{\mu(T_a)} = \overline{\mu}(\overline{T_a}).$$

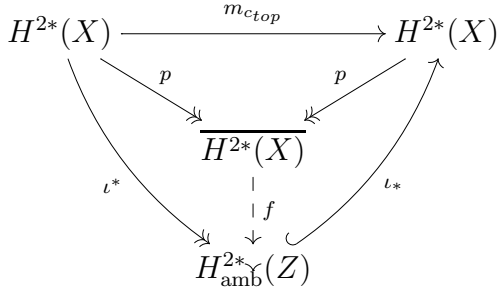
It follows that, for any $\gamma \in H^{2*}(X)$,

$$\overline{\nabla\gamma} = \overline{\nabla}\overline{\gamma}. \tag{2.20}$$

and $\overline{\nabla}$ is flat since ∇ is.

As for the second point, consider the following diagram, where we make use of notations of §. 2.2.

(2.21)



The morphism f is well defined by $f : \overline{\gamma} \mapsto \iota^*\gamma$. By the decomposition (2.13), f is an isomorphism.

This diagram and Corollary 2.16 gives the required isomorphism between vector bundles ; Formula (2.20) ensures that the connections are compatible. □

Remark 2.22. — The reduced quantum D-module does exist even if the assumption 2.11 is not satisfied ; that is if the subvariety Z is not well defined. It is used in **[RS15]**.

We now come to the reduced fundamental solutions.

Lemma 2.23. — *For any (q, z) in $\mathbf{D} \times \mathbb{C}$, we have :* $L^{\mathrm{tw}}(q, z)(\ker m_{c_{top}}) = \ker m_{c_{top}}$.

Proof. — Let γ be in $\ker m_{c_{top}}$ and $\alpha \in H^{2*}(X)$. Since $L^{\mathrm{tw}}(q, z)$ is an automorphism of $H^{2*}(X)$ and $\ker m_{c_{top}}$ is the kernel of the twisted pairing $(\cdot, \cdot)^{\mathrm{tw}}$ we find, using Proposition 2.10:

$$\big(\alpha, L^{\mathrm{tw}}(q, z)\gamma\big)^{\mathrm{tw}} = \big(L^{\mathrm{tw}}(q, -z)^{-1}\alpha, \gamma\big)^{\mathrm{tw}} = 0.$$

Then $L^{\mathrm{tw}}(q, z)\gamma$ belongs to $\ker m_{c_{top}}$. □

This lemma implies that we can define a reduced L function : for any $(q, z) \in \mathbf{D} \times \mathbb{C}$ put

$$\overline{L}(q, z)\overline{\gamma} = \overline{L^{\mathrm{tw}}(q, z)\gamma} \tag{2.24}$$

The following corollary follows from Proposition 2.10.

Corollary 2.25. — *We have the following properties.*

1. *A fundamental solution of $\overline{\nabla}$ is given by $\overline{L}(q, z)z^{-\overline{\mu}}z^{\overline{c_1(\mathcal{T}_X)}-\overline{c_1(\mathcal{E})}}$.*
2. *For any $\gamma_1, \gamma_2 \in H^{2*}(X)$, we have*

$$(\overline{L}(q, -z)\overline{s}_1, \overline{L}(q, z)\overline{s}_2)^{\mathrm{red}} = (\overline{s}_1, \overline{s}_2)^{\mathrm{red}}$$

3. Batyrev algebras for toric varieties with a split vector bundle

From now on, the smooth projective variety X is a toric variety. In **[Bat93]**, Batyrev constructs an algebra from the fan of a smooth toric projective variety. If the variety is Fano, this algebra is its quantum cohomology ring. In this section, we define and study similar objects for toric varieties endowed with a split vector bundle.

3.1. Fan for the total space of a split vector bundle. — Denote by N a n-dimensional lattice and by M its dual lattice. Consider a fan Σ of $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and denote by $\Sigma(l)$ the set of l-dimensional cones of Σ. The set of rays is $\Sigma(1) = \{\theta_1, \ldots, \theta_m\}$, and for any $\theta \in \Sigma(1)$ we denote by w_θ the generator of $\theta \cap N$.

Let X be the variety defined by Σ. We assume that X is smooth and projective.

Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be k globally generated line bundles on X. Put $\mathcal{E} = \oplus_{i=1}^k \mathcal{L}_i$. Let $L_1, \ldots, L_k$ be k toric divisors of X, such that $\mathcal{L}_i \simeq \mathcal{O}(L_i)$. To any ray $\theta \in \Sigma(1)$, there is an associated toric Weil divisor denoted by D_θ ; we write, in a unique way:

$$L_i = \sum_{\theta \in \Sigma(1)} \ell_\theta^i D_\theta, \quad \ell_\theta^i \in \mathbb{Z},\ i = 1, \ldots, k$$

Consider the $n+k$ dimensional lattice $N' := N \oplus \mathbb{Z}^k$. Let $(\epsilon_1, \ldots, \epsilon_k)$ be the canonical basis of $\mathbb{Z}^k$. Denote by :

$$\phi\ : N' = N \times \mathbb{Z}^k \longrightarrow N$$

the natural projection. Define a fan Δ in $N'_{\mathbb{R}} := N' \otimes \mathbb{R}$ in the following way :

– The rays of Δ are indexed by $\Sigma(1) \cup \{1, \ldots, k\}$:

$$\begin{cases} \text{For } \theta \in \Sigma(1), & \text{put } v_\theta := (w_\theta, 0) + \sum_{i=1}^k \ell_\theta^i (0, \epsilon_i), \\ \text{For } i \in \{1, \ldots, k\}, & \text{put } v_i := (0, \epsilon_i). \end{cases}$$

Then,

$$\Delta(1) := \{\rho_\theta := \mathbb{R}^+ v_\theta, \theta \in \Sigma(1)\} \cup \{\rho_i := \mathbb{R}^+ v_i,\ i \in \{1, \ldots, k\}\}.$$

– a strongly convex polyhedral cone σ is in Δ if and only if $\phi(\sigma) \in \Sigma$.

Notation 3.1. — In the following, for any $\rho \in \Delta(1)$, we denote by $v_\rho \in N$ the generator of ρ. It will be convenient to make the distinction between rays ρ_θ coming from the base variety X, and rays ρ_i coming from the split vector bundle $\mathcal{E}$. We put :

$$\Delta(1)^{base} = \{\rho_\theta, \theta \in \Sigma(1)\},\ \Delta(1)^{bund} = \{\rho_1, \ldots, \rho_k\} \quad \text{so that } \Delta(1) = \Delta(1)^{base} \sqcup \Delta(1)^{bund}.$$

Let Y be the toric variety associated to the fan Δ. As X is smooth, Y is also smooth. The scheme morphism induced by the projection $\phi\ : N' \longrightarrow N$ is denoted by the same letter $\phi : Y \longrightarrow X$.

Proposition 3.2 **([CLS11], Proposition 7.3.1 and Exercise 7.3.3)**
The toric variety Y is the total space of the vector bundle $\mathcal{E}^\vee$, dual of $\mathcal{E} = \oplus_{i=1}^k \mathcal{L}_i$. The natural projection is the toric morphism $\phi\ : Y \to X$. □

We will make use of the following easy result about cohomology classes:

Proposition 3.3. — *The projection $\phi\ : Y \longrightarrow X$ induces an isomorphism:*

$$\phi^*\ : H^*(X) \xrightarrow{\sim} H^*(Y).$$

For $i \in \{1, \ldots, k\}$, let D_{ρ_i} be the toric divisor of Y corresponding to the ray ρ_i, then

$$\phi^*[L_i] = [-D_{\rho_i}] \text{ in } H^2(Y).$$

To any toric Weil divisor $D = \sum a_\theta D_\theta$ of X, there is an associated piecewise linear function ψ_D, defined on the support $|\Sigma| = N_{\mathbb{R}}$ of Σ and linear on each cone, such that $\psi_D(w_\theta) = -a_\theta$. Since the line bundles $\mathcal{L}_i$ are globally generated, the functions ψ_{L_i} are concave. This gives :

Lemma 3.4. — *The support $|\Delta| = \cup_{\sigma \in \Delta} \sigma$ of the fan Δ in $N'_{\mathbb{R}}$ is convex.*

Proof. — First assume for simplicity that $k = 1$ *i.e.*, $N' = N \times \mathbb{Z}$ and $L = \sum_{\theta\in\Sigma(1)} \ell_\theta D_\theta$. Let ψ_L the concave piecewise linear function such that $\psi_L(w_\theta) = -\ell_\theta$. Notice that $v_\theta = (w_\theta, \ell_\theta) = (w_\theta, -\psi_L(w_\theta))$.

Let σ be a cone of Δ of the form $\sigma = \sum_{\theta\in\tau(1)} \rho_\theta + \rho_1$, where τ is the cone of Σ obtained by projection of $\sigma : \tau = \phi(\sigma)$. A points p of σ can be written in $N_\mathbb{R} \times \mathbb{R}$ as :

$$\begin{aligned} p &= \sum_{\theta\in\tau(1)} t_\theta \left(w_\theta, -\psi_L(w_\theta)\right) + t_1(0_{N_\mathbb{R}}, 1), \qquad t_\theta, t_1 \in \mathbb{R}^+ \\ &= \left(\sum_{\theta\in\tau(1)} t_\theta w_\theta, -\psi_L\left(\sum_{\theta\in\tau(1)} w_\theta\right) + t_1\right) \quad \text{(by linearity of } \psi_L \text{ on } \tau) \end{aligned}$$

so that

$$\sigma = \{(p_N, p_1) \in N_\mathbb{R} \times \mathbb{R} \mid p_N \in \phi(\sigma), \quad p_1 \geq -\psi_L(p_N)\}.$$

By definition, the support of Δ is the union of such cones σ. Since $|\Sigma| = N_\mathbb{R}$, one get :

$$|\Delta| = \{(p_N, p_1) \in \mathbb{N}_\mathbb{R} \times \mathbb{R} \mid p_1 \geq -\psi_L(p_N)\}.$$

Now, consider $p = (p_N, p_1) \in N, q = (q_N, q_1)$ two points in $|\Delta|$ and $t \in [0, 1]$. Since ψ_L is concave, we have : $tp_1 + (1-t)q_1 \geq -\psi_L(tp_N + (1-t)q_N)$, and $(tp + (1-t)q) \in |\Delta|$ as required.

In case $k \geq 2$, we get

$$|\Delta| = \{(p_N, p_1, \ldots, p_k) \in \mathbb{N}_\mathbb{R} \times \mathbb{R}^k \mid p_1 \geq -\psi_{L_1}(p_N), \ldots, p_k \geq -\psi_{L_k}(p_N)\}.$$

and $|\Delta|$ is also convex. □

Example 3.5. — Consider the fan of $\mathbb{P}^1$ given by $(N = \mathbb{Z}, w_1 = 1, w_2 = -1)$, $\mathcal{L} = \mathcal{O}(2)$ and $L = 2D_{\theta_1}$. The fan Δ is given by the rays $v_{\theta_1} = (1, 2), v_{\theta_2} = (-1, 0)$ an $v_L = (0, 1)$ (*cf.* Figure 1).

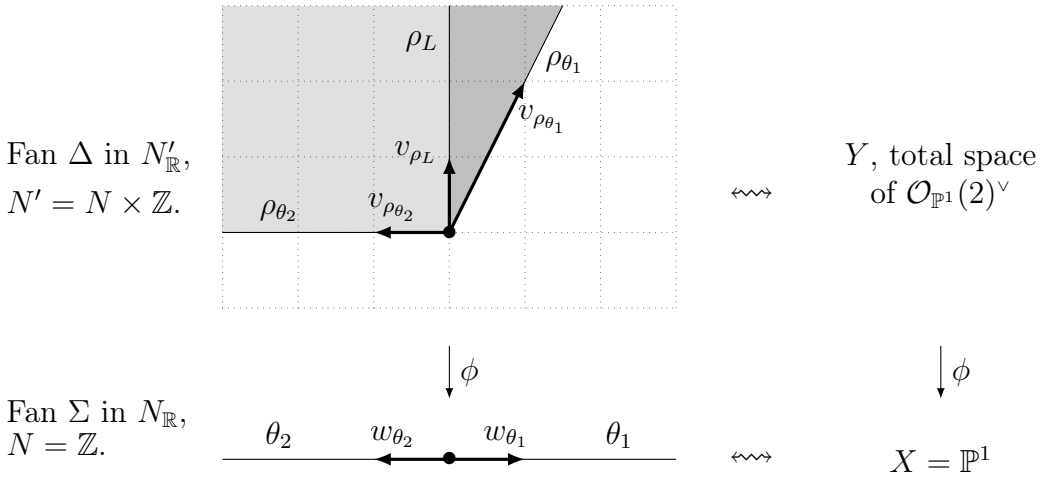


FIGURE 1. Fans Σ and Δ associated to $X = \mathbb{P}^1$, $L = 2D_{\theta_1}$

3.2. Definition and properties of Batyrev algebras associated to $(X, \mathcal{E})$.—

3.2.a. Mori cone. — Let X, Y and $\mathcal{E}$ be as in section 3.1. Using Proposition 3.3, we will identify $H^2(X)$ and $H^2(Y)$, as well as the Mori cones of X and Y.

Notation 3.6. — For any class d of $H_2(Y,\mathbb{Z})$ and ray ρ of $\Delta(1)$ corresponding to the weil divisor D_ρ, we put

$$d_\rho := D_\rho.d = \int_d D_\rho \in \mathbb{Z}.$$

There is an exact sequence :

$$0 \longrightarrow H_2(Y,\mathbb{Z}) \longrightarrow \mathbb{Z}^{\Delta(1)} \longrightarrow N' \longrightarrow 0, \tag{3.7}$$

Where $N' = N \oplus \mathbb{Z}^k$ is the lattice defined in section 3.1 and where the image of $d \in H_2(Y,\mathbb{Z})$ is $(d_\rho)_{\rho\in\Delta(1)} \in \mathbb{Z}^{\Delta(1)}$. We identify $H_2(Y,\mathbb{Z})$ and its image in $\mathbb{Z}^{\Delta(1)}$.

For any real number a, we put $a^+ = \max(a,0), a^- = \max(-a,0)$. Also put, for any $d \in H_2(Y,\mathbb{Z})$, $d^+ = (d_\rho^+)_{\rho\in\Delta(1)}$ and $d^- = (d_\rho^-)_{\rho\in\Delta(1)}$. With the identification above, we have :

$$d = d^+ - d^-.$$

If a is an element of $H_2(Y,\mathbb{Z}) \subset \mathbb{Z}^{\Delta(1)}$, we say that a is *supported by a cone* if the set $\{\rho \in \Delta(1) \mid a_\rho \neq 0\}$ is contained in a cone of Δ.

We will use the following facts :

Lemma 3.8. — *Let d be in $H_2(Y,\mathbb{Z})$.*

1. *If d^+ is supported by a cone, then $-d \in NE(Y)$.*
2. *If $d \in NE(Y) \setminus \{0\}$, then d^+ is not supported by a cone.*

Proof. — 1. We have to show that, for any nef toric divisor T, $T.(-d) \geq 0$. Let T be such a divisor and let ψ be the piecewise linear concave function associated to T :

$$\begin{aligned} T.d &= \sum_\rho -\psi(v_\rho)d_\rho^+ - \sum_\rho -\psi(v_\rho)d_\rho^- \\ &= -\psi(\sum_\rho v_\rho d_\rho^+) + \sum_\rho d_\rho^-\psi(v_\rho) && (d^+ \text{ supported by } \sigma) \\ &\leq -\psi(\sum_\rho d_\rho^+ v_\rho) + \psi(\sum_\rho d_\rho^+ v_\rho) = 0 && (\psi \text{ concave and } \sum_\rho d_\rho^+ v_\rho = \sum_\rho d_\rho^- v_\rho). \end{aligned}$$

2. If d^+ is supported by a cone, then $-d \in \mathrm{NE}(Y)$ and $d \in -\mathrm{NE}(Y) \cap \mathrm{NE}(Y) = 0$.

□

3.2.b. Twisted Batyrev algebra of $(X,\mathcal{E})$. — Let Λ be the semi-group algebra of $\mathrm{NE}(X)$, as defined in Notation 2.4. Since the Mori cones of X and Y are identified, we have :

$$\Lambda = \mathbb{C}[\mathrm{NE}(Y)] = \mathbb{C}[Q^d, d \in \mathrm{NE}(Y)]. \tag{3.9}$$

Fix a set of indeterminates $(x_\rho)_{\rho\in\Delta(1)}$. We put :

$$\Lambda[x_\rho] := \Lambda[x_\rho, \rho \in \Delta(1)].$$

For any $d \in H_2(Y,\mathbb{Z})$ denote by R_d the polynomial :

$$R_d := x^{d^+} - Q^d x^{d^-} = \prod_{\rho\in\Delta(1)} x_\rho^{d_\rho^+} - Q^d \prod_{\rho\in\Delta(1)} x_\rho^{d_\rho^-}.$$

Let M' be the dual lattice of $N' = N \oplus \mathbb{Z}^k$. For any $u \in M'$ denote by Z_u the linear polynomial :

$$Z_u := \sum_{\rho\in\Delta(1)} \langle u, v_\rho\rangle x_\rho.$$

Definition 3.10. — Consider the ring $\Lambda[x_\rho]$ defined above. The *quantum Stanley-Reisner* ideal of $\Lambda[x_\rho]$ is the ideal QSR generated by the polynomials R_d :

$$\text{QSR} := \left\langle R_d = x^{d^+} - Q^d x^{d^-},\ d \in \text{NE}(Y) \right\rangle \tag{3.11}$$

The *linear ideal* of $\Lambda[x_\rho]$, is the ideal Lin generated by the polynomials Z_u :

$$\text{Lin} := \left\langle Z_u = \sum_{\rho \in \Delta(1)} \langle u, v_\rho \rangle x_\rho,\ u \in M' \right\rangle \tag{3.12}$$

The *twisted Batytrev algebra of* $(X, \mathcal{E})$ is the Λ-algebra :

$$B := \Lambda[x_\rho]/(\text{QSR} + \text{Lin}).$$

Remark 3.13. — 1. Up to isomorphism, B is well defined since it does not depend on the specific choice of the fan Δ (*i.e.*, choices of the fan Σ and toric divisors L_i).
2. For any fan defining a smooth quasi-projective variety Y, we can define, as in Definition 3.10, the (untwisted) *Batyrev algebra of* Y. However, for Proposition 3.20 and first point of Theorem 3.18 to be true, the support of the fan must be convex (in our case, this is equivalent to each L_i being nef) of maximal dimension, and the anticanonical divisor $-K_Y$ must be nef.

The quantum Stanley-Reisner ideal QSR defined above is a deformation, parametrized by $\text{Spec}(\Lambda)$, of the following ideal :

$$\text{SR} = \left\langle x^{d^+}, d \in \text{NE}(Y) \right\rangle\ ; \tag{3.14}$$

SR is the Stanley-Reisner ideal associated to the simplicial complex defined by Δ (see **[BH93]**). We have :

Proposition 3.15. — *There is a natural isomorphism*

$$\begin{aligned} \mathbb{C}[x_\rho]/(\text{SR} + \text{Lin}) &\xrightarrow{\sim} H^{2*}(Y, \mathbb{C}) = H^{2*}(X, \mathbb{C}) \\ x_\rho &\longmapsto [D_\rho] \end{aligned}$$

where $[D_\rho] \in H^2(Y)$ *is the class of the toric divisor* D_ρ.

