

ORBIFOLD QUANTUM COHOMOLOGY OF WEIGHTED PROJECTIVE SPACES

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Abstract

In this article, we prove the following results.

- We show a mirror theorem: the Frobenius manifold associated to the orbifold quantum cohomology of weighted projective spaces is isomorphic to the one attached to a specific Laurent polynomial.
- We show a reconstruction theorem; that is, we can reconstruct in an algorithmic way the full genus 0 Gromov-Witten potential from the 3-point invariants.

1. Introduction

Motivated by the works of physicists E. Witten, R. Dijkgraaf, E. Verlinde and H. Verlinde, B. Dubrovin defined in [Dub96] a Frobenius structure on a complex manifold. Frobenius manifolds are complex manifolds endowed with a flat metric and a product on the tangent bundle which satisfies some compatibility conditions.

In 2001, S. Barannikov showed in [Bar00] that the Frobenius manifold coming from the quantum cohomology of the complex projective space of dimension n is isomorphic to the Frobenius manifold associated to the Laurent polynomial $x_1 + \cdots + x_n + 1/x_1 \cdots x_n$.

The goal of this article is to generalize this correspondence to weighted projective spaces. For this purpose we use the theory of orbifolds and the related constructions. In [CR02] and [CR04], W. Chen and Y. Ruan define the orbifold cohomology ring via the orbifold Gromov-Witten invariants. The orbifold cup product is defined as the degree zero part of the orbifold quantum product, and one computes it