Proof. — Since Δ is convex (Lemma 3.4) and Y is quasi-projective, the proof of **[Ful93]** in the complete case can be adapted to our case, which shows that there is a well defined isomorphism $\mathbb{Z}[x_\rho]/(\text{SR} + \text{Lin}) \xrightarrow{\sim} H^{2*}(Y, \mathbb{Z})$ sending x_ρ to $[D_\rho]$. □

3.2.c. Residual Batyrev algebra of $(X, \mathcal{E})$. — From Proposition 3.3 there exists an isomorphism $H^2(X) \simeq H^2(Y)$; via this isomorphism, we have, for any toric divisor L_i and its corresponding ray ρ_{L_i}, $[L_i] = [-D_{\rho_{L_i}}]$.

Notation 3.16. — Put :

$$\begin{aligned} c_{\text{top}} &:= \prod_{i=1}^{k} [L_i] = \prod_{\rho \in \Delta(1)^{bund}} [-D_\rho] \in H^{2k}(X) \\ x_{\text{top}} &:= \prod_{\rho \in \Delta(1)^{bund}} (-x_\rho) \in \Lambda[x_\rho]. \end{aligned}$$

Then c_{top} is the top Chern class of the fiber bundle $\mathcal{E} = \oplus_{i=1}^{k} \mathcal{L}_i$, and x_{top} is sent to c_{top} via the morphism defined in Proposition 3.15.

Definition 3.17. — Consider the algebra $\Lambda[x_\rho]$ and the ideals QSR and Lin defined in 3.10. Put $G = (\mathrm{QSR} + \mathrm{Lin})$.

The *quotient ideal* of G by x_{top} is :

$$(G : x_{\mathrm{top}}) := \{P \in \Lambda[x_\rho], \quad x_{\mathrm{top}} P \in G\}.$$

The *residual Batyrev algebra* of $(X, \mathcal{E})$ is the Λ-algebra :

$$B^{\mathrm{res}} := \Lambda[x_\rho]/(G : x_{\mathrm{top}}),$$

3.2.d. Main Theorem for Batyrev algebras. — The main properties of twisted and residual Batyrev algebras of $(X, \mathcal{E})$ are summed up in the following result :

Theorem 3.18. — *Let X be a toric smooth projective variety endowed with a split vector bundle $\mathcal{E} = \oplus_{i=1}^k \mathcal{L}_i$. Assume that each line bundle $\mathcal{L}_i$ is nef, as well as $\omega_X \otimes {\mathcal{L}_1}^\vee \otimes \cdots \otimes {\mathcal{L}_k}^\vee$.*

Put $\Lambda = \mathbb{C}[Q^d, d \in NE(X)]$, $\mathbf{S} := \operatorname{Spec} \Lambda$ and denote by $\mathbf{0}$ the maximal ideal $\langle Q^d, d \neq 0 \rangle$. Let c_{top} be the top chern class of $\mathcal{E}$ and let $m_{c_{top}}$ be the morphism of multiplication by c_{top} in $H^{2*}(X)$.

There exists a Zariski neighbourhood $\mathbf{V}$ of $\mathbf{0} \in \mathbf{S}$ such that :

1. *Over $\mathbf{V}$, the twisted Batyrev algebra B of $(X, \mathcal{E})$ is a locally free Λ-module of rank $\dim H^{2*}(X)$.*
2. *Over $\mathbf{V}$, if the line bundles $\mathcal{L}_i$ are ample, then the residual Batyrev algebra B^{res} of $(X, \mathcal{E})$ is a locally free Λ-module of rank $(\dim H^{2*}(X) - \dim \ker m_{c_{top}})$.*

Remark 3.19. — A convenient neighbourhood $\mathbf{V}$ will be defined in Lemma 3.35 and could be explicitly computed by elimination algorithm (see 6.2). If Y is Fano, $\mathbf{V}$ is the whole scheme $\mathbf{S}$.

The proof of Theorem 3.18 will be given in section 3.4. We will actually rephrase its first part and show that the scheme morphism $\operatorname{Spec} B \to \mathbf{S}$ is finite, flat, of degree $\dim H^{2*}(X)$ over $\mathbf{V}$.

Let us first study the quotient by the ideal QSR defined in 3.11.

3.3. Quotient by the Quantum Stanley Reisner ideal. — In this section, we show :

Proposition 3.20. — *Put $\mathbf{Q} := \operatorname{Spec}(\Lambda[x_\rho]/\,\mathrm{QSR})$. Under assumptions of Theorem 3.18, the morphism $\mathbf{Q} \to \mathbf{S}$ is flat of relative dimension $\dim X + k = \dim Y$. The schemes $\mathbf{Q}$ and $\mathbf{S}$ are Cohen-Macaulay.*

We will prove this proposition by performing a Gröbner degeneration of the Quantum Stanley-Reisner ideal. For that, we first need to consider a graded version of this ideal and define a weight function on the monomials of the graded algebra. We then compute the initial ideal corresponding to this weight function in term of primitive classes introduced by Batyrev in **[Bat93]**.

3.3.a. Graded QSR *ideal.* — Consider a new variable h and define the graded Λ-algebra $\Lambda[x_\rho, h]$ with the grading given by $\deg(h) = 1$ and $\deg(x_\rho) = 1$.

Let P be a polynomial in $\Lambda[x_\rho]$. The homogenisation of P in $\Lambda[x_\rho, h]$ is :

$$P^h := h^{\deg P} P\left(\frac{x_\rho}{h}\right) \in \Lambda[x_\rho, h].$$

Recall that the toric divisor $K_Y = -\sum_{\rho \in \Delta(1)} D_\rho$ is a canonical divisor of Y. For any $d \in H_2(Y, \mathbb{Z})$, we have $\deg(x^{d^+}) - \deg(x^{d^-}) = \sum_\rho D_\rho.d = -K_Y.d$. It follows that, for any d in $H_2(Y, \mathbb{Z})$,

$$R_d^h = x^{d^+} h^{k^+} - Q^d h^{k^-} x^{d^-}, \quad \text{where } k = K_Y.d.$$

Definition 3.21. — The *graded quantum Stanley-Reisner* ideal of $\Lambda[x_\rho, h]$ is the homogeneous ideal QSR^h generated by the polynomials R_d^h.

Remark 3.22. — 1. If $-K_Y$ is nef, we get :

$$\mathrm{QSR}^h := \left\langle R_d^h = x^{d^+} - Q^d h^{-K_Y.d} x^{d^-},\ d \in \mathrm{NE}(Y) \right\rangle$$

2. The graded ideal QSR^h could be different from the ideal generated by the whole set of homogeneous polynomials $\{P^h, P \in \mathrm{QSR}\}$. Under our assumptions, we conjecture that they are actually equal.

3.3.b. Weight function and monomial order. — Fix, once and for all, a strictly concave piecewise-linear function φ of $|\Delta|$, rational on N'. Since Δ is quasi-projective, such a function exists, corresponding to an ample $\mathbb{Q}$-divisor $A_\varphi = \sum_{\rho\in\Delta(1)} -\varphi(v_\rho) D_\rho$.

Define a weight function ω on the monomials of $\Lambda[x_\rho, h]$ by setting, for any monomial $x^a h^k := \prod_{\rho\in\Delta(1)} x_\rho^{a_\rho} h^k$:

$$\omega(x^a h^k) = \sum_{\rho\in\Delta(1)} -a_\rho \varphi(v_\rho).$$

In particular, $\omega(h^k) = 0$ for any integer k. For convenience, we extend this function to any polynomial P by setting :

$$\omega(P) = \max\{\omega(x^a h^k), x^a h^k \text{ monomial of } P\}.$$

The *initial form* of a polynomial $P = \sum_i \alpha_i x^{a_i} h^{k_i}$ is

$$\mathrm{in}_\omega(P) = \sum_{i:\omega(x^{a_i}h^{k_i})=\omega(P)} \alpha_i x^{a_i} h^{k_i}.$$

This is not a term in general. The initial ideal $\mathrm{in}_\omega(I)$ of an ideal I is the ideal generated by initial forms of elements of I.

Also define a new monomial order $\preceq$ on the variable x_ρ, h by setting :

$$x^a h^k \preceq x^{a'} h^{k'} \iff \begin{cases} \omega(x^a h^k) < \omega(x^{a'} h^{k'}) \\ \text{or} \\ \omega(x^a h^k) = \omega(x^{a'} h^{k'}) \text{ and } x^a h^k \trianglelefteq x^{a'} h^{k'} \end{cases}$$

Where $\trianglelefteq$ is any fixed monomial order on the variables $\{x_\rho, h\}$. The leading monomial of a polynomial P for the order $\preceq$ will be denoted by $\mathrm{Lm}(P)$.

Lemma 3.23. — *For any d in the Mori cone of Y,* $\mathrm{Lm}(R_d^h) = \mathrm{in}_\omega(R_d^h) = x^{d^+}$.

Proof. — We compute :

$$\omega(x^{d^+}) - \omega(h^{-K_Y.d} x^{d^-}) = \sum_{\rho\in\Delta(1)} -d_\rho^+ \varphi(v_\rho) - \sum_{\rho\in\Delta(1)} -d_\rho^- \varphi(v_\rho) = \sum_{\rho\in\Delta(1)} -d_\rho \varphi(v_\rho) = A_\varphi.d > 0.$$

□

3.3.c. Primitive collections and classes. — Primitive classes are specific elements in $H_2(Y, \mathbb{Z})$ that generate the Mori cone of Y. They were introduced in ([**Bat93**] and [**CvR09**]).

Definition 3.24. — A subset $\{\rho_1, \ldots, \rho_l\}$ of $\Delta(1)$ is called a primitive collection for Δ if $\{\rho_1, \ldots, \rho_l\}$ is not contained in a single cone of Δ but every proper subset is.

Let $C = \{\rho_1, \ldots, \rho_l\}$ be a primitive collection, and $v_1, \ldots, v_l$ be the generating vectors of $\rho_1 \cap N', \ldots, \rho_l \cap N'$. Let σ be the minimal cone of Δ containing $v = \sum_{i=1}^l v_i$. Denote by $\rho_1', \ldots, \rho_s'$ the rays of σ and $v_1', \ldots, v_s'$ the primitive vectors of the ρ_i'. Since σ is the minimal cone of Δ containing v, the vector v is in the relative interior of σ and there exists $a_1, \ldots, a_s$, real positive numbers, such that : $v = a_1 v_1' + \cdots + a_s v_s'$. Moreover, since v is in N' and the v_j' are part of a basis of N' (Y is non singular), the a_j's are uniquely defined in $\mathbb{N}_{>0}$.

Remark 3.25. — With the above notations : $\{v_1, \dots, v_l\} \cap \{v'_1, \dots, v'_s\} = \varnothing$. ([**CvR09**], proposition 1.9).

Let $C = \{\rho_1, \dots, \rho_l\}$ be a primitive collection and $v = \sum_{i=1}^{l} v_i = a_1 v'_1 + \cdots + a_s v'_s$ be as above. Since $\sum_{i=1}^{l} v_i - \sum_{j=1}^{s} a_j v'_j = 0$, the exact sequence (3.7) shows that there exists a well defined element $d^C \in H_2(Y, \mathbb{Z})$ such that :

$$d^C_\rho = \begin{cases} 1 & \text{if } \rho \in C, \\ -a_j & \text{if } \rho = \mathbb{R}^+ v'_j,\ j \in \{1, \dots, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Notation 3.26. — A *primitive class* is a class $d^C \in H_2(Y, \mathbb{Z})$ corresponding to a primitive collection as above. We denote by :

$$\mathcal{P} := \{d^C \in H_2(Y, \mathbb{Z}), C \text{ primitive collection}\}$$

the set of primitive classes.

***Proposition 3.27* ([CvR09], Propositions 1.9. and 1.10)**. — *Each primitive class is contained in the Mori cone NE(Y). The Mori cone is generated by primitive classes.*

3.3.d. Initial ideal of the graded QSR *ideal.*—

Lemma 3.28. — *Assume that the anticanonical divisor of Y is nef. Let F be the fraction field of* Λ. *The set* $\{R^h_c, c \in \mathcal{P}\}$ *is a Gröbner basis, for the order* $\preceq$, *of the ideal generated by* $\{R^h_d, d \in NE(Y)\}$ *in* $F[x_\rho, h]$.

Proof. — First prove that, for any $d \in H_2(Y, \mathbb{Z})$, there exists a set of homogeneous polynomials $\{B_c \in F[x_\rho, h], c \in \mathcal{P}\}$ such that :

$$R^h_d = \sum_{c \in \mathcal{P}} B_c R^h_c, \quad \text{and} \quad \forall c \in \mathcal{P},\ \mathrm{Lm}(B_c R^h_c) \preceq \mathrm{Lm}(R^h_d). \tag{3.29}$$

Let E be the set of polynomials R^h_d which can not be expressed as in (3.29). Assume that E is not empty and consider $R^h_d \in E$, whose leading monomial is minimal. Write $R^h_d = x^{d^+} h^{k^+} - Q^d x^{d^-} h^{k^-}$ where $k = K_Y.d$. Two cases may occur :

a) $\mathrm{Lm}(R^h_d) = x^{d^+} h^{k^+}$. If d^+ is supported by a cone, then $-d \in \mathrm{NE}(Y)$ by Lemma 3.8. By Lemma (3.23), $\mathrm{Lm}(R_{-d}) = x^{(-d)^+} = x^{d^-}$; and $x^{d^-} \prec x^{d^+}$ which does not satisfy the assumption.

Then d^+ is not supported by a cone and there exists a primitive collection C contained in the support of d^+. Denote by c the class of C and put $a = d^+ - c^+ \in \mathbb{N}^{\Delta(1)}$. Notice that $\min(d^-, a + c^-) + (d - c)^+ = d^+ - c^+ + c^- = a + c^-$ and $\min(d^-, a + c^-) + (d - c)^- = d^-$, which gives :

$$R^h_d - x^a h^{k^+} R^h_c = Q^c h^{k'} x^{\min(d^-, a + c^-)} R^h_{d-c}$$

where k' is the integer ensuring homogeneity. We also have :

$$\mathrm{Lm}(x^{\min(d^-, a+c^-)} R^h_{d-c}) = \max(x^{d^+ - c^+ + c^-}, x^{d-})$$

But $x^{c^-} \prec x^{c^+}$ by Lemma (3.23) and $x^{d^-} \prec x^{d^+}$ by assumption. It follows that :

$$\mathrm{Lm}(x^{\min(d^-, a+c^-)} R^h_{d-c}) \prec \mathrm{Lm}(R^h_d)$$

and $\mathrm{Lm}(R^h_{d-c}) \prec \mathrm{Lm}(R^h_d)$. By the minimality assumption, R^h_{d-c} admit a standard expression with zero remainder (3.29). This gives in return such an expression for R^h_d which contradicts $R_d \in E$.

b) $\mathrm{Lm}(R^h_d) = x^{d^-} h^{k^-}$. Since $R^h_{-d} = -Q^{-d} R^h_d$ and $\mathrm{Lm}(R^h_{-d}) = \mathrm{Lm}(R^h_d)$ one may replace d by $-d$ and apply the first case.

By (3.29), $\{R^h_c, c \in \mathcal{P}\}$ is a set of generators of the ideal and we can apply the Buchberger's criterion : Let c_1, c_2 be two primitive classes. We have :

$$\begin{aligned} S(R^h_{c_1}, R^h_{c_2}) :=& \frac{\mathrm{Lm}(R^h_{c_1})}{\gcd\left(\mathrm{Lm}(R^h_{c_1}), \mathrm{Lm}(R^h_{c_2})\right)} R^h_{c_2} - \frac{\mathrm{Lm}(R^h_{c_2})}{\gcd\left(\mathrm{Lm}(R^h_{c_1}), \mathrm{Lm}(R^h_{c_2})\right)} R^h_{c_1} \\ =& x^{c_1^+ - \min(c_1^+, c_2^+)} R_{c_2} - x^{c_2^+ - \min(c_1^+, c_2^+)} R_{c_1} = x^{\min(c_1^-, c_2^-)} Q^{c_1} R^h_{c_2 - c_1}. \end{aligned}$$

But (3.29) gives a normal expression with a zero remainder for $R_{c_2-c_1}$ and then for $S(R_{c_1}, R_{c_2})$. □

Proposition 3.30. — *The initial ideal of* QSR^h *for the weight function* ω *is :*

$$\mathrm{in}_\omega(\mathrm{QSR}^h) = \langle \mathrm{in}_\omega(R^h_c), c \in \mathcal{P}\rangle = \langle x^{c^+}, c \in \mathcal{P}\rangle = \langle\ x^a,\ a \text{ is not supported by a cone}\rangle.$$

Proof. — Let a be in $\mathbb{N}^{\Delta(1)}$ not supported by a cone. There exists a primitive class $c \in \mathcal{P}$ such that the support of c^+ is contained in the support of a. Then $a - c^+ \in \mathbb{N}^{\Delta(1)}$ and the leading form of $x^{a-c^+} R^h_c \in \mathrm{QSR}^h$ is x^a. This, and Lemma 3.23, proves the two equalities on the right, and the inclusion $\langle \mathrm{in}_\omega(R^h_c), c \in \mathcal{P}\rangle \subset \mathrm{in}_\omega(\mathrm{QSR}^h)$. It remains to show that $\mathrm{in}_\omega(\mathrm{QSR}^h) \subset \langle \mathrm{in}_\omega(R^h_c), c \in \mathcal{P}\rangle$.

Let P be in QSR^h. Using Lemma 3.28, we can write :

$$P = \sum_{c \in \mathcal{P}} Q_c R^h_c$$

where, for any $c \in \mathcal{P}$, $Q_c \in F[x_\rho, h]$ ($F = \mathrm{Frac}(\Lambda)$) and $\mathrm{Lm}(Q_c R^h_c) \preceq \mathrm{Lm}(P)$; this implies $\omega(Q_c R^h_c) \leq \omega(P)$. The initial form of P is :

$$\mathrm{in}_\omega(P) = \sum_{c \in \mathcal{P}, \omega(Q_c R^h_c) = \omega(P)} \mathrm{in}_\omega(Q_c R^h_c) = \sum_{c \in \mathcal{P}, \omega(Q_c R^h_c) = \omega(P)} \mathrm{in}_\omega(Q_c) x^{c^+}.$$

it follows that $\mathrm{in}_\omega(P)$ is in $\Lambda[x_\rho, h] \cap (\sum_{c \in \mathcal{P}} F[x_\rho, h] x^{c^+})$ which is equal to $\sum_{c \in \mathcal{P}} \Lambda[x_\rho, h] x^{c^+}$ since the x^{c^+} are monomials. □

The initial form of R^h_c are unitary terms. This gives :

Corollary 3.31. — *Let* p *be any closed point of* $\mathrm{Spec}\,\Lambda$ *and* $\kappa \simeq \mathbb{C}$ *be its residual field. Let* $\overline{\mathrm{QSR}}^h$ *be the images of* QSR^h *in* $\kappa[x_\rho, h]$*. The initial ideal of* $\overline{\mathrm{QSR}}^h$ *for the weight function* ω *is :*

$$\mathrm{in}_\omega(\overline{\mathrm{QSR}}^h) = \langle x^{c^+}, c \in \mathcal{P}\rangle = \langle\ x^a,\ a \text{ is not supported by a cone}\rangle.$$

Forgetting the variable h, we can restrict the weight function ω to $\Lambda[x_\rho]$; we still denote it by ω. Proposition 3.30 above, and specialization to $h = 1$ gives :

Corollary 3.32. — *The initial ideal of* QSR *for the weight function* ω *is :*

$$\mathrm{in}_\omega(\mathrm{QSR}) = \langle \mathrm{in}_\omega(R_c), c \in \mathcal{P}\rangle = \langle x^{c^+}, c \in \mathcal{P}\rangle = \langle\ x^a,\ a \text{ is not supported by a cone}\rangle.$$

3.3.e. Proof of Proposition 3.20. — We first show that $\mathbf{Q} \to \mathbf{S}$ is flat, then show, by Groebner degeneration, that each fiber is Cohen-Macaulay.

Flatness. Let us prove that $\Lambda[x_\rho]/\,\mathrm{QSR}$ is a free Λ-module : for any P in $\Lambda[x_\rho]$, denote by $\overline{P}$ its image in $\Lambda[x_\rho]/\,\mathrm{QSR}$. Let A be the set of monomials of $\Lambda[x_\rho]$ not contained in $\mathrm{in}_\omega(\mathrm{QSR})$. By Corollary 3.32 $A = \{x^a, a \in \mathbb{N}^{\Delta(1)} \mid a \text{ is supported by a cone}\}$. We claim that $\overline{A} = \{\overline{x^a}, x^a \in A\}$ is a base of $\Lambda[x_\rho]/\,\mathrm{QSR}$.

Let $x^{a_1}, \ldots, x^{a_l}$ be in A and $\alpha_1, \ldots, \alpha_l$ be in Λ. If $\sum_i \alpha_i \overline{x^{a_i}} = 0$, then $\sum_i \alpha_i x^{a_i} \in \mathrm{QSR}$ and $\mathrm{in}_\omega(\sum_i \alpha_i x^{a_i}) \in \mathrm{in}_\omega(\mathrm{QSR})$. Since every a_i is supported by a cone, $\alpha_i = 0$ for any i, and $\overline{A}$ is free over Λ.

Suppose now that $\overline{A}$ does not generate $\Lambda[x_\rho]/\,\mathrm{QSR}$ as a Λ-module. Let x^a be the smallest monomial for $\preceq$ such that $\overline{x^a} \notin \Lambda.\overline{A}$. Then a is not supported by a cone. There exists a primitive class d, and $b \in \mathbb{N}^{\Delta(1)}$ such that $a = b + d^+$ and $x^a = x^b R_d + Q^d x^{b+d^-}$. We deduce

that $\overline{x^a} = Q^d\overline{x^{b+d^-}}$. Since $x^{b+d^-} \prec x^a$, the class $\overline{x^{b+d^-}}$ belongs to $\Lambda.\overline{A}$, hence $\overline{x^a} \in \Lambda.\overline{A}$; this is a contradiction.

Cohen-Macaulayness and relative dimension. Consider the graded ring $\Lambda[x_\rho, h]$ and the ideal QSR^h defined in 3.34. Put $n' = n + k = \dim Y$. We first prove that $\Lambda[x_\rho, h]/\mathrm{QSR}^h$ is Cohen-Macaulay of relative dimension $n'+1$ over Λ.

Since $\mathbf{S}$ is a toric affine variety, it is a Cohen-Macaulay scheme. By **[BH93]**, Theorem 2.1.7, it is sufficient to show that over every closed point p of $\mathbf{S}$, the fiber $\mathbf{Q}_p$ of $\mathbf{Q} \to \mathbf{S}$ is a Cohen-Macaulay scheme of dimension $n'+1$.

Let p be a closed point of $\mathbf{S}$, $\kappa \simeq \mathbb{C}$ its residual field, and denote by $\overline{\mathrm{QSR}}^h$ the image of QSR^h in $\kappa[x_\rho, h]$. By Proposition 3.30, the initial ideal of $\overline{\mathrm{QSR}}^h$ is the Stanley-Reisner ideal $\mathrm{SR} = \langle x^a, a \in \mathbb{N}^{\Delta(1)}, \text{ not supported by a cone}\rangle$ defined in 3.14. Since $\overline{\mathrm{QSR}}^h$ is a graded ring, we can perform Gröbner degeneration, *i.e.*, construct a flat and proper family over $\mathbb{A}_1 = \operatorname{Spec}\mathbb{C}[t]$ whose fibre over 0 is $\operatorname{Proj}(\mathbb{C}[x_\rho, h]/\mathrm{SR})$ and whose fibre over any other point is $\operatorname{Proj}(\mathbb{C}[x_\rho, h]/\overline{\mathrm{QSR}}^h)$. From **[BH93]**, Theorem 5.1.4, and Corollary 5.4.6, we know that $\mathbb{C}[x_\rho]/\mathrm{SR}$ is a Cohen-Macaulay ring of dimension n'. Then $\mathbb{C}[x_\rho, h]/\mathrm{SR}$ and $\mathbb{C}[x_\rho, h]/\overline{\mathrm{QSR}}^h$ both are Cohen-Macaulay rings of dimension $n'+1$.

Since QSR^h is homogeneous, the polynomial $h-1$ is not a zero divisor of $\Lambda[x_\rho, h]/\mathrm{QSR}^h$. Then $\Lambda[x_\rho]/\mathrm{QSR} = \Lambda[x_\rho, h]/(\mathrm{QSR}^h, h-1)$ is a Cohen-Macaulay ring. □

3.4. Proof of Theorem 3.18. — Put $\mathbf{B} = \operatorname{Spec} B$ and consider the scheme morphism $f : \mathbf{B} \to \mathbf{S}$. We first study the fiber of f over $\mathbf{0}$. We then define a convenient neighbourhood $\mathbf{V}$ of $\mathbf{0}$ with help of a graded version of the Batyrev algebra and show that $\mathbf{B} \to \mathbf{S}$ is finite, flat, of degree $\dim H^{2*}(X)$ over $\mathbf{V}$; this prove the Theorem for the twisted Batyrev algebra. We finally prove the Theorem for the residual Batyrev algebra.

3.4.a. Fibre of $\mathbf{B} \to \mathbf{S}$ over $\mathbf{0}$.— Recall the definition of the Stanley-Reisner ideal of Δ (*cf.* 3.14). By Proposition 3.15 we have :

$$B \otimes (\Lambda/\mathbf{0}) \xrightarrow{\sim} \mathbb{C}[x_\rho]/(\mathrm{SR} + \mathrm{Lin}) \xrightarrow{\sim} H^{2*}(Y, \mathbb{C}) = H^{2*}(X, \mathbb{C}) \tag{3.33}$$
$$x_\rho \longmapsto [D_\rho]$$

3.4.b. Definition of a neighbourhood of $\mathbf{0}$. — First define a graded version of the Batyrev algebra of Y :

Definition 3.34. — Assume that the canonical divisor $-K_Y$ is nef, and consider the graded Λ-algebra $\Lambda[x_\rho, h]$ and the graded quantum Stanley-Reisner ideal QSR^h defined in 3.21.

The *linear ideal* of $\Lambda[x_\rho, h]$, is the homogeneous ideal Lin generated by the polynomials $Z_u = \sum_{\rho\in\Delta(1)}\langle u, v_\rho\rangle x_\rho$, $u \in M'$. The *graded Batyrev algebra* of Δ is the Λ-algebra :

$$B^h := \Lambda[x_\rho, h]/(\mathrm{QSR}^h + \mathrm{Lin}).$$

Put :

- $\mathbf{Q} := \operatorname{Spec}(\Lambda[x_\rho]/\mathrm{QSR})$.
- $\mathbf{P} := \operatorname{Proj}(\Lambda[x_\rho, h])$ and $\pi : \mathbf{P} \to \mathbf{S}$, the natural projective morphism.
- $\mathbf{H} \subset \mathbf{P}$, the relative hyperplane at infinity, defined by $h = 0$.
- $\mathbf{B}^h := \operatorname{Proj}(\Lambda[x_\rho, h]/(\mathrm{QSR}^h + \mathrm{Lin}))$.

By definition, $\mathbf{B} = \mathbf{B}^h \cap (\mathbf{P} \setminus \mathbf{H})$.

Lemma 3.35. — *Set :*

$$\mathbf{V} := \mathbf{S} \setminus \pi(\mathbf{B}^h \cap \mathbf{H}).$$

then $\mathbf{V}$ is an open Zariski neighbourhood of $\mathbf{0}$.

Proof. — Since π is projective, $\mathbf{V}$ is an open subset of $\mathbf{S}$. Let us show that $\mathbf{0} \in \mathbf{V}$. The intersection $(\mathbf{B}^h \cap \mathbf{H}) \cap (\pi^{-1}(\mathbf{0}))$ is defined by the homogeneous ideal $\langle h \rangle + \mathrm{SR} + \mathrm{Lin}$ in $\mathbb{C}[x_\rho, h]$. By Proposition 3.15, $\mathbb{C}[x_\rho, h]/(\langle h \rangle + \mathrm{SR} + \mathrm{Lin}) \xrightarrow{\sim} \mathbb{C}[x_\rho]/(\mathrm{SR} + \mathrm{Lin})$ is isomorphic to $H^{2*}(Y, \mathbb{C})$. Then the ideal $\langle h \rangle + \mathrm{SR} + \mathrm{Lin}$ defines a zero dimensional scheme supported by the origin of $\operatorname{Spec} \mathbb{C}[x_\rho, h]$; its radical is the irrelevant ideal of the graded ring $\mathbb{C}[x_\rho, h]$, and $(\mathbf{B}^h \cap \mathbf{H}) \cap (\pi^{-1}(\mathbf{0})) = \emptyset$. □

3.4.c. Local freeness and rank of the twisted Batyrev algebra of $(X, \mathcal{E})$.— Let $\mathbf{B}^h_{\mathbf{V}}$ be the pull-back of $\mathbf{B}^h$ by the open inclusion $\mathbf{V} \hookrightarrow \mathbf{S}$; we make use of the same notation for any other scheme defined over $\mathbf{S}$.

By Definition of $\mathbf{V}$, $\mathbf{B}^h_{\mathbf{V}}$ does not meet the relative hyperplane $\mathbf{H}_{\mathbf{V}}$, hence $\mathbf{B}_{\mathbf{V}} = \mathbf{B}^h_{\mathbf{V}}$. Moreover, as a closed subscheme of the projective bundle $\mathbf{P}_{\mathbf{V}}$ which do not meet a relative hyperplane, $\mathbf{B}_{\mathbf{V}}$ has relative dimension zero. Thus, $\mathbf{B}_{\mathbf{V}} \to \mathbf{V}$ is a finite and proper morphism.

By Proposition 3.20. $\mathbf{Q}_{\mathbf{V}} \longrightarrow \mathbf{V}$ is a flat morphism of relative dimension $n' = \dim Y$ between Cohen-Macaulay schemes. One get the following diagram :

$$\begin{array}{ccc} \mathbf{B}_{\mathbf{V}} & \hookrightarrow & \mathbf{Q}_{\mathbf{V}} \\ & \searrow^{\text{rel. dim. } 0} & \downarrow \text{rel. dim. } n' = \dim Y \\ & & \mathbf{V} \end{array}$$

Let $(e_1, \ldots, e_{n'})$ be a basis of $M' = \mathrm{Hom}(N \oplus \mathbb{Z}^k, \mathbb{Z})$. Let p be a closed point of $\mathbf{V}$ and denote by $\overline{Z}_i$ the image of $Z_i := Z_{e_i}$ in the quotient of $\Lambda[x_\rho]$ by the maximal ideal defining p. In the Cohen-Macaulay fiber $\mathbf{Q}_p$ over p, the scheme $\mathbf{B}_p$ has codimension n' and is defined by a sequence of the same length n' (namely $(\overline{Z}_1, \ldots, \overline{Z}_{n'})$). Then, by **[BH93]**, theorem 2.1.2, $(\overline{Z}_1, \ldots, \overline{Z}_{n'})$ is a regular sequence.

Since $\mathbf{Q}_{\mathbf{V}} \to \mathbf{V}$ is flat, and $(\overline{Z}_1, \ldots, \overline{Z}_{n'})$ is a regular sequence over any point of $\mathbf{V}$, the morphism $\mathbf{B}_{\mathbf{V}} \to \mathbf{V}$ is flat ([**Mat86**] Theorem 22.5 and Corollary). The degree of this finite morphism can be computed as the length of the fibre $\mathbf{B}^h_{\mathbf{0}}$ over $\mathbf{0}$. From isomorphism 3.33, it is equal to $\dim H^{2*}(Y)$. □

3.4.d. Local freeness and rank of the residual Batyrev algebra of $(X, \mathcal{E})$.— Denote by $\overline{x}_{\mathrm{top}}$ the image of x_{top} in $B = \Lambda[x_\rho]/(\mathrm{QSR} + \mathrm{Lin})$, and by $m_{\overline{x}_{\mathrm{top}}} : B \longrightarrow B$ the morphism of multiplication by $\overline{x}_{\mathrm{top}}$ in B. This multiplication induces an isomorphism :

$$B^{\mathrm{res}} = \Lambda[x_\rho]/(G : x_{\mathrm{top}}) \xrightarrow{\sim} \overline{x}_{\mathrm{top}} B = \mathrm{Im}(m_{\overline{x}_{\mathrm{top}}}),$$

which gives an exact sequence :

$$0 \longrightarrow B^{\mathrm{res}} \longrightarrow B \longrightarrow B/\overline{x}_{\mathrm{top}} B \longrightarrow 0 \tag{3.36}$$

Let d be a class of $\mathrm{NE}(Y)$, and ρ_i be the ray of $\Delta(1)^{bund}$ corresponding to a line bundle $\mathcal{L}_i$. Since $\mathcal{L}_i$ is ample and the Chern class of $\mathcal{L}_i$ is $[-D_{\rho_i}]$, we have $d_{\rho_i} = D_{\rho_i}.d < 0$. Then, we have :

$$R_d = x^{d^+} - Q^d x_{\mathrm{top}} x^{d^- - \epsilon}, \tag{3.37}$$

where $\epsilon = (\epsilon_\rho)_{\rho \in \Delta(1)}, \epsilon_\rho = 1$ if $\rho \in \Delta(1)^{bund}$, $\epsilon_\rho = 0$ if $\rho \in \Delta(1)^{base}$.

As a consequence, the image of R_d in $B/\overline{x}_{\mathrm{top}} B = \Lambda[x_\rho]/(\mathrm{QSR} + \mathrm{Lin} + \langle x_{\mathrm{top}} \rangle)$, is x^{d^+} and we have :

$$\begin{aligned} B/\overline{x}_{\mathrm{top}} B &\xrightarrow{\sim} \Lambda[x_\rho]/(\langle x^{d^+}, d \in \mathrm{NE}(Y) \rangle + \langle Z_u, u \in M' \rangle + \langle x_{\mathrm{top}} \rangle) \\ &\xrightarrow{\sim} \Lambda \otimes \Big(\mathbb{C}[x_\rho]/(\mathrm{SR} + \mathrm{Lin} + \langle x_{\mathrm{top}} \rangle) \Big) \end{aligned}$$

Using Proposition 3.15, we get :

$$\Lambda[x_\rho]/(\mathrm{QSR} + \mathrm{Lin} + \langle x_{\mathrm{top}} \rangle) \xrightarrow{\sim} \Lambda \otimes \big(H^{2*}(X, \mathbb{C}) / \langle c_{\mathrm{top}} \rangle \big).$$

Thus, $B/\overline{x}_{\mathrm{top}}B$ is a free Λ-module of rank $\dim_{\mathbb{C}} H^{2*}(Y) \,/\, c_{\mathrm{top}}H^{2*}(Y) = \dim_{\mathbb{C}} \ker m_{c_{top}}$.

Restricting the exact sequence (3.36) to $\mathbf{V}$, and using Theorem 3.18, we find that $(B^{\mathrm{res}})|_{\mathbf{V}}$ is a locally free Λ-module of rank $(\dim H^{2*}(Y) - \dim \ker m_{c_{top}})$ over $\mathbf{V}$. □

4. GKZ modules for toric varieties with a split vector bundle

GKZ systems were defined and studied by Gelfand-Kapranov-Zelevinskiĭ in the end of the eighties (*cf.* **[GGZ87]**, **[GZK88]**, **[GZK89]** and **[GKZ90]**). Our approach is closer to the one of **[Giv95]**, **[Giv98]**, **[CK99**, §5.5.3 and §11.2] or **[Iri09]**.

4.1. Definition and main Theorem for GKZ-modules. — For Batyrev algebras, the natural base ring is $\Lambda = \mathbb{C}[\mathrm{NE}(X)] = \mathbb{C}[\mathrm{NE}(Y)]$ as defined in Notation 2.4 or 3.9. When dealing with differential operators, we need to work over a smooth subvariety of $\mathbf{S} = \operatorname{Spec}\Lambda$.

Put, as in 3.10, $\Lambda = \mathbb{C}[Q^d, d \in \mathrm{NE}(Y)]$. Consider the ring $\mathbb{C}[Q^d, d \in H_2(Y,\mathbb{Z})]$, which is the localization of Λ where the Q^d are made invertible. Let $(B_1, \dots, B_r)$ be the fixed base of $H_2(X,\mathbb{Z})$ and $(T_1, \dots, T_r)$ be its dual base in $H^2(X,\mathbb{Z})$ (*cf.* 2.1). Put $q_i = Q^{B_i}$.

Notation 4.1. — Set :

$$\mathbb{C}[q_a^{\pm}] := \mathbb{C}[q_1^{\pm 1}, \dots, q_r^{\pm 1}] = \mathbb{C}[Q^d, d \in H_2(Y,\mathbb{Z})],$$
$$\mathbf{T} := \operatorname{Spec}\mathbb{C}[q_a^{\pm}] \qquad \mathbf{U} := \mathbf{V} \cap \mathbf{T}$$

where $\mathbf{V}$ is the neighbourhood of $\mathbf{0}$ defined in Lemma 3.35 ; $\mathbf{V}$ is the locus over which the Batyrev algebra is ensured to be locally free, and $\mathbf{U}$ will play the same role for differential modules. We have :

$$\begin{array}{ccccc} \mathbf{0} & \in & \mathbf{V} & \subset & \mathbf{S} \\ & & \cup & & \cup \\ & & \mathbf{U} & \subset & \mathbf{T}. \end{array} \qquad \text{and } \mathbf{0} \notin \mathbf{T}.$$

For any $d = \sum_{a=1}^r d_a B_a \in H_2(X,\mathbb{Z})$ we write :

$$q^d := \prod_{a=1}^r q_a^{d_a} \in \mathbb{C}[q_a^{\pm}].$$

Let z be another variable ; we write $\mathbb{C}_z$ for $\operatorname{Spec}\mathbb{C}[z]$, or $\mathbb{C}$ when no confusion can occur. Consider the non-commutative ring :

$$\mathbb{D} := \mathbb{C}[q_1^{\pm 1}, \dots, q_r^{\pm 1}, z]\langle z\delta_{q_1}, \dots, z\delta_{q_r}, z\delta_z\rangle = \mathbb{C}[q_a^{\pm}, z]\langle z\delta_q, z\delta_z\rangle, \tag{4.2}$$

where the non commutative relations are $(z\delta_{q_i})q_i = q_i(z\delta_{q_i}) + zq_i$ and $(z\delta_z)z = z(z\delta_z) + z^2$.

Notation 4.3. — 1. *Quantisation:* To any class $\tau = \sum_{a=1}^r t_a T_a \in H^2(X)$ we associate the operator

$$\widehat{\tau} := \sum_{a=1}^r t_a z\delta_{q_a} \in \mathbb{D}$$

If $\mathcal{L}$ is a line bundle or a divisor on X we also write $\widehat{\mathcal{L}} := \widehat{c_1(\mathcal{L})}$. Finally put :

$$\widehat{c}_{top} := \prod_{i=1}^k \widehat{\mathcal{L}}_i \in \mathbb{D}.$$

2. *Pochhammer symbol with a variable z:* For any element a of a $\mathbb{Z}[z]$-algebra, and any $k \in \mathbb{N}$ define :

$$[a]_0 = 1, \qquad [a]_k := a(a-z)\cdots(a-(k-1)z) \quad \text{if } k > 0. \tag{4.4}$$

This is a variant of the traditional Pochhammer symbol.

Definition 4.5. — For any $d \in H_2(X,\mathbb{Z})$ put :

$$\Box_d := \prod_{i=1}^{k} \left[\widehat{L}_i + z d_i^-\right]_{d_i^-} \prod_{\theta\in\Sigma(1)} \left[\widehat{D}_\theta\right]_{d_\theta^+} - q^d \prod_{i=1}^{k} \left[\widehat{L}_i + z d_i^+\right]_{d_i^+} \prod_{\theta\in\Sigma(1)} \left[\widehat{D}_\theta\right]_{d_\theta^-}.$$

where $d_i = L_i.d$, $d_\theta = D_\theta.d$. Also define the Euler field :

$$\widehat{\mathfrak{E}} := z\delta_z + \widehat{c_1(\mathcal{T}_X)} + \widehat{c_1(\mathcal{E}^\vee)}.$$

1. The *GKZ-ideal* $\mathbb{G}$ of $\mathbb{D}$ associated to $(\Sigma, L_1, \ldots, L_k)$ is the left ideal generated by the operators $\Box_d$ and $\widehat{\mathfrak{E}}$:
$$\mathbb{G} := \langle \widehat{\mathfrak{E}}, \Box_d, d \in H_2(X,\mathbb{Z})\rangle$$
2. The *quotient ideal* $(\mathbb{G} : \widehat{c}_{top})$ *of* $\mathbb{G}$ *with respect to* $\widehat{c}_{top}$, is the left ideal of $\mathbb{D}$ generated by : $\{P \in \mathbb{D} \mid \widehat{c}_{top} P \in \mathbb{G}\}$.
$$(\mathbb{G} : \widehat{c}_{top}) := \langle P \in \mathbb{D} \mid \widehat{c}_{top} P \in \mathbb{G}\rangle.$$

Remark 4.6. — 1. The set $\{P \in \mathbb{D} \mid \widehat{c}_{top} P \in \mathbb{G}\}$ contains the ideal $\mathbb{G}$; however, unlike the commutative case, it is not an ideal of $\mathbb{D}$ in general, but only a $\mathbb{C}[z]$-module (as an example, in $\mathbb{C}[q]\langle\delta_q\rangle$, fix $\mathbb{I} = \langle\delta_q\rangle$; then $q \in (\mathbb{I} : \delta_q)$ but $\delta_q q \notin \mathbb{I}$).

2. If $\rho \in \Delta(1)^{bund}$ corresponds to a divisor L_i, we have $[-D_\rho] = [L_i]$. This enables us to write :

$$\Box_d = \prod_{\rho\in\Delta(1)^{bund}} \left[-\widehat{D_\rho} + z d_\rho^+\right]_{d_\rho^+} \prod_{\rho\in\Delta(1)^{base}} \left[\widehat{D_\rho}\right]_{d_\rho^+} - q^d \prod_{\rho\in\Delta(1)^{bund}} \left[-\widehat{D_\rho} + z d_\rho^-\right]_{d_\rho^-} \prod_{\rho\in\Delta(1)^{base}} \left[\widehat{D_\rho}\right]_{d_\rho^-}$$

Note that, in this writing, the sign in front of $\widehat{D_\rho}$ differs for rays coming from the base X or from the line bundles $\mathcal{L}_i$. We follow here the conventions of **[CK99]**, taking account of their "Erratum to Proposition 5.5.4".

Definition 4.7. — Let $\mathbb{D} = \mathbb{C}[q_a^\pm, z]\langle z\delta_q, z\delta_z\rangle$ be the non commutative ring defined above. Let $\mathcal{D}$ be the corresponding sheaf of $\mathcal{O}_{\mathbf{T}\times\mathbb{C}}$-algebras.

1. The *twisted GKZ module* associated to $(\Sigma, L_1, \ldots, L_k)$ is the left $\mathbb{D}$-module
$$\mathbb{M} := \mathbb{D}/\mathbb{G},$$
the corresponding sheaf of $\mathcal{D}$-modules is denoted by $\mathcal{M}$.
2. The *residual GKZ module* $\mathbb{M}^{\text{res}}$ is the left $\mathbb{D}$-module
$$\mathbb{M}^{\text{res}} := \mathbb{D}/(\mathbb{G} : \widehat{c}_{top}),$$
the corresponding sheaf of $\mathcal{D}$-modules is denoted by $\mathcal{M}^{\text{res}}$.

Remark 4.8. — Up to isomorphism, $\mathbb{M}$ and $\mathbb{M}^{\text{res}}$ does not depend on the specific choices of the fan Σ and toric divisors L_i.

Indeed, the GKZ system is defined from the following exact sequence

$$0 \longrightarrow H_2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}^{\#\Delta(1)} \xrightarrow{\beta} N \times \mathbb{Z}^k \longrightarrow 0$$

where $\beta(e_\theta) = w_\theta$. If one use an other fan, than we have an isomorphic exact sequence which gives an isomorphic GKZ system. Notice that this exact sequence just depends on the rays of Δ and not on the higher dimensional cone of Δ.

Remark 4.9. — We will need alternative definitions of the GKZ modules :

(1) *Removing* $z\delta_z$ *:* Put $\mathbb{D}' := \mathbb{C}[q_a^\pm, z]\langle z\delta_q\rangle$ and $\mathbb{G}' = \langle \Box_d,\ d \in H_2(X,\mathbb{Z})\rangle \subset \mathbb{D}'$. The Euler operator $\widehat{\mathfrak{E}}$ of the ideal $\mathbb{G}$ enables us to remove $z\delta_z$ in the quotient, which gives two isomorphisms of $\mathbb{C}[q_a^\pm, z]$-module :

$$\mathbb{M} \xrightarrow{\sim} \mathbb{D}'/\mathbb{G}' \qquad\qquad \mathbb{M}^{\text{res}} \xrightarrow{\sim} \mathbb{D}'/(\mathbb{G}' : \widehat{c}_{top}) \tag{4.10}$$

(2) *Differential operators indexed by rays of* Δ *:* For any $\rho \in \Delta(1)$ write, in a unique way :

$$[D_\rho] = \sum_{a=1}^{r} D_\rho^a T_a \in H^2(Y, \mathbb{Z}). \tag{4.11}$$

Consider a set of indeterminates $\lambda_\rho, \rho \in \Delta(1)$. Put $\mathbb{D}'' := \mathbb{C}[q_a^\pm, z]\langle z\delta_{\lambda_\rho}, \rho \in \Delta(1)\rangle$, where the non commutative relations are : $z\delta_{\lambda_\rho}.q_a = q_a.z\delta_{\lambda_\rho} + D_\rho^a z q_a$. For any $d \in H_2(X, \mathbb{Z})$, put :

$$\Box_d'' := \prod_{\rho\in\Delta(1)^{bund}} [-z\delta_{\lambda_\rho} + zd_\rho^+]_{d_\rho^+} \prod_{\rho\in\Delta(1)^{base}} [z\delta_{\lambda_\rho}]_{d_\rho^+} - q^d \prod_{\rho\in\Delta(1)^{bund}} [-z\delta_{\lambda_\rho} + zd_\rho^-]_{d_\rho^-} \prod_{\rho\in\Delta(1)^{base}} [z\delta_{\lambda_\rho}]_{d_\rho^-}, \tag{4.12}$$

$$\mathcal{Z}_u'' := \sum_{\rho\in\Delta(1)} \langle u, v_\rho\rangle z\delta_{\lambda_\rho}, u \in M'. \tag{4.13}$$

Put $\mathbb{G}'' := \langle \Box_d'', \mathcal{Z}_u''\rangle$. Then there is an isomorphism of $\mathbb{C}[q_a^\pm, z]$-modules :

$$\begin{aligned} f \ : \ \mathbb{D}''/\mathbb{G}'' &\longrightarrow \mathbb{D}'/\mathbb{G}' \simeq \mathbb{M} \\ z\delta_{\lambda_\rho} &\longmapsto \sum_{a=1}^{r} D_\rho^a z\delta_{q_a} \end{aligned} \tag{4.14}$$

The previous isomorphism f induces an isomorphism f^{res} between the residual's modules, that is

$$\begin{array}{ccc} \mathbb{D}''/\mathbb{G}'' & \xrightarrow[\sim]{f} & \mathbb{D}'/\mathbb{G}' \simeq \mathbb{M} \\ \downarrow & & \downarrow \\ \mathbb{D}''/\left(\mathbb{G}'' : \prod_{\rho\in\Delta(1)^{bund}} -z\delta_{\lambda_\rho}\right) & \xrightarrow[\sim]{f^{\mathrm{res}}} & \mathbb{D}'/(\mathbb{G}' : \widehat{c}_{top}) \simeq \mathbb{M}^{\mathrm{res}} \end{array}$$

The main property of GKZ sheaves of $\mathcal{D}$-modules is given by :

Theorem 4.15. — *Let X be a toric smooth projective variety endowed with a split vector bundle $\mathcal{E} = \oplus_{i=1}^k \mathcal{L}_i$. Assume that $\omega_X \otimes {\mathcal{L}_1}^\vee \otimes \cdots \otimes {\mathcal{L}_k}^\vee$ and each line bundle $\mathcal{L}_i$ is nef. Let c_{top} be the top chern class of $\mathcal{E}$ and let $m_{c_{top}}$ be the morphism of multiplication by c_{top} in $H^{2*}(X)$.*

Let $\mathcal{M}$ and $\mathcal{M}^{\mathrm{res}}$ be the twisted and residual GKZ sheaf of $\mathcal{D}$-modules associated to $(X, \mathcal{E})$, as defined in 4.7. Let $\mathbf{U}$ be the open subset of $\mathbf{T}$ defined in 4.1. We have :

1. *Over $\mathbf{U} \times \mathbb{C}$, $\mathcal{M}$ is a locally free $\mathcal{O}_{\mathbf{U}\times\mathbb{C}}$-modules of rank* $\dim H^{2*}(X)$.
2. *Over $\mathbf{U} \times \mathbb{C}$, $\mathcal{M}^{\mathrm{res}}$ is a locally free $\mathcal{O}_{\mathbf{U}\times\mathbb{C}}$-modules of rank* $(\dim H^{2*}(X) - \dim\ker m_{c_{top}})$.

Proof. — This theorem follows from Proposition 4.16 ($\mathcal{M}_{|\mathbf{U}\times\mathbb{C}}$ and $\mathcal{M}^{\mathrm{res}}_{|\mathbf{U}\times\mathbb{C}}$ are coherent), Proposition 4.18 ($\mathcal{M}_{|\mathbf{U}\times\mathbb{C}}$ is locally free of the expected rank) and Proposition 4.23 below. In this last proposition we only prove that $\mathcal{M}^{\mathrm{res}}_{|\mathbf{U}\times\mathbb{C}}$ is locally free over $\mathbf{U} \times \mathbb{C}^*$ (that is on $z \neq 0$) and isomorphic to the residual Batyrev algebra on $z = 0$. By Nakayama's Lemma, this only gives an inequality on the dimension of $\mathcal{M}^{\mathrm{res}}_{|\mathbf{U}\times\mathbb{C}^*}$.

We are left to show that $\mathcal{M}^{\mathrm{res}}_{|\mathbf{U}\times\mathbb{C}^*}$ has the expected rank over $z \neq 0$. This point follows from Mirror symmetry and will be proved in section 5 (*cf.* Remark 5.22). □

4.2. Coherence of GKZ sheaves associated to $(X, \mathcal{E})$. —

Proposition 4.16. — *Under assumptions of Theorem 4.15, $\mathcal{M}_{|\mathbf{U}\times\mathbb{C}}$ and $\mathcal{M}^{\mathrm{res}}_{|\mathbf{U}\times\mathbb{C}}$ are coherent sheaves of $\mathcal{O}_{\mathbf{U}\times\mathbb{C}}$-modules.*

Proof. — If $\mathcal{M}$ is coherent then the surjective morphism $\mathcal{M} \to \mathcal{M}^{\mathrm{res}}$ implies that $\mathcal{M}^{\mathrm{res}}$ is finitely generated. Hence, it is sufficient to show that $\mathcal{M}$ is coherent over $\mathbf{U} \times \mathbb{C}$.

A usual proof of coherence for a differential module, consists in finding a good filtration and proving that the characteristic variety is supported by the zero section of the cotangent bundle

(*cf.* [**RS15**, §3] and [**Sab05**, Proposition 1.2.8]). In our case, $\mathbb{M}$ is not a $\mathbb{C}[q_a^{\pm}, z]\langle\partial_q, \partial_z\rangle$-module but only a $\mathbb{C}[q_a^{\pm}, z]\langle z\delta_q\rangle$-module ; we must adapt the classical proof (*cf.* for instance [**HTT08**, Proposition 2.2.5]) :

We make use of notations of Remark 4.9.(1) and put $\mathbb{M}' := \mathbb{D}'/\mathbb{G}'$. Let us define the following increasing filtration of $\mathbb{D}'$:

$$F_k\mathbb{D}' := \left\{ P \in \mathbb{D}' \mid P(q, z, z\delta_q) = \sum_{\substack{\alpha\in\mathbb{N}^r \\ |\alpha|\leq k}} P_\alpha(q, z)(z\delta_q)^\alpha \right\}$$

where $(z\delta_q)^\alpha := (z\delta_{q_1})^{\alpha_1} \cdots (z\delta_{q_r})^{\alpha_r}$. Let $\operatorname{gr}\mathbb{D}'$ be the graduated ring of $\mathbb{D}'$ defined by this filtration. Denote by y_a the class of $z\delta_{q_a}$ in $\operatorname{gr}\mathbb{D}'$, then $\operatorname{gr}\mathbb{D}'$ is isomorphic to $\mathbb{C}[q_a^{\pm}, z][y_1, \ldots, y_r]$. We define the *symbol* of an element $P = \sum_{\alpha\in\mathbb{N}^r} P_\alpha(q, z)(z\delta_q)^\alpha$ of $\mathbb{D}'$ by :

$$\sigma(P) = \sum_{\substack{\alpha\in\mathbb{N}^r \\ |\alpha|=\deg P}} P_\alpha(q, z)y^\alpha.$$

We also define an increasing filtration on $\mathbb{M}'$ by

$$F_k\mathbb{M}' := F_k\mathbb{D}'/\mathbb{G}'_k, \quad \text{where} \quad \mathbb{G}'_k := F_k\mathbb{D}' \cap \mathbb{G}'.$$

One can check that $(F_k\mathbb{M}')_{k\geq 0}$ satisfies the properties of a good filtration ; in particular, for any k in $\mathbb{N}$, $F_k\mathbb{M}'$ is a coherent $\mathbb{C}[q_a^{\pm}, z]$-module. We have $\operatorname{gr}\mathbb{M}' = \operatorname{gr}\mathbb{D}'/\operatorname{gr}\mathbb{G}'$, which shows that the annihilator ideal of $\operatorname{gr}\mathbb{M}'$ in $\operatorname{gr}\mathbb{D}'$ is $\operatorname{gr}\mathbb{G}'$. Recall that the characteristic variety of $\mathbb{M}'$ is the subscheme of $\operatorname{Spec}\operatorname{gr}\mathbb{D}'$ defined by the radical of the annihilator of $\operatorname{gr}\mathbb{M}'$. Put $\mathbf{A}^r_{\mathbf{T}\times\mathbb{C}} = \operatorname{Spec}\operatorname{gr}\mathbb{D}' = \operatorname{Spec}\mathbb{C}[q_a^{\pm}, z][y_i]$; denote by $\mathbf{C} \subset \mathbf{A}^r_{\mathbf{T}\times\mathbb{C}}$ the characteristic variety of $\mathbb{M}'$ defined by the ideal $\sqrt{\operatorname{Ann}\operatorname{gr}\mathbb{M}'}$. Let $\mathbf{U}$ be the open subset of $\mathbf{T}$ defined in Notations 4.1 and $\mathbf{C}_{\mathbf{U}\times\mathbb{C}} \subset \mathbf{A}^r_{\mathbf{U}\times\mathbb{C}}$ be the pull-back of $\mathbf{C}$ by the open immersion $\mathbf{U}\times\mathbb{C} \hookrightarrow \mathbf{T}\times\mathbb{C}$.

Lemma 4.17. — *The characteristic variety $\mathbf{C}_{\mathbf{U}\times\mathbb{C}}$ is the image of the zero section of the trivial bundle $\mathbf{A}^r_{\mathbf{U}\times\mathbb{C}} \to \mathbf{U}\times\mathbb{C}$. It is defined by the ideal $\langle y_1, \ldots, y_r\rangle$.*

Proof. — By definition of the symbol, the characteristic variety is contained in the closed subscheme of $\mathbf{A}^r_{\mathbf{T}\times\mathbb{C}}$ defined by the ideal

$$J = \langle \sigma(\Box_d), d \in H_2(X, \mathbb{Z})\rangle \subset \mathbb{C}[q_a^{\pm}, z][y_1, \ldots, y_r].$$

Consider the Batyrev Λ-algebra B defined in 3.34. After localisation of Λ and tensorization by $\mathbb{C}[z]$, one get a $\mathbb{C}[q_a^{\pm}, z]$-graded algebra. There is a natural surjective morphism :

$$\begin{aligned} \alpha : \mathbb{C}[q_a^{\pm}, z][x_\rho, h] &\longrightarrow \operatorname{gr}\mathbb{D}' = \mathbb{C}[q_a^{\pm}, z][y_1, \ldots, y_r] \\ h &\longmapsto 0 \\ x_\rho &\longmapsto \begin{cases} \sum_{a=1}^r D_\rho^a y_a & \text{if } \rho \in \Delta(1)^{base} \\ -(\sum_{a=1}^r D_\rho^a y_a) & \text{if } \rho \in \Delta(1)^{bund} \end{cases} \end{aligned}$$

where the integers D_ρ^a are defined by : $[D_\rho] = \sum_{a=1}^r D_\rho^a T_a$ (cf. 4.11). One check that, taking the quotients, the morphism α gives an isomorphism :

$$\mathbb{C}[q_a^{\pm}, z][x_\rho, h]/(\mathrm{QSR}^h + \mathrm{Lin} + \langle h\rangle) \simeq \mathbb{C}[q_a^{\pm}, z][y_1, \ldots, y_r]/J.$$

Let p be a closed point of $\mathbf{U}$, and $\kappa \simeq \mathbb{C}$ its residual field. Let $\overline{\mathrm{QSR}}^h$ be the image of QSR in $\kappa[x_\rho, h]$, and $\overline{J}$ be the image of J in $\kappa[z][y_1, \ldots, y_r]$. By definition of $\mathbf{V}$ (Lemma 3.35), the radical of $(\overline{\mathrm{QSR}}^h + \overline{\mathrm{Lin}} + \langle h\rangle)$ is the "irrelevant" ideal $\langle h, x_\rho, \rho \in \Delta(1)\rangle$. This shows that the radical of $\overline{J}$ is equal to $\langle \alpha(x_\rho), \rho \in \Delta(1)\rangle = \langle y_1, \ldots, y_r\rangle$. □

Denote by $\mathcal{D}'$ and $\mathcal{G}'$ the sheaves associated to $\mathbb{D}'$ and $\mathbb{G}'$. Consider the sheaf of ideals $\mathcal{I}$ in $\operatorname{gr}\mathcal{D}'$, generated by $\{y_1,\dots,y_r\}$. By Lemma 4.17 above, there exists $m_0\in\mathbb{N}$ such that $\mathcal{I}^{m_0}_{|\mathbf{U}\times\mathbb{C}}\subset\operatorname{gr}\mathcal{G}'_{|\mathbf{U}\times\mathbb{C}}$ (one may take $m_0=m_1+\cdots+m_r$ where $y_i^{m_i}\in\operatorname{gr}\mathcal{G}'_{|\mathbf{U}\times\mathbb{C}}$). We have :

$$F_{m_0+k}\mathcal{M}'_{|\mathbf{U}\times\mathbb{C}}=F_{m_0}\mathcal{D}'_{|\mathbf{U}\times\mathbb{C}}\cdot F_k\mathcal{M}'_{|\mathbf{U}\times\mathbb{C}}$$

which shows that the increasing filtration $F_k\mathcal{M}'_{|\mathbf{U}\times\mathbb{C}}$ is stationary after m_0. But we know that $F_k\mathcal{M}'$ is a coherent $\mathcal{O}_{\mathbf{T}\times\mathbb{C}}$-module. □

4.3. Local freeness and rank of the twisted GKZ sheaf associated to $(X,\mathcal{E})$. —

Proposition 4.18. — *Under assumptions of Theorem 4.15, the $\mathcal{O}_{\mathbf{U}\times\mathbb{C}}$-module $\mathcal{M}_{|\mathbf{U}\times\mathbb{C}}$ is locally free of rank* $\dim H^{2*}(X)$.

Proof. — The following proof is inspired from Theorem 2.14 of [**RS15**], with modifications taking into account the twisting by $\mathcal{E}$ and the use of q_a variables instead of $(\lambda_\rho)_{\rho\in\Delta(1)}$.

Step 1. *$\mathcal{M}/z\mathcal{M}$ is locally free of rank $H^{2*}(X)$.*

Let B be the Batyrev algebra $\Lambda[x_\rho]/\langle\mathrm{QSR}+\mathrm{Lin}\rangle$ defined in 3.10. Localizing Λ by inverting Q^d $(d\neq 0)$ gives $\mathbb{C}[q_a^\pm]$. There is an isomorphism of $\mathbb{C}[q_a^\pm]$-algebra :

$$B\otimes\mathbb{C}[q_a^\pm]\simeq\mathbb{C}[q_a^\pm,x_\rho]/\langle\mathrm{QSR}+\mathrm{Lin}\rangle\longrightarrow\mathbb{M}/z\mathbb{M}=\mathbb{D}'/(\langle z\rangle+\mathbb{G}') \tag{4.19}$$

$$x_\rho\longmapsto\begin{cases}\sum_{a=1}^r D_\rho^a z\delta_{q_a} & \text{if } \rho\in\Delta(1)^{base}\\ -\sum_{a=1}^r D_\rho^a z\delta_{q_a} & \text{if } \rho\in\Delta(1)^{bund}\end{cases}$$

By Theorem 3.18, B is locally free of rank $\dim H^{2*}(X)$ over $\mathbf{V}$; then $\mathcal{M}/z\mathcal{M}$ is locally free of rank $\dim H^{2*}(X)$ over $\mathbf{U}=\mathbf{V}\cap\mathbf{T}$.

Step 2. *$\mathcal{M}$ is locally free over $\mathbf{U}\times\mathbb{C}^*$.*

By Proposition 4.16, $\mathcal{M}|_{\mathbf{U}\times\mathbb{C}}$ is a coherent $\mathcal{O}_{\mathbf{U}\times\mathbb{C}}$-modules. If z is invertible, Theorem 1.4.10 of [**HTT08**] shows that the coherent sheaf $\mathcal{M}$ is actually locally free.

Step 3. *Up to a pull-back, $\mathbb{M}$ is a GKZ-module studied in Adolphson's article* [**Ado94**].

Let $\{\lambda_\rho,\rho\in\Delta(1)\}$, be a set of indeterminates. Put $\mathbb{D}^1=\mathbb{C}[\lambda_\rho^\pm]\langle\partial_{\lambda_\rho}\rangle$, with the usual relations $\partial_{\lambda_\rho}.\lambda_\rho=\lambda_\rho.\partial_{\lambda_\rho}+1$. For any $d\in H_2(X,\mathbb{Z})$, put $\square_d^1=\partial_\lambda^{d^+}-\partial_\lambda^{d^-}$. Consider the vector $\beta=(0_N,-1,\dots,-1)\in N\times\mathbb{Z}^k$ and for any $u\in M'$ put $\mathcal{Z}_u^1=\sum_\rho\langle u,v_\rho\rangle\lambda_\rho\partial_{\lambda_\rho}-\langle u,\beta\rangle$. Then the $\mathbb{D}^1$-module $\mathbb{D}^1/\langle\square_d^1,\mathcal{Z}_u^1\rangle$ is studied in [**Ado94**].

Let φ be the injective morphism :

$$\varphi:\mathbb{C}[q_a^\pm]\hookrightarrow\mathbb{C}[\lambda^\pm] \tag{4.20}$$

$$q_a\longmapsto\prod_{\rho\in\Delta(1)^{bund}}(-\lambda_\rho)^{D_\rho^a}\prod_{\rho\in\Delta(1)^{base}}\lambda_\rho^{D_\rho^a}, \tag{4.21}$$

where the D_ρ^a are defined in Remark 4.9.(2). viewing $\mathbb{C}[\lambda^\pm]$ as a $\mathbb{C}[q_a^\pm]$-algebra, we claim that there exist an isomorphism :

$$\mathbb{M}\otimes_{\mathbb{C}[q_a^\pm]}\mathbb{C}[\lambda_\rho^\pm]\xrightarrow{\sim}\mathbb{D}^1/\langle\square_d^1,\mathcal{Z}_u^1\rangle\otimes_{\mathbb{C}}\mathbb{C}[z^\pm] \tag{4.22}$$

To construct this isomorphism, put $\mathbb{D}^2=\mathbb{C}[\lambda_\rho^\pm,z]\langle z\partial_{\lambda_\rho}\rangle$, with the relations $z\partial_{\lambda_\rho}.\lambda_\rho=\lambda_\rho.z\partial_{\lambda_\rho}+z$. For any $d\in H_2(X,\mathbb{Z})$, put $\square_d^2=(z\partial_\lambda)^{d^+}-(z\partial_\lambda)^{d^-}$. Consider as above the vector $\beta=(0_N,-1,\dots,-1)\in N\times\mathbb{Z}^k$ and for any $u\in M'$ put $\mathcal{Z}_u^2=\sum_\rho\langle u,v_\rho\rangle\lambda_\rho z\partial_{\lambda_\rho}-\langle u,\beta\rangle z$. Sending λ_ρ to $z\lambda_\rho$, $z\partial_{\lambda_\rho}$ to ∂_{λ_ρ} and z to z, one get an isomorphism :

$$\mathbb{D}^2/\langle\square_d^2,\mathcal{Z}_u^2\rangle\xrightarrow{\sim}\mathbb{D}^1/\langle\square_d^1,\mathcal{Z}_u^1\rangle\otimes_{\mathbb{C}}\mathbb{C}[z^\pm].$$

Then consider as in remark 4.9.(2) the module $\mathbb{D}''/\langle \Box''_d, Z''_u\rangle$, isomorphic to $\mathbb{M}$. Put, in $\mathbb{D}^2$, $\ell := \prod_{\rho\in\Delta(1)^{bund}} \lambda_\rho$. There is an injective morphism of non commutative $\mathbb{C}[z^\pm]$-algebras :

$$\begin{aligned}
\mathbb{D}'' &\hookrightarrow \mathbb{D}^2\\
z\delta_{\lambda_\rho} &\longmapsto \ell^{-1}(\lambda_\rho.z\partial_{\lambda_\rho})\ell = \begin{cases}\lambda_\rho.z\partial_{\lambda_\rho} & \text{if } \rho\in\Delta(1)^{base}\\ \lambda_\rho.z\partial_{\lambda_\rho}+z & \text{if } \rho\in\Delta(1)^{bund}\end{cases}\\
q_a &\longmapsto \prod_{\rho\in\Delta(1)^{bund}}(-\lambda_\rho)^{D^a_\rho}\prod_{\rho\in\Delta(1)^{base}}\lambda_\rho^{D^a_\rho},
\end{aligned}$$

which gives :

$$\mathbb{M}\otimes_{\mathbb{C}[q_a^\pm]}\mathbb{C}[\lambda_\rho^\pm]\simeq \mathbb{D}''/\langle\Box''_d,\mathcal{Z}''_u\rangle\otimes_{\mathbb{C}[q_a^\pm]}\mathbb{C}[\lambda_\rho^\pm]\xrightarrow{\sim}\mathbb{D}^2/\langle\Box^2_d,\mathcal{Z}^2_u\rangle.$$

Step 4. *The rank of $\mathcal{M}$ over $\mathbf{U}\times\mathbb{C}^*$ is* $\dim H^{2*}(X,\mathbb{C})$.

The morphism φ defined in 4.20 is injective ; this gives a surjective morphism $h : \operatorname{Spec}\mathbb{C}[\lambda_\rho^\pm]\to \mathbf{T} = \operatorname{Spec}\mathbb{C}[q_a^\pm]$, and $\mathbf{O} = h^{-1}(\mathbf{U})$ is a dense open subset of the irreducible smooth variety $\operatorname{Spec}\mathbb{C}[\lambda_\rho^\pm]$.

The isomorphism 4.22 ensures that, over $\mathbf{O}$, the differential module $\mathbb{D}^1/\langle\Box^1_d,\mathcal{Z}^1_u\rangle$ is locally free of rank equals to the generic rank of $\mathcal{M}$. Moreover, by Corollary 5.11 of [**Ado94**] the rank of $\mathbb{D}^1/\langle\Box^1_d,\mathcal{Z}^1_u\rangle$ is $(n+k)!\mathrm{Vol}(\Gamma_\Delta)$ where Γ_Δ is the convex hull of the points $\{0, v_\rho, \rho\in\Delta(1)\}$ in $N'_{\mathbb{R}}$.

Since all the L_i are nef, the fan Δ is convex, and 0 is not in the interior of this convex hull. Since the divisor $-K_X-\sum_{i=1}^k L_i$ is nef, the vectors $(v_1,\dots,v_k)\in N\times\mathbb{Z}^k$ defined by the toric divisors L_i all are either vertices or contained in faces of Γ_Δ which do not contain 0. Hence, Γ_Δ is a "disjoint" (except for faces) union of the simplexes $\Gamma_\Delta(\tau) := (v_1,\dots,v_k,(v_{\rho_\theta})_{\theta\in\tau})$ where τ is any simplex defined by $0\in N_{\mathbb{R}}$ and generating vectors of rays of Σ (we make use of notations of Section 3.1). Let Γ_Σ be the convex hull of the points $\{0, w_\theta, \theta\in\Sigma(1)\}$ in $N_{\mathbb{R}}$. We have :

$$\begin{aligned}
\mathrm{rk}(\mathcal{M}) = (n+k)!\mathrm{Vol}(\Gamma_\Delta) &= \sum_{\tau,\text{ simplex of }\Sigma}(n+k)!\mathrm{Vol}(\Gamma_\Delta(\tau))\\
&= \sum_{\tau,\text{ simplex of }\Sigma}|\det(v_1,\dots,v_k,(v_{\rho_\theta})_{\theta\in\tau})| = \sum_{\tau,\text{ simplex of }\Sigma}|\det((w_\theta)_{\theta\in\tau})|\\
&= \sum_{\tau,\text{ simplex of }\Sigma} n!\mathrm{Vol}(\Gamma_\Sigma(\tau)) = n!\mathrm{Vol}(\Gamma_\Sigma) = \dim H^{2*}(X).
\end{aligned}$$

□

4.4. Local freeness and rank of the residual GKZ sheaf associated to $(X,\mathcal{E})$. —

Proposition 4.23. — *Under assumptions of Theorem 4.15.*

1. *On $z=0$, the $\mathcal{O}_{\mathbf{U}}$-module $(\mathcal{M}^{\mathrm{res}}/z\mathcal{M}^{\mathrm{res}})|_{\mathbf{U}}$ is locally free of rank* $\dim_{\mathbb{C}}\overline{H^{2*}(X)} = \dim_{\mathbb{C}}H^{2*}(X) - \dim_{\mathbb{C}}\ker(m_{c_{top}})$.
2. *On $z\neq 0$, the $\mathcal{O}_{\mathbf{U}\times\mathbb{C}^*}$-module $\mathcal{M}^{\mathrm{res}}|_{\mathbf{U}\times\mathbb{C}^*}$ is locally free of rank less than* $\dim_{\mathbb{C}}\overline{H^{2*}(X)}$.

Proof. — On $z\neq 0$, $\mathcal{M}^{\mathrm{res}}|_{\mathbf{U}\times\mathbb{C}^*}$ is locally free by Theorem 1.4.10 of [**HTT08**], as as in Step 2 of the proof of Proposition 4.18. By Nakayama's lemma, it is enough to prove the first statement.

Consider the residual Batyrev Λ-algebra $B^{\mathrm{res}} = \Lambda[x_\rho]/(G : x_{\mathrm{top}})$ defined in Subsection 3.2.c. By Proposition 3.18.2, B^{res} is a locally free module of rank $\dim_{\mathbb{C}}H^{2*}(X) - \dim_{\mathbb{C}}\ker(m_{c_{top}})$ over the open subscheme $\mathbf{V}\subset\mathbf{S}$ defined in Lemma 3.35. Proposition 4.23 follows from the Lemma below. □

Lemma 4.24. — *Consider the alternative definition of $\mathbb{M}$ given in Remark 4.9.(2) : $\mathbb{M} = \mathbb{D}''/\mathbb{G}''$, where $\mathbb{D}'' = \mathbb{C}[q_a^\pm, z]\langle z\delta_{\lambda_\rho}\rangle$ and $\mathbb{G}'' := \langle\Box''_d,\mathcal{Z}''_u\rangle$. Put $\widehat{c}_{top} = \prod_{\rho\in\Delta(1)^{bund}}(-z\delta_{\lambda_\rho})\in\mathbb{D}''$.*

Then $\mathbb{M}^{\rm res} \simeq \mathbb{D}''/(\mathbb{G}'' : \widehat{c}_{top})$ *and the following application is a well defined isomorphism of commutative* $\mathbb{C}[q_a^{\pm}]$*-algebras :*

$$\begin{aligned} \mathbb{M}^{\rm res}/z\mathbb{M}^{\rm res} &\longrightarrow B^{\rm res} \otimes_\Lambda \mathbb{C}[q_a^{\pm}] \\ z &\longmapsto 0 \\ z\delta_{\lambda_\rho} &\longmapsto \begin{cases} x_\rho & \text{if } \rho \in \Delta(1)^{base} \\ -x_\rho & \text{if } \rho \in \Delta(1)^{bund} \end{cases} \end{aligned} \tag{4.25}$$

Proof. — The first isomorphism $\mathbb{M}^{\rm res} \simeq \mathbb{D}''/(\mathbb{G}'' : \widehat{c}_{top})$ is immediate. We make use of the same notation, $\widehat{c}_{top}$, in $\mathbb{D}$ or in $\mathbb{D}''$.

Consider the morphism of $\mathbb{C}[q_a^{\pm}]$-algebras :

$$\begin{aligned} h : \mathbb{C}[q_a^{\pm}, z]\langle z\delta_{\lambda_\rho}\rangle &\longrightarrow \mathbb{C}[q_a^{\pm}][x_\rho] \\ z &\longmapsto 0 \\ z\delta_{\lambda_\rho} &\longmapsto \begin{cases} x_\rho & \text{if } \rho \in \Delta(1)^{base} \\ -x_\rho & \text{if } \rho \in \Delta(1)^{bund}, \end{cases} \end{aligned}$$

well defined since z is sent to 0. For any $d \in \mathrm{NE}(Y)$ and $u \in M'$ we have :

$$h(\square_d'') = R_d, \quad h(\mathcal{Z}_u'') = Z_u.$$

To prove that 4.25 is a well defined isomorphism, we must show that each polynomial $P \in (G : x_{\rm top})$ in $\mathbb{C}[q_a^{\pm}][x_\rho]$ possesses an antecedent for h in $(\mathbb{G} : \widehat{c}_{top})$.

Let us choose a section of h as a morphism of $\mathbb{C}[q_a^{\pm}]$-module. First consider the following isomorphism of $\mathbb{C}$-algebras :

$$\begin{aligned} \widehat{\ } : \mathbb{C}[x_\rho] &\longrightarrow \mathbb{C}[z\delta_{\lambda_\rho}] \\ x_\rho &\longmapsto \widehat{x_\rho} = \begin{cases} z\delta_{\lambda_\rho} & \text{if } \rho \in \Delta(1)^{base}, \\ -z\delta_{\lambda_\rho} & \text{if } \rho \in \Delta(1)^{bund}, \end{cases} \end{aligned}$$

and extend it $\mathbb{C}[q_a^{\pm}]$-linearly to $\widehat{\ } : \mathbb{C}[q_a^{\pm}][x_\rho] \to \mathbb{C}[q_a^{\pm}, z]\langle z\delta_{\lambda_\rho}\rangle$. For any $P \in \mathbb{C}[q_a^{\pm}][x_\rho]$, one check that $h\big(\widehat{P}\big) = P$.

Let P be in $(G : x_{\rm top})$. Recall that the ideal QSR is generated by polynomials $R_d, d \in \mathcal{P}$, where $\mathcal{P}$ is the set of primitive classes. Let $(u_i, i \in I = \{1, \dots, k\})$ be a base of the dual lattice of $N' = N \oplus \mathbb{Z}^k$. Put $Z_i := \sum_{\rho \in \Delta(1)} \langle u_i, v_\rho\rangle x_\rho$. The ideal Lin is generated by polynomials $\{Z_i, i \in I\}$. Then we can write :

$$x_{\rm top}P = \sum_{d \in \mathcal{P}} A_d R_d + \sum_{i \in I} B_i Z_i, \qquad A_d, B_i \in \mathbb{C}[q_a^{\pm}][x_\rho]. \tag{4.26}$$

We need to find $\widetilde{P} \in (\mathbb{G}'' : \widehat{c}_{top})$ such that $h\big(\widetilde{P}\big) = P$. For that, we may assume that $x_{\rm top}$ does not divide any monomial of A_d or B_i. If not we have, for any $d \in \mathcal{P}$ or $i \in I$ a unique decomposition :

$$A_d = A_{d,1} + x_{\rm top}A_{d,2}, \quad B_i = B_{i,1} + x_{\rm top}B_{i,2}.$$

where $x_{\rm top}$ does not divide any monomial of $A_{d,1}$ or $B_{i,1}$. Put $P_2 = \sum A_{d,2}R_d + \sum B_{i,2}Z_i$ and $\widetilde{P_2} = \sum \widehat{A_{d,2}}\square_d'' + \sum \widehat{B_{i,2}}\mathcal{Z}_i''$. Since $\widetilde{P_2}$ is in $\mathbb{G}''$, it is also in $(\mathbb{G}'' : \widehat{c}_{top})$; moreover, $h\big(\widetilde{P_2}\big) = P_2$. If we find $\widetilde{P_1} \in (\mathbb{G}'' : \widehat{c}_{top})$ such that $h\big(\widetilde{P_1}\big) = P - x_{\rm top}P_2$, then $\widetilde{P} = \widetilde{P_1} + \widetilde{P_2}$ is an antecedent of P in $(\mathbb{G}'' : \widehat{c}_{top})$.

Assume now that $x_{\rm top}$ does not divide any monomial of A_d or B_i.

Since each primitive class is in the Mori cone, and each line bundle $\mathcal{L}_i$ is ample, $x_{\rm top} = \prod_{\rho \in \Delta(1)^{bund}} x_\rho$ divides x^{d^-} for any $d \in \mathcal{P}$ and we can write :

$$R_d = x^{d^+} - Q^d x_{\rm top} x^{d^- - \epsilon},$$

where $\epsilon = (\epsilon_\rho)_{\rho\in\Delta(1)}, \epsilon_\rho = 1$ if $\rho \in \Delta(1)^{bund}$, $\epsilon_\rho = 0$ if $\rho \in \Delta(1)^{base}$. In the same way, for any $i \in I$, we write

$$Z_i = Z_i' + a_i x_{\text{top}},$$

where $a_i \in \mathbb{C}$, and x_{top} does not divide any term of Z_i'. Since $\deg x_{\text{top}} = k$, $a_i = 0$ if $k > 1$, and $a_i = \langle u_i, v_{\rho_{\text{top}}}\rangle$ if $k = 1$.

Finally, for any $d \in \mathcal{P}$, $i \in I$, we write :

$$A_d = \sum_{\alpha\in\mathbb{Z}^r} q^\alpha A_{d,\alpha}, \quad B_i = \sum_{\alpha\in\mathbb{Z}^r} q^\alpha B_{i,\alpha}, \quad \text{where } A_{d,\alpha}, B_{i,\alpha} \in \mathbb{C}[x_\rho]$$

In the quotient ring $\mathbb{C}[q_a^\pm][x_\rho]/(x_{\text{top}})$ we obtain from (4.26), for any $\alpha \in \mathbb{Z}^r$:

$$\sum_{d\in\mathcal{P}} A_{d,\alpha} x^{d^+} + \sum_{i\in I} B_{i,\alpha} Z_i' = 0. \tag{4.27}$$

For any $\alpha \in \mathbb{Z}^r$, put

$$\widetilde{P}_\alpha = -\sum_{d\in\mathcal{P}} \widehat{A_{d,\alpha}} q^d \prod_{\rho\in\Delta(1)^{bund}} [-z\delta_{\lambda_\rho} + zd_\rho^- - z]_{d_\rho^- - 1} \prod_{\rho\in\Delta(1)^{base}} [z\delta_{\lambda_\rho}]_{d_\rho^-} + \sum_{i\in I} \widehat{B_{i,\alpha}} a_i,$$

where we make use of the Pochammer symbol defined in Notation 4.3. Set :

$$\widetilde{P} = \sum_{\alpha\in\mathbb{Z}^r} q^\alpha \widetilde{P}_\alpha.$$

Then $h(\widetilde{P}_\alpha) = -\sum_{d\in\mathcal{P}} A_{d,\alpha} q^d x^{\gamma_d} + \sum_{i\in I} B_{i,\alpha} a_i$, and (4.27) gives :

$$x_{\text{top}}(P - h(\widetilde{P})) = \sum_{\alpha\in\mathbb{Z}^r} q^\alpha \left(\sum_{d\in\mathcal{P}} A_{d,\alpha} x^{d^+} + \sum_{i\in I} B_{i,\alpha} Z_i' \right) = 0$$

Which proves that $P = h(\widetilde{P})$.

We claim that $\widetilde{P} \in (\mathbb{G}'' : \widehat{c}_{top})$. By definition of the quotient ideal $(\mathbb{G}'' : \widehat{c}_{top})$, it is sufficient to show that, for any $\alpha \in \mathbb{Z}^r$, $\widehat{c}_{top}\widetilde{P}_\alpha \in \mathbb{G}$.

Recall that, for any $\rho \in \Delta(1)$, $z\delta_{\lambda_\rho}.q_a = q_a.z\delta_{\lambda_\rho} + D_\rho^a z q_a$ (Remark 4.9.(2)). Then we have, for $d \in \mathrm{NE}(Y)$:

$$\widehat{c}_{top} q^d = \prod_{\rho\in\Delta(1)^{bund}} (-z\delta_{\lambda_\rho}) \prod_a q_a^{T_a.d} = \prod_{\rho\in\Delta(1)^{bund}} q^d (-z\delta_{\lambda_\rho} - z\sum_{a=1}^r (T_a.d) D_\rho^a) = q^d \prod_{\rho\in\Delta(1)^{bund}} (-z\delta_{\lambda_\rho} + zd_\rho^-) \tag{4.28}$$

since $d_\rho = d_\rho^-$ for any $d \in \mathrm{NE}(Y)$ and $\rho \in \Delta(1)^{bund}$.

Applying morphism $\widehat{\cdot}$ to equality (4.27), which does not contain any variable q_a, we have :

$$\sum_{d\in\mathcal{P}} \widehat{A_{d,\alpha}}\widehat{x^{d^+}} + \sum_{i\in I} \widehat{B_{i,\alpha}} \widehat{Z}_i' = 0.$$

Moreover, since d is a primitive class, coefficients of $d^+ = (d_\rho^+)_{\rho\in\Delta(1)}$ are either equal to 0 or 1, and $d_\rho^+ = 0$ if $\rho \in \Delta(1)^{bund}$. Thus $\widehat{x^{d^+}} = \prod_{\rho\in\Delta(1)} (z\delta_{\lambda_\rho})^{d_\rho^+} = \prod_{\rho\in\Delta(1)^{bund}} [-z\delta_{\lambda_\rho} + zd_\rho^+]_{d_\rho^+} \prod_{\rho\in\Delta(1)^{base}} [z\delta_{\lambda_\rho}]_{d_\rho^+}$, which gives :

$$\sum_{d\in\mathcal{P}} \widehat{A_{d,\alpha}} \prod_{\rho\in\Delta(1)^{bund}} [-z\delta_{\lambda_\rho} + zd_\rho^+]_{d_\rho^+} \prod_{\rho\in\Delta(1)^{base}} [z\delta_{\lambda_\rho}]_{d_\rho^+} + \sum_{i\in I} \widehat{B_{i,\alpha}} \widehat{Z}_i' = 0. \tag{4.29}$$

Finally, equalities (4.29) and (4.28) gives :

$$\begin{aligned}\widehat{c}_{top}\widetilde{P_\alpha} &= -\sum_{d\in\mathcal{P}}\widehat{c}_{top}\widehat{A}_{d,\alpha}q^d\prod_{\rho\in\Delta(1)^{bund}}[-z\delta_{\lambda_\rho}+d_\rho^- -1]_{d_\rho^- -1}\prod_{\rho\in\Delta(1)^{base}}[z\delta_{\lambda_\rho}]_{d_\rho^-}+\sum_{i\in I}\widehat{c}_{top}\widehat{B}_{i,\alpha}a_i\\ &= \sum_{d\in\mathcal{P}}\widehat{A_{d,\alpha}}\Box''_d+\sum_{i\in I}\widehat{B_{i,\alpha}}\mathcal{Z}''_i\in\mathbb{G}''.\end{aligned}$$

□

5. Isomorphisms between quantum $\mathcal{D}$-modules and GKZ modules

5.1. The mirror Theorem of Givental and Lian-Liu-Yau. — The mirror theorem was proved by Givental (*cf.* [**Giv98**, Theorem 0.1] and [**CG07**, Corrolary 5]) and by Lian-Liu-Yau [**LLY99**]. Our technics are closed to the work of Givental that we recall now. As before, X is a smooth toric projective variety endowed with k globally generated line bundles $\mathcal{L}_1,\dots,\mathcal{L}_k$ such that $(\omega_X\otimes\mathcal{L}_1\otimes\cdots\otimes\mathcal{L}_k)^\vee$ is nef. We put $\mathcal{E}=\oplus_{i=1}^k\mathcal{L}_i$.

Denote by t_0 the coordinate on $H^0(X)$ associated to $T_0=\mathbf{1}$. In the definition below, we denote by $\mathcal{E}_{0,1,d}(1)$ the vector bundle on $X_{0,1,d}$ defined in Subsection 2.1.a.

Definition 5.1. — We define the cohomological multi-valued function J^{tw} by :

$$J^{\mathrm{tw}}(t_0,q,z):=e^{t_0/z}q^{T/z}\left(1+z^{-1}\sum_{\substack{d\in H_2(X,\mathbb{Z})\\ d\neq 0}}q^d e_{1*}\left(\frac{c_{\mathrm{top}}(\mathcal{E}_{0,1,d}(1))}{z-\psi}\cap[X_{0,1,d}]^{\mathrm{vir}}\right)\right)$$

where q is in the domain of convergence of the quantum product $\mathbf{D}\subset\mathbf{T}$, z is in $\mathbb{C}$ and $q^{T/z}=\prod_{a=1}^r q_a^{T_a/z}:=e^{z^{-1}\sum_{a=1}^r T_a\log(q_a)}$.

The proposition below is the twisted version of Lemma 10.3.3 of [**CK99**].

Proposition 5.2. — *Let L^{tw} be the multivalued section of* $\mathrm{Hom}(F,F)$ *defined in (2.9). In $H^{2*}(X)$, we have*

$$c_{\mathrm{top}}(\mathcal{E})J^{\mathrm{tw}}(t_0,q,z)=c_{\mathrm{top}}(\mathcal{E})(e^{-t_0/z}L^{\mathrm{tw}}(q,z))^{-1}\mathbf{1}$$

In the reduced cohomology ring $H^{2*}(X)/\ker m_{c_{top}}$ we have

$$\overline{J^{\mathrm{tw}}}(t_0,q,z)=(e^{-t_0/z}\overline{L}(q,z))^{-1}\overline{\mathbf{1}}.$$

Remark 5.3. — Notice that $c_{\mathrm{top}}(\mathcal{E})J^{\mathrm{tw}}(t_0,q,z)$ is exactly $J_{\mathcal{V}}$ of [**CK99**, p.358].

Proof of Proposition 5.2. — The first equalities is obtained by repeating the proof of Lemma 10.3.3 in [**CK99**] where one changes the standard Gromov-Witten axioms by the twisted axioms (see Appendix A). This first equality implies that $\overline{J^{\mathrm{tw}}(t_0,q,z)}=e^{t_0/z}\overline{(L^{\mathrm{tw}}(q,z))^{-1}\mathbf{1}}$ which is $(\overline{L}(q,z))^{-1}\overline{\mathbf{1}}$ by definition of $\overline{L}$ (*cf.* Formula (2.24)). □

Recall notations from section 3.1 and 3.2 : to a ray $\theta\in\Sigma(1)$, we associate a toric divisor denoted by D_θ. For any class $d\in H_2(X,\mathbb{Z})$ and any $i\in\{1,\dots,k\}$ we put

$$d_\theta:=\int_d D_\theta\quad\text{and}\quad d_i:=\int_d L_i$$

We define a cohomological multi-valued function by

$$I(q,z):=q^{T/z}\sum_{d\in H_2(X,\mathbb{Z})}q^d\prod_{i=1}^k\frac{\prod_{m=-\infty}^{d_i}([L_i]+mz)}{\prod_{m=-\infty}^{0}([L_i]+mz)}\prod_{\theta\in\Sigma(1)}\frac{\prod_{m=-\infty}^{0}([D_\theta]+mz)}{\prod_{m=-\infty}^{d_\theta}([D_\theta]+mz)}\tag{5.4}$$

where $q^{T/z}:=e^{z^{-1}\sum_{a=1}^r T_a\log(q_a)}$.

We develop the I-function in power series in z^{-1} and a direct computation gives :

$$I(q,z) = F(q)\mathbf{1} + z^{-1}G(q) + O(z^{-2}) \tag{5.5}$$

where F is an invertible univariate scalar function and G takes value in $H^{\leq 2}(X)$.

There exists a natural map $\alpha : H^2(X,\mathbb{C}) \to \mathbf{T}$ defined by :

$$\begin{aligned} \alpha : H^2(X,\mathbb{C}) &\longrightarrow \mathbf{T} = \operatorname{Spec}\mathbb{C}[H_2(X,\mathbb{Z})] \\ \tau &\longmapsto q := \left[d \mapsto q^d = \exp\left(2i\pi \int_d \tau\right)\right], \end{aligned}$$

so that $\alpha(\sum_{a=1}^r t_a T_a) = (e^{2i\pi t_a})_{a\in\{1,\dots,r\}}$.

Definition 5.6. — The *mirror map* of $(X,\mathcal{E})$ is the composite map

$$\begin{aligned} \mathrm{Mir} : \mathbf{T} &\longrightarrow H^0(X)\times\mathbf{T} \\ q &\longmapsto (\mathrm{Id}\times\alpha)\left(\frac{G(q)}{F(q)}\right) \end{aligned} \tag{5.7}$$

where $\alpha : H^2(X,\mathbb{C}) \to \mathbf{T}$ is defined above, and F, G are the functions appearing in (5.4). One can check that the mirror map is univariate.

The mirror theorem of Givental (*cf.* [**Giv98**, Theorem 0.1] and [**CG07**, Corrolary 5] ; see also [**CK99**, Theorem 11.2.16] or Lian-Liu-Yau [**LLY99**]) tells us the following.

Theorem 5.8. — [**CG07**, Corrolary 7] *Let* Mir *be the mirror map defined in 5.6. There exists an open subset*

$$\mathbf{W} = \{(q_a)_{a\in\{1,\dots,r\}}, |q_a| < \delta, \delta \in \mathbb{R}_{>0}\}$$

of $\mathbf{T}$ *such that*

1. $\mathrm{Mir}(\mathbf{W})$ *is contained in* $H^0(X)\times\mathbf{D}$ *where* $\mathbf{D}\subset\mathbf{T}$ *is the convergence domain of the quantum product (see Notation 2.7),*
2. $\mathrm{Mir}(q) = (0,q) + O(q)$,
3. $J^{\mathrm{tw}}(\mathrm{Mir}(q),z) = I(q,z)/F(q)$.

5.2. Quantum $\mathcal{D}$-module of a toric complete intersection in terms of residual GKZ system.— In order to relate the GKZ modules defined in section 4 and quantum $\mathcal{D}$-modules defined in section 2 we make use of the mirror map. As the target of this map is not $\mathbf{D}$ but $H^0(X)\times\mathbf{D}$ (Theorem 5.8), we first need to extend the base space of the various quantum $\mathcal{D}$-modules defined over $\mathbf{D}$. We will keep the same notations for these extended $\mathcal{D}$-modules :

– The twisted quantum $\mathcal{D}$-module $\mathrm{QDM}(X,\mathcal{E})$ is the trivial bundle F^{tw} with fibre $H^{2*}(X)$ over $H^0(X)\times\mathbf{D}\times\mathbb{C}_z$ endowed with the connection :

$$\nabla_{\delta_z} = \delta_z - \frac{1}{z}\mathfrak{E}\bullet_q^Z + \mu, \quad \nabla_{\partial_{t_0}} = \partial_{t_0} + \frac{1}{z}\mathbf{1}\bullet_q^{\mathrm{tw}} \qquad \text{and} \quad \forall a\in\{1,\dots,r\}, \nabla_{\delta_a} = \delta_a + \frac{1}{z}T_a\bullet_q^{\mathrm{tw}},$$

where $\mathfrak{E} = c_1(\mathcal{T}_X) - c_1(\mathcal{E}) + t_0\mathbf{1}$ and μ is the unchanged endomorphism of $H^{2*}(X)$ defined in 2.8. The fundamental solution L^{tw} is also extended in :

$$L^{\mathrm{tw}}(t_0,q,z) := e^{-t_0/z}L^{\mathrm{tw}}(q,z)$$

– The reduced quantum $\mathcal{D}$-module $\overline{\mathrm{QDM}}(X,\mathcal{E})$ is extended over $H^0(X)\times\mathbf{D}\times\mathbb{C}_z$ by taking the quotients of $\mathrm{QDM}(X,\mathcal{E})$. We do the same for the fundamental solution, which gives $\overline{L}(t_0,q,z) := e^{-t_0/z}\overline{L}(q,z)$.

– The ambient quantum $\mathcal{D}$-module $\overline{\mathrm{QDM}}_{\mathrm{amb}}(Z,\mathcal{E})$ is the trivial bundle F^Z_{amb} with fiber $H^{2*}_{\mathrm{amb}}(X)$ over $H^0(X)\times\mathbf{D}\times\mathbb{C}_z$ endowed with the connection :

$$\nabla^Z_{\delta_z}=\delta_z-\frac{1}{z}\mathfrak{E}^Z\bullet^Z_q+\mu^Z,\quad \nabla^Z_{\partial_{t_0}}=\partial_{t_0}+\frac{1}{z}\mathbf{1}\bullet^Z_q\quad\text{and}\quad\forall a\in\{1,\dots,r\},\nabla^Z_{\delta_a}=\delta_a+\frac{1}{z}T_a\bullet^Z_q,$$

where $\mathfrak{E}^Z:=c_1(\mathcal{T}_Z)+t_0\mathbf{1}$ and $\mu^Z(\psi_a)=\psi_a(\deg(\psi_a)-\dim_{\mathbb{C}}Z)/2$.

We have :

Theorem 5.9. — *Let X be a projective smooth toric variety endowed with k line bundles $\mathcal{L}_1,\dots,\mathcal{L}_k$; put $\mathcal{E}:=\oplus_{i=1}^k\mathcal{L}_i$. Assume that each $\mathcal{L}_i$ is globally generated and that $(\omega_X\otimes\mathcal{L}_1\otimes\ldots\otimes\mathcal{L}_k)^\vee$ is nef.*

Consider the mirror map Mir *and the open subset* $\mathbf{W}$ *of* $\mathbf{T}$ *defined in Theorem 5.8. For* $\varepsilon\in\mathbb{R}_{>0}$*, put*

$$\mathbf{W}_\varepsilon:=\{(q_1,\dots,q_r)\in\mathbf{W}\mid 0<|q_a|<\varepsilon\}.$$

There exists ε *in* $\mathbb{R}_{>0}$ *such that*

1. *Let $\mathcal{M}$ be the GKZ sheaf and* $\mathrm{QDM}(X,\mathcal{E})$ *the twisted quantum $\mathcal{D}$-module. Over* $\mathbf{W}_\varepsilon\times\mathbb{C}$*, the morphism*

$$\begin{aligned}\mathcal{M}&\xrightarrow{\sim}(\mathrm{Mir}\times\mathrm{id})^*\,\mathrm{QDM}(X,\mathcal{E})\\ 1&\longmapsto L^{\mathrm{tw}}(\mathrm{Mir}(q),z)I(q,z)\end{aligned}\tag{5.10}$$

   is a well-defined isomorphism of $\mathcal{D}$-modules.

2. *Let $\mathcal{L}_1,\dots,\mathcal{L}_k$ be k ample line bundles. Over* $\mathbf{W}_\varepsilon\times\mathbb{C}$*, the morphism*

$$\begin{aligned}\mathcal{M}^{\mathrm{res}}|_{\mathbf{W}_\varepsilon\times\mathbb{C}}&\xrightarrow{\sim}(\mathrm{Mir}\times\mathrm{id})^*\overline{\mathrm{QDM}}(X,\mathcal{E})\simeq(\mathrm{Mir}\times\mathrm{id})^*\mathrm{QDM}_{\mathrm{amb}}(Z)\\ 1&\longmapsto\overline{L}(\mathrm{Mir}(q),z)\overline{I(q,z)}\end{aligned}\tag{5.11}$$

   is a well defined isomorphism of $\mathcal{D}$-modules.

Remark 5.12. — 1. The first point of Theorem 5.9 should be known by specialists. However, we did not find a precise reference in our settings.

2. The second point constitutes our main result. It answers to the question addressed in the [**CK99**, p.94-95 and p.101]: "What differential equations shall we add to $\mathcal{G}$ to get an isomorphism with $\mathrm{QDM}_{\mathrm{amb}}(Z)$?".

   This result should permit us to compute algorithmically a finite system of differential equation defining $\mathrm{QDM}_{\mathrm{amb}}(Z)$. We present it in the Remark 6.2.

Lemma 5.13. — *Under the assumption of Theorem 5.9.1, the morphism of $\mathcal{D}$-modules*

$$\begin{aligned}\mathcal{M}&\xrightarrow{\sim}(\mathrm{Mir}\times\mathrm{id})^*\,\mathrm{QDM}(X,\mathcal{E})\\ 1&\longmapsto L^{\mathrm{tw}}(\mathrm{Mir}(q),z)I(q,z)\end{aligned}$$

is well defined over $\mathbf{W}$*.*

Proof of Lemma 5.13. — A direct computation shows that

$$L^{\mathrm{tw}}(\mathrm{Mir}(q),z)I(q,z)=L^{\mathrm{tw}}(\mathrm{Mir}(q),z)J^{\mathrm{tw}}(\mathrm{Mir}(q),z)/F(q)$$

is univariate.

We make use of notations of Definitions 4.7 and 4.5 ; we have $\mathcal{M}$ the sheaf associated to $\mathbb{D}/\mathbb{G}$, where

$$\mathbb{D}:=\mathbb{C}[q_a^{\pm},z]\langle z\delta_q,z\delta_z\rangle,\quad\mathbb{G}:=\left\langle\widehat{\mathfrak{E}},\Box_d,d\in H_2(X,\mathbb{Z})\right\rangle.$$

It is sufficient to prove that, for any $d\in H_2(X,\mathbb{Z})$:

$$\begin{aligned}\Box_d\left(z^{-c_1(\mathcal{T}_X)-c_1(\mathcal{E})}z^\mu I(q,z)\right)&=0,\\ \text{and}\quad\widehat{\mathfrak{E}}\left(z^{-c_1(\mathcal{T}_X)-c_1(\mathcal{E})}z^\mu I(q,z)\right)&=0.\end{aligned}$$

Put

$$A_d(z) := \prod_{i=1}^{k} \frac{\prod_{m=-\infty}^{d_i}([L_i]+mz)}{\prod_{m=-\infty}^{0}([L_i]+mz)} \prod_{\theta\in\Sigma(1)} \frac{\prod_{m=-\infty}^{0}([D_\theta]+mz)}{\prod_{m=-\infty}^{d_\theta}([D_\theta]+mz)}.$$

For any $\alpha \in H^2(X)$, we have $[\mu,\alpha]=\alpha$. This implies that

$$z^\mu \frac{\alpha}{z} = \alpha z^\mu. \tag{5.14}$$

From this we deduce that $z^\mu A_d(z) = z^{-d_{\mathcal{T}_X}+d_{\mathcal{E}}} A_d(1)$ where we set, as usual : $d_{\mathcal{T}_X} = \int_d c_1(\mathcal{T}_X)$ and $d_{\mathcal{E}} = \int_d c_1(\mathcal{E})$. Using the definition (5.4) of the function I we find :

$$z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})} z^\mu I(q,z) = \sum_{d\in H_2(X,\mathbb{Z})} q^{T+d} z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}}} A_d(1). \tag{5.15}$$

For any class $\alpha \in H^2(X)$, a direct computation shows that

$$\widehat{\alpha} q^{T+d} = q^{T+d} z(\alpha + d_\alpha), \tag{5.16}$$

$$z\delta_z(z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}}}) = z(-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}}) z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}}}. \tag{5.17}$$

We deduce that

$$z\delta_z\left(q^{T+d} z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}}}\right) = \left(-\widehat{c_1(\mathcal{T}_X)} + c_1(\widehat{\mathcal{E}})\right)\left(q^{T+d} z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}}}\right).$$

which gives : $\widehat{\mathfrak{E}}\left(z^{-c_1(\mathcal{T}_X)-c_1(\mathcal{E})} z^\mu I(q,z)\right) = 0$.

Using Formula (5.16), the equality $\square_d(z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})} z^\mu I(q,z)) = 0$ reduces to the following relation :

$$\begin{aligned} A_{d-d'}(1) \prod_{i=1}^{k}\prod_{\nu=1}^{d_i^+}([L_i]+(d-d')_{L_i}+\nu) \prod_{\theta\in\Sigma(1)}\prod_{\nu=0}^{d_\theta^+-1}([D_\theta]+(d-d')_\theta-\nu) \\ = A_d(1) \prod_{i=1}^{k}\prod_{\nu=1}^{d_i^-}([L_i]+d_i+\nu) \prod_{\theta\in\Sigma(1)}\prod_{\nu=0}^{d_\theta^--1}([D_\theta]+d_\theta-\nu). \end{aligned} \tag{5.18}$$

This formula can be proved for any $d, d' \in H_2(X,\mathbb{Z})$ by direct computation. □

Lemma 5.19. — *Under the assumption of Theorem 5.9.1, the morphism of $\mathcal{D}$-modules*

$$\begin{aligned} \mathcal{M}^{\mathrm{res}} &\xrightarrow{\sim} (\mathrm{Mir}\times\mathrm{id})^*\overline{\mathrm{QDM}}(X,\mathcal{E}) \\ 1 &\longmapsto L^{\mathrm{tw}}(\mathrm{Mir}(q),z) I(q,z) \end{aligned}$$

is well defined over $\mathbf{W}$.

Proof of Lemma 5.19. — Let $R(q,z,z\delta_q,z\delta_z) \in \mathbb{D}$ be in the quotient ideal $(\mathbb{G} : \widehat{c}_{top})$. We have to show that the cohomological valued function $R(q,z,z\delta_q,z\delta_z) z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})} z^\mu I(q,z)$ belongs to $\ker m_{c_{top}}$ where $m_{c_{top}}$ is the endomorphism of $H^{2*}(X) : \alpha \mapsto c_{\mathrm{top}}(\mathcal{E}) \cup \alpha$.

It is enough to prove it when R is a generator of the ideal $(\mathbb{G} : \widehat{c}_{top})$ *i.e.*, $\widehat{c}_{top} R \in \mathbb{G}$. From Formulas (5.16) and (5.17), we deduce that

$$\begin{aligned} &R(q,z,z\delta_q,z\delta_z) q^{T+d} z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}}} \\ &= R\left(q,z,z(T+d),z(-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}})\right) q^{T+d} z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_{\mathcal{E}}}. \end{aligned} \tag{5.20}$$

We decompose

$$R(q,z,z\delta_q,z\delta_z) = \sum_{\substack{d'\in H_2(X,\mathbb{Z}) \\ \text{finite}}} q^{d'} R_{d'}(z,z\delta_q,z\delta_z).$$

From Equalities (5.15) and (5.20), we deduce that

$$R(q,z,z\delta_q,z\delta_z)z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})}z^\mu I(q,z)=\sum_{d\in H_2(X,\mathbb{Z})}q^{d+T}z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_\mathcal{E}}B_d(z)$$

where

$$B_d(z):=\sum_{\substack{d'\in H_2(X,\mathbb{Z})\\ \text{finite}}}R_{d'}\left(z,z(T+d),z(-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_\mathcal{E})\right)A_{d-d'}(1).$$

To prove the lemma, it is enough to show that $c_{\text{top}}(\mathcal{E})B_d(z)=0$ for all $d\in H_2(X,\mathbb{Z})$. By $\widehat{c}_{top}R\in\mathbb{G}$ and Lemma 5.13, we have

$$\begin{array}{ll}\widehat{c}_{top}R(q,z,z\delta_q,z\delta_z)z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})}z^\mu I(q,z) & = 0\\ \sum_{d\in H_2(X,\mathbb{Z})}q^{d+T}z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})-d_{\mathcal{T}_X}+d_\mathcal{E}}\left(\prod_{i=1}^k z\left([L_i]+d_i\right)\right)B_d(z) & = 0.\end{array}$$

As $c_{\text{top}}(\mathcal{E})B_d:\mathbb{C}\to H^*(X)$ is a polynomial function in z, it is enough to prove that it vanishes on $\mathbb{C}^*$. Assume $z\in\mathbb{C}^*$. As $q\in(\mathbb{C}^*)^r$, we deduce that q^T and $z^{-c_1(\mathcal{T}_X)+c_1(\mathcal{E})}$ are invertible in $H^*(X)$. Denote by $I_d:=\{i\in\{1,\dots,k\}\mid d_i=0\}$ and I_d^c its complementary set. For $i\in I_d^c$, the class $[L_i]+d_i$ is invertible in $H^*(X)$. So we deduce that

$$\left(\prod_{i\in I_d}[L_i]\right)B_d(z)=0.$$

This implies that $c_{\text{top}}(\mathcal{E})B_d(z)=0$ as $c_{\text{top}}(\mathcal{E})=\prod_{i=1}^k[L_i]$.

□

Proof of Theorem 5.9. — Let us first prove that φ is an isomorphism. By Theorem 4.15, $\operatorname{rk}\mathcal{M}=\operatorname{rk}F$, so it is enough to prove that the morphism φ is surjective in a neighbourhood of $\mathbf{0}$. From [**CK99**, Proof of Proposition 5.5.4 p.100] we deduce that the "d" term in the definition of the I function (see (5.4)) vanishes when $d\notin\operatorname{NE}(X)$, so that we have:

$$I(q,z)=q^{T/z}\sum_{d\in\operatorname{NE}(X)}q^dA_d(z). \tag{5.21}$$

Then from (5.16) we have, for any $\alpha\in H^2(X)$:

$$\widehat{\alpha}I(q,z)=q^{T/z}(\alpha+O(q)).$$

As $H^{2*}(X)$ is generated by $H^2(X)$, we deduce that for any $a\in\{0,\dots,s-1\}$, there exists an operator $P_a(q,z,z\delta_q)$ such that

$$P_a(q,z,z\delta_q)I(q,z)F(q)^{-1}=q^{T/z}(T_a+O(q))$$

where $F(q)$ is defined in (5.5) ; notice that we do not need $z\delta_z$ in the operator P_a. From the definition of the function $L^{\text{tw}}(t_0,q,z)$ (*cf.* Equality (2.9)), we deduce that

$$L^{\text{tw}}(t_0,q,z)\gamma=e^{-t_0/z}q^{-T/z}(\gamma+O(q)).$$

By the mirror Theorem 5.8 we have that

$$\operatorname{Mir}(q)=q+O(q).$$

Putting the last three arguments together, for any $a\in\{0,\dots,s-1\}$ we have

$$\varphi(P_a(q,z,z\delta_q))=L^{\text{tw}}(\operatorname{Mir}(q),z)q^{T/z}(T_a+O(q))=T_a+o(1).$$

This proves the surjectivity of φ near the point $\mathbf{0}$. As it is an open condition, it is true in a neighbourhood of $q=0$.

Let us prove that φ' is an isomorphism. First, the surjectivity of φ implies the surjectivity of $\pi\circ\varphi$. We deduce that φ' is also surjective. On $z\neq0$, Proposition 4.23 implies that the

rank of $\mathcal{M}^{\mathrm{res}}$ is less than $\operatorname{rk}\overline{F}$. Hence the surjectivity implies that its rank is $\operatorname{rk}\overline{F}$. This also implies that $\mathcal{M}^{\mathrm{res}}$ is locally free on $\mathbf{U}\times\mathbb{C}$ of rank $\dim\overline{H^{2*}(X)}_{\mathbb{C}} = \operatorname{rk}\overline{F}$. We deduce that φ' is an isomorphism. □

Remark 5.22. — The last point of this proof is the missing argument to finish the proof of Theorem 4.15.2.

6. Examples: hypersurface in $\mathbb{P}^n$ and in $\mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^n$

In the following examples, we want to give explicit computations of the quotient ideal $(\mathbb{G} : \widehat{c}_{top})$. The first example is $\mathbb{P}^n$ with the line bundle $\mathcal{O}(a)$ and the second one is the blow up of $\mathbb{P}^n$ at one point with an appropriate bundle (see below). In a forthcoming paper, we will prove the following general statement

Theorem 6.1. — *Let X be a smooth projective toric variety with $\mathcal{L}_1,\dots,\mathcal{L}_k$ nef line bundles on X such that $\omega_X\otimes\mathcal{L}_1^\vee\otimes\cdots\otimes\mathcal{L}_k^\vee$ is nef. Put $\mathbb{D}' := \mathbb{C}[q_a^{\pm},z]\langle z\delta_q\rangle$ and $\mathbb{G}'$ the left ideal generated by $\Box_d$ for $d\in H_2(X,\mathbb{Z})$ (see Remark 4.9). Let $P\in\mathbb{G}'$, we can write*

$$P=\sum_{c\in\mathcal{P}} B_c\Box_c,\quad \deg(B_c\Box_c)\le\deg(P).$$

where the degree means the degree as differential operators in $\mathbb{D}'$ and $\mathcal{P}$ is the set of primitive classes (see Notation 3.26 and Definition 3.24).

Remark 6.2. — Let us explain how one could use this theorem to get an algorithm to compute the residual ideal $\langle\mathbb{G}' : \widehat{c}_{top}\rangle$ in order to get, via the isomorphism of Theorem 5.9, a presentation of $\mathrm{QDM}_{\mathrm{amb}}(Z)$.

1. First, Theorem 6.1 implies that the generators of the ideal $\mathbb{G}'$ can be indexed by the primitive classes, *i.e.*, $\mathbb{G}' = \langle\Box_c,\quad c\in\mathcal{P}\rangle$.
2. As the line bundle $\mathcal{L}_i$ are ample, for any $d\in\mathrm{NE}(X)$, we see that the operator $\Box_d$ is of the form $P_1-\widehat{c}_{top}q^dP_2$ where P_1,P_2 are two operators in $\mathbb{D}'$ (see (3.37) for a similar statement in the commutative case). Let c_1,c_2 be two primitive classes. Using the same ideas that S-polynomials for Groebner basis (in the commutative case), we can find three operators $T,U,V\in\mathbb{D}'$ such that
$$U\Box_{c_1}-V\Box_{c_2}=\widehat{c}_{top}T_{c_1,c_2}$$
This means that for each pair of primitive class c_1,c_2, we get an operator T_{c_1,c_2} in the residual ideal $(\mathbb{G}' : \widehat{c}_{top})$.
3. We think that the residual ideal is generated by the $\Box_c$ for $c\in\mathcal{P}$ and by T_{c_1,c_2} for $c_1,c_2\in\mathcal{P}$. At this point, we do not have a complete proof of this statement. We hope that an induction, like in Proposition 6.5, could work.

Ideas of proof of Theorem 6.1. — We only give some ideas for a proof because it is quite long and technical.
Calabi-Yau case *i.e.*, $\omega_X\otimes\mathcal{L}_1^\vee\otimes\cdots\otimes\mathcal{L}_k^\vee=\mathcal{O}_X$:The theorem follows immediately from the homogeneity of the operator $\Box_d$ for any $d\in H_2(X,\mathbb{Z})$.
Non Calabi-Yau case: This case is more difficult. Let's recall some notations of (4.14). We use the following isomorphism

$$\begin{aligned} f\ :\ \mathbb{D}''/\langle\Box''_d,\mathcal{Z}''_u\rangle &\longrightarrow \mathbb{D}'/\mathbb{G}'\simeq\mathbb{M}\\ z\delta_{\lambda_\rho} &\longmapsto \sum_{a=1}^{r} D_\rho^a z\delta_{q_a}\end{aligned}$$

where $\mathbb{D}'' := \mathbb{C}[q_a^{\pm}, z]\langle z\lambda q_\rho, \rho \in \Delta(1)\rangle$ and

$$\Box''_d := \prod_{\rho\in\Delta(1)^{bund}} [-z\delta_{\lambda_\rho} + zd_\rho^+]_{d_\rho^+} \prod_{\rho\in\Delta(1)^{base}} [z\delta_{\lambda_\rho}]_{d_\rho^+} - q^d \prod_{\rho\in\Delta(1)^{bund}} [-z\delta_{\lambda_\rho} + zd_\rho^-]_{d_\rho^-} \prod_{\rho\in\Delta(1)^{base}} [z\delta_{\lambda_\rho}]_{d_\rho^-},$$

$$\mathcal{Z}''_u := \sum_\rho \langle u, v_\rho\rangle z\delta_{\lambda_\rho}, u \in M'.$$

Using a suitable monomial order, we can prove a first result. Let $P \in \langle \Box''_d\rangle \subset \mathbb{D}''$, we can write

$$P = \sum_{c\in\mathcal{P}} B_c \Box''_c, \quad \deg(B_c\Box''_c) \leq \deg(P).$$

Then, we have to incorporate the $\mathcal{Z}''_u$ operators into the picture. This is the tricky part. We consider the ideals generated by the symbols which is a monomial ideals in a commutative ring. We use the Taylor's complex (see [**Lyu88**]) which plays the role of the Koszul resolution for monomial ideals. Then we pass to the ideal $\langle \Box''_d, \mathcal{Z}_u\rangle$ and use the isomorphism f to conclude. □

6.1. The projective space $X = \mathbb{P}^n$ and the invertible sheaf $\mathcal{L} = \mathcal{O}(a)$.— We have $H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$. Denote by h the homology class of a line in $\mathbb{P}^n$, and by H the Chern class of $\mathcal{O}(1)$. They both generates their respective group, and we have $\int_h H = 1$. The nef cone in $H^2(X, \mathbb{Z})$ is $\mathbb{N}.H$ and its dual, the Mori cone in $H_2(X, \mathbb{Z})$, is equal to $\mathbb{N}.h$. The ring Λ is $\mathbb{C}[Q^d, d \in \mathrm{NE}(X)] \simeq \mathbb{C}[q]$ where we set $q := Q^h$. The ring Π is $\Pi = \mathbb{C}[Q^d, d \in H_2(X, \mathbb{Z})] \simeq \mathbb{C}[q^{\pm}] := \mathbb{C}[q, q^{-1}]$. We put $\mathcal{L} = \mathcal{O}(a)$ for $a \in \mathbb{Z}$. The sheaf $\mathcal{O}(a)$ is ample if and only if $a > 0$. The sheaf $\omega_{\mathbb{P}^n}^\vee \otimes \mathcal{O}(a)^\vee = \mathcal{O}(n+1-a)$ is nef if and only if $n+1-a \geq 0$. We have $0 < a \leq n+1$. The different cases are :

Calabi-Yau	$a = n + 1$.
Fano	$1 \leq a \leq n$, where $(\omega_X \otimes \mathcal{L})^\vee = \mathcal{O}(n + 1 - a)$ is ample.

We make use of the notations of Subsection 3.1. Let us choose a fan for $\mathbb{P}^n$: Denote by N the lattice $\mathbb{Z}^n$ and by $(e_1, \ldots, e_n)$ its canonical basis. Put $w_1 := e_1, \ldots w_n := e_n$, $w_{n+1} := -e_1 - \cdots - e_n$. These are the lattice generators of the rays θ_i, where $\theta_i = \mathbb{R}^+ w_i$ for any $i \in \{1, \ldots, n + 1\}$. We set $\Sigma(1) := \{\theta_1, \ldots, \theta_{n+1}\}$. The set of maximal cones is

$$\Sigma(n) = \{\text{every (necessarily convex) cone generated by } n \text{ vectors in } \Sigma(1)\}.$$

Denote by D_θ the toric divisor associated to the ray $\theta \in \Sigma(1)$. We have $[D_\theta] = H$ in $H^{2*}(X)$. There is only one primitive collection (see §.3.3.c) $P = \{\theta_1, \ldots, \theta_{n+1}\} = \Sigma(1)$. The primitive class is $\mathcal{P} = \{h\}$.

Let us compute the quotient ideal. We will use the alternative definition 4.9.1 of the GKZ module, that is $\mathbb{M} = \mathbb{D}'/\mathbb{G}'$, with $\mathbb{D}' = \mathbb{C}[q^{\pm}]\langle z\delta_q\rangle$ and $\mathbb{G}' = \langle \Box_h\rangle$. We have $c_{\mathrm{top}} = c_1(\mathcal{L}) = c_1(\mathcal{O}(a)) = aH$ and $\widehat{c}_{top} = az\delta_q$.

Proposition 6.3. — *We have:*

$$(\mathbb{G}' : \widehat{c}_{top}) = \left\langle \Box_h, \frac{1}{a}(z\delta_q)^n - q(az\delta_q + z).\ldots.(az\delta_q + (a-1)z) \right\rangle.$$

Proof. — The operator $P_0 = \frac{1}{a}(z\delta_q)^n - q(az\delta_q + 2z).\ldots.(az\delta_q + az)$ is in $(\mathbb{G}' : \widehat{c}_{top})$: since $z\delta_q.q = q(z\delta_q + z)$, we have $\widehat{c}_{top}.P = a\Box_h$. We prove now, by induction on the degree, that any operator P in $(\mathbb{G}' : \widehat{c}_{top})$ is in $\langle P_0\rangle$. First notice that $a.y\sigma(P_0) = \sigma(\Box_h)$, even in the Calabi-Yau case where σ is the symbol. Let P be in $(\mathbb{G}' : \widehat{c}_{top})$.

If $\deg P = 0$ (and $P \neq 0$). We have, $az\delta_q.P = Q.\Box_h$, where $Q \in \mathbb{D}$ and $\deg(az\delta_q.P) = 1$, $\deg Q.\Box_h = \deg Q + \deg \Box_h = \deg Q + (n + 1)$ (recall that $a \leq n + 1$). It follows that $n = 0$, which is impossible.

Assume it is true for $\deg P = l$. If $\deg P = l + 1$, we still have $az\delta_q.P = Q.\Box_h$. Passing to the symbol we get : $ay\sigma(P) = \sigma(Q).\sigma(\Box_h) = ay\sigma(Q).\sigma(P_0)$. It follows that the polynomials P and

QP_0 both are in $\mathrm{Quot}(\widehat{c}_{top}, \mathbb{G}')$ and have the same symbol. Hence, $P - QP_0$ is in $\mathrm{Quot}(\widehat{c}_{top}, \mathbb{G}')$ and has degree strictly less than l. By induction, $P - QP_0 \in \langle P_0 \rangle$ and $P \in \langle P_0 \rangle$. □

6.2. The blown-up plane $X = \mathrm{Bl}_{\mathrm{pt}}\,\mathbb{P}^n$ and the sheaf $\mathcal{L} = \mathcal{O}(aH + bE)$.— Denote by $N = \mathbb{Z}^n$ the lattice and by $(e_1, \ldots, e_n)$ the canonical basis of N. The fan Σ of X is given by the rays

$$v_0 = -e_n, \quad \forall i \in \{1, \ldots, n\},\ v_i = e_i,\ v_{n+1} = (-1, \ldots, -1).$$

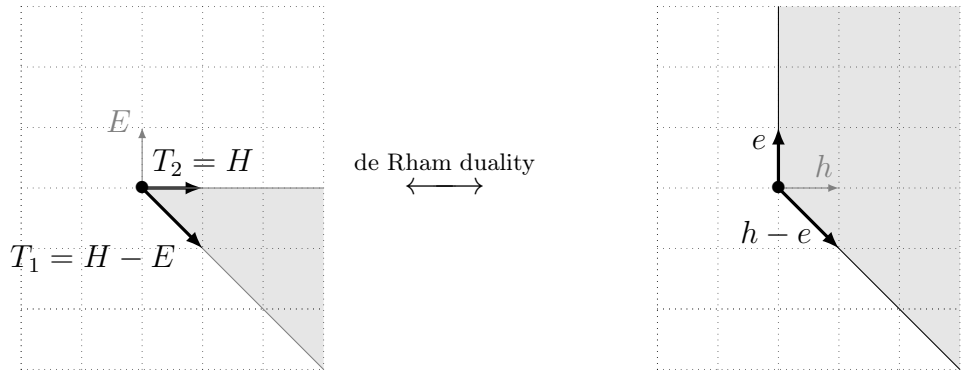
The maximal cone in $N \otimes \mathbb{R}$ are

$$\forall i \in \{1, \ldots, n+1\} \setminus \{n\},\ \sigma_i = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \mathbb{R}^+ v_j,\ \text{and } \sigma_{n,i} = \mathbb{R}^+ v_0 + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \mathbb{R}^+ v_j$$

We have $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^2$ and $H_2(X, \mathbb{Z}) \cong \mathbb{Z}^2$. Let E be the exceptional divisor, and H the strict transform by the blown-up of an hyperplane of $\mathbb{P}^n$ which does not meet the blown-up point. We also denote by E and H their Chern classes. Denote by e the homology class of E and h the homology class of H. We choose the following bases which are dual to each others :

- Base of $H^2(X, \mathbb{Z})$: $(T_1 = H - E, T_2 = H)$.
- Base of $H_2(X, \mathbb{Z})$: $(B_1 = e, B_2 = h - e)$.

Notice that $c_1(\omega_X) = (n+1)H - (n-1)E$. We denote by D_θ the toric divisor associated to $\theta \in \Sigma(1)$ and $[D_\theta]$ its class in $H^2(X, \mathbb{Z})$. We have for $i \in \{1, \ldots, n+1\} \setminus \{n\}, [D_i] = H - E,\ [D_n] = H,\ [D_0] = E$. There are two primitive collections, $P_1 = \{\theta_0, \theta_n\}$ and $P_2 = \{\theta_i, i \notin \{0, n\}\}$. The primitive classes are $\mathcal{P} = \{e, h - e\}$. The nef cone in $H^2(X, \mathbb{Z})$ is $\mathbb{R}^+ H + \mathbb{R}^+(H - E)$, an its dual, the Mori cone in $H_2(X, \mathbb{Z})$ is equal to $\mathbb{R}^+ e + \mathbb{R}^+(h - e)$ (see Figure 2). Following the choice of



The nef cone in $H^2(\mathrm{Bl}_p\,\mathbb{P}^2, \mathbb{Z})$. The Mori cone in $H_2(Bl_{\mathrm{pt}}\mathbb{P}^2, \mathbb{Z})$.

FIGURE 2. Nef and Mori cone of $\mathrm{Bl}_p\,\mathbb{P}^2$.

our base, we put $q_1 := Q^e$, $q_2 := Q^{h-e}$. The ring Λ is $\mathbb{C}[q_1, q_2]$. We want $\mathcal{O}(aH + bE)$ to be ample and $\omega^\vee_{Bl_p\mathbb{P}^n} \otimes \mathcal{O}(aH + bE)^\vee = \mathcal{O}((n + 1 - a)H - (n - 1 + b)E)$ to be nef. This leads to the following cases

$$\{(a, b) \in \mathbb{Z}^2 \mid b \in \{-1, \ldots, 1 - n\},\ a + b \in \{1, 2\}\}$$

The Calabi-Yau case is $(a, b) = (n + 1, 1 - n)$. We have

$$c_{\mathrm{top}} = -bT_1 + (a + b)T_2, \quad \widehat{c}_{\mathrm{top}} = -b(z\delta_{q_1}) + (a + b)(z\delta_{q_2})$$

In the differential ring $\mathbb{D}' := \mathbb{C}[z, q_1^{\pm}, q_2^{\pm}]\,\langle z\delta_{q_1}, z\delta_{q_2}\rangle$ we consider the GKZ ideal $\mathbb{G}' = \langle \Box_e, \Box_{h-e}\rangle$, where

$$\left\{ \begin{array}{ll} \Box_e & = (z\delta_{q_1})^n - q_1(z\delta_{q_2} - z\delta_{q_1}) \prod_{\nu=1}^{-b}(-bz\delta_{q_1} + (a + b)z\delta_{q_2} + \nu z), \\ \Box_{h-e} & = (z\delta_{q_2})(z\delta_{q_2} - z\delta_{q_1}) - q_2 \prod_{\nu=1}^{a+b}(-bz\delta_{q_1} + (a + b)z\delta_{q_2} + \nu z). \end{array} \right\} \tag{6.4}$$

The following proposition gives the generator of the quotient ideal.

Proposition 6.5. — *The quotient ideal of the GKZ ideal by $\widehat{c}_{top}$ is:*

$$(\mathbb{G}' : \widehat{c}_{top}) = \langle P_0, \Box_e, \Box_{h-e} \rangle$$

where

$$\begin{aligned} P_0 := & -a(z\delta_{q_1})^{n-1} + (a+b)(z\delta_{q_1})^{n-2}(z\delta_{q_2}) + \\ & - abq_1(z\delta_{q_2} - z\delta_{q_1}) \prod_{\nu=1}^{-b-1} (-b(z\delta_{q_1}) + (a+b)(z\delta_{q_2}) + \nu z) \\ & - (a+b)^2 q_2 (z\delta_{q_1})^{n-2} \prod_{\nu=1}^{a+b-1} (-b(z\delta_{q_1}) + (a+b)(z\delta_{q_2}) + \nu z) \end{aligned}$$

Proof of Proposition 6.5. — First, we have $\widehat{c}_{\mathrm{top}} P_0 \in \mathbb{G}'$ as

$$ab\Box_e + (a+b)^2 (z\delta_{q_1})^{n-2} \Box_{h-e} = \widehat{c}_{\mathrm{top}} P_0 \tag{6.6}$$

Let us prove by induction on the degree of the operator $P \in \mathbb{D}'$ that if $\widehat{c}_{\mathrm{top}} P \in \mathbb{G}'$ then $P \in \langle P_0, \Box_e, \Box_{h-e} \rangle$. From Theorem 6.1, we have

$$\widehat{c}_{\mathrm{top}} P = R_1 \Box_e + R_2 \Box_{h-e} \tag{6.7}$$

where the degree of the operators $R_1\Box_e$ and $R_2\Box_{h-e}$ are less or equal to $\deg(P) + 1$.

Taking the symbol of 6.7 we get

$$\sigma(\widehat{c}_{\mathrm{top}})\sigma(P) = S_1\sigma(\Box_e) + S_2\sigma(\Box_{h-e})$$

where S_i are either the symbol of R_i or 0. Replacing $\sigma(\Box_e)$ by Equality (6.6), we get the following equality in $\mathbb{Q}[q_1, q_2, y_1, y_2]$

$$\sigma(\widehat{c}_{\mathrm{top}})\sigma(P) = \sigma(\widehat{c}_{\mathrm{top}}) \frac{S_1\sigma(P_0)}{ab} + \sigma(\Box_{h-e}) \left(-\frac{(a+b)^2}{ab} S_1 y_1^{n-2} + S_2 \right) \tag{6.8}$$

From (6.4), we have

$$\Box_{h-e} = (z\delta_{q_2})(z\delta_{q_2} - z\delta_{q_1}) - \widehat{c}_{\mathrm{top}} q_2 \prod_{\nu=1}^{a+b-1} (-bz\delta_{q_1} + (a+b)z\delta_{q_2} + \nu z). \tag{6.9}$$

In $\mathbb{Q}[q_1, q_2, y_1, y_2]/\sigma(\widehat{c}_{\mathrm{top}})$, we get from (6.9)

$$0 = \overline{y}_2(\overline{y}_2 - \overline{y}_1) \left(-\frac{(a+b)^2}{ab} \overline{S}_1 \overline{y}_1^{n-2} + \overline{S}_2 \right)$$

As $\overline{y}_2(\overline{y}_2 - \overline{y}_1) \neq 0$, there exists $Q \in \mathbb{Q}[q_1 q_2, y_1, y_2]$ such that

$$-\frac{(a+b)^2}{ab} S_1 y_1^{n-2} + S_2 = Q\sigma(\widehat{c}_{\mathrm{top}})$$

By the degree conditions on $R_1\Box_e$ and $R_2\Box_{h-e}$, we have that $\deg Q = \deg P - 2$. Putting this in (6.8), we get

$$\sigma(\widehat{c}_{\mathrm{top}})\sigma(P) = \sigma(\widehat{c}_{\mathrm{top}}) \left(\frac{S_1\sigma(P_0)}{ab} + \sigma(\Box_{h-e}) Q \right) \tag{6.10}$$

As $\deg S_1\sigma(P_0) = \deg \sigma(\Box_{h-e})Q = \deg P$, we have

$$\sigma(P) = \sigma \left(-\frac{R_1}{a} P_0 - \frac{\widehat{Q}}{a} \Box_{h-e} \right)$$

where $\widehat{Q}$ is any operator having symbol Q. The operator

$$P' := P - \left(-\frac{R_1}{a} P_0 - \frac{\widehat{Q}}{a} \Box_{h-e} \right)$$

satisfies that $\widehat{c}_{\text{top}} P' \in \mathbb{G}'$ and has degree strictly inferior to the degree of P. By induction, we deduce that $P' \in \langle P_0, \Box_e, \Box_{h-e} \rangle$ that is P is in $\langle P_0, \Box_e, \Box_{h-e} \rangle$. □

A
Twisted Axioms for Gromov-Witten invariants

In this Appendix, we will state (without proof) the twisted axioms for twisted Gromov-Witten invariants. For the "untwisted" axioms, we refer to two papers of Behrend and Manin ([**BM96**] and [**Beh97**]). As explained in §.2.1.b, the twisted axioms are the non-equivariant limit of the equivariant twisted axioms.

Recall from Notation 2.1 and $T_0, \ldots, T_{s-1}$ be a basis of $H^{2*}(X)$. We denote by T^a the Poincaré dual of T_a for $a \in \{0, \ldots, s-1\}$. Let d be in $H_2(X, \mathbb{Z})$. Let $\gamma_1, \ldots, \gamma_\ell$ be in $H^{2*}(X)$, $m_1, \ldots, m_\ell$ be in $\mathbb{N}$, for any $\sigma \in S_\ell$ and j be in $\{1, \ldots, \ell\}$.

A.1. (Twisted S_ℓ-invariance)

$$\begin{aligned}
&\left\langle \tau_{m_1}(\widetilde{c_1(\mathcal{E})} \cup \gamma_1), \ldots, \tau_{m_\ell}(\gamma_\ell) \right\rangle_{0,\ell,d} \\
&= \left\langle \tau_{m_{\sigma(1)}}(\gamma_{\sigma(1)}), \ldots, \tau_{m_{\sigma(j)}}(\widetilde{c_1(\mathcal{E})} \cup \gamma_{\sigma(j)}), \ldots, \tau_{m_{\sigma(\ell)}}(\gamma_{\sigma(\ell)}) \right\rangle_{0,\ell,d}
\end{aligned}$$

A.2. (Twisted Fundamental class equation / string equation)

$$\begin{aligned}
&\left\langle \tau_{m_1}(\gamma_1), \ldots, \widetilde{\tau_{m_k}(\gamma_k)}, \ldots, \tau_{m_\ell}(\gamma_\ell), \mathbf{1} \right\rangle_{0,\ell+1,d} \\
&= \sum_{i | m_i > 0} \left\langle \tau_{m_1}(\gamma_1), \ldots, \tau_{m_i - 1}(\gamma_i), \ldots, \widetilde{\tau_{m_k}(\gamma_k)}, \ldots, \tau_{m_\ell}(\gamma_\ell) \right\rangle_{0,\ell,d}
\end{aligned}$$

A.3. (Consequence of the two above)

$$\begin{aligned}
&\left\langle \tau_{m_1}(\gamma_1), \ldots, \widetilde{\tau_{m_k}(\gamma_k)}, \ldots, \tau_{m_\ell}(\gamma_\ell), \mathbf{1} \right\rangle_{0,\ell+1,d} \\
&= \sum_{i | m_i > 0} \left\langle \tau_{m_1}(\gamma_1), \ldots, \tau_{m_i - 1}(\gamma_i), \ldots, \widetilde{\tau_{m_k}(\gamma_k)}, \ldots, \tau_{m_\ell}(\gamma_\ell) \right\rangle_{0,\ell,d}
\end{aligned}$$

A.4. (Twisted Divisor axiom)

$$\begin{aligned}
&\left\langle \tau_{m_1}(\gamma_1), \ldots, \widetilde{\tau_{m_k}(\gamma_k)}, \ldots, \tau_{m_\ell}(\gamma_\ell), \gamma \right\rangle_{0,\ell+1,d} \\
&= \left(\int_d \gamma \right) \langle \tau_{m_1}(\gamma_1), \ldots, \widetilde{\tau_{m_k}(\gamma_k)}, \ldots, \tau_{m_\ell}(\gamma_\ell) \rangle_{0,\ell,d} \\
&+ \sum_{i : m_i > 0} \left\langle \tau_{m_1}(\gamma_1), \ldots, \widetilde{\tau_{m_i - 1}(\gamma \cup \gamma_i)}, \ldots, \tau_{m_\ell}(\gamma_\ell) \right\rangle_{0,\ell,d}
\end{aligned}$$

A.5. (Twisted Dilaton equation)

$$\begin{aligned}
&\langle \tau_{m_1}(\gamma_1), \ldots, \widetilde{\tau_{m_j}(\gamma_j)}, \ldots, \tau_{m_\ell}(\gamma_\ell), \tau_1(\mathbf{1}) \rangle_{0,\ell+1,d} \\
&= (-2+n) \langle \tau_{m_1}(\gamma_1), \ldots, \widetilde{\tau_{m_j}(\gamma_j)}, \ldots, \tau_{m_\ell}(\gamma_\ell) \rangle_{0,\ell,d}
\end{aligned}$$

A.6. (Twisted TRR *i.e.*, Topological Recursion Relation)

$$\left\langle\!\left\langle \tau_{m_1+1}(\gamma_1), \tau_{m_2}(\gamma_2), \widetilde{\tau_{m_3}(\gamma_3)} \right\rangle\!\right\rangle_0 = \sum_{a=0}^{s-1} \left\langle\!\left\langle \tau_{m_2}(\gamma_2), \widetilde{\tau_{m_3}(\gamma_3)}, T^a \right\rangle\!\right\rangle_0 \left\langle\!\left\langle \tau_{m_1}(\gamma_1), \widetilde{T}_a \right\rangle\!\right\rangle_0$$

$$\left\langle\!\left\langle \widetilde{\tau_{m_1+1}(\gamma_1)}, \tau_{m_2}(\gamma_2), \tau_{m_3}(\gamma_3) \right\rangle\!\right\rangle_0 = \sum_{a=0}^{s-1} \left\langle\!\left\langle \tau_{m_2}(\gamma_2), \tau_{m_3}(\gamma_3), \widetilde{T}^a \right\rangle\!\right\rangle_0 \left\langle\!\left\langle \widetilde{\tau_{m_1}(\gamma_1)}, T_a \right\rangle\!\right\rangle_0$$

using the notation

$$\text{(A.1)} \qquad \left\langle\!\left\langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_\ell}(\gamma_\ell) \right\rangle\!\right\rangle_0 := \sum_{\ell \geq 0} \sum_{d \in H_2(X,\mathbb{Z})} \frac{1}{\ell!} \left\langle \tau_{m_1}(\gamma_1), \dots, \tau_{m_\ell}(\gamma_\ell), \tau, \dots, \tau \right\rangle_{0,\ell+n,d}$$

A.7. (Twisted WDVV equations)

$$\sum_{a=0}^{s-1} \left\langle\!\left\langle \tau_{m_1}(\gamma_1), \tau_{m_2}(\gamma_2), \widetilde{T}_a \right\rangle\!\right\rangle_0 \left\langle\!\left\langle \tau_{m_3}(\gamma_3), \widetilde{\tau_{m_4}(\gamma_4)}, T^a \right\rangle\!\right\rangle_0$$
$$= \sum_{a=0}^{s-1} \left\langle\!\left\langle \tau_{m_1}(\gamma_1), \tau_{m_3}(\gamma_3), \widetilde{T}_a \right\rangle\!\right\rangle_0 \left\langle\!\left\langle \tau_{m_2}(\gamma_2), \widetilde{\tau_{m_4}(\gamma_4)}, T^a \right\rangle\!\right\rangle_0$$

$$\sum_{a=0}^{s-1} \left\langle\!\left\langle \tau_{m_1}(\gamma_1), \tau_{m_2}(\gamma_2), T_a \right\rangle\!\right\rangle_0 \left\langle\!\left\langle \tau_{m_3}(\gamma_3), \widetilde{\tau_{m_4}(\gamma_4)}, \widetilde{T}^a \right\rangle\!\right\rangle_0$$
$$= \sum_{a=0}^{s-1} \left\langle\!\left\langle \tau_{m_1}(\gamma_1), \tau_{m_3}(\gamma_3), T_a \right\rangle\!\right\rangle_0 \left\langle\!\left\langle \tau_{m_2}(\gamma_2), \widetilde{\tau_{m_4}(\gamma_4)}, \widetilde{T}^a \right\rangle\!\right\rangle_0$$

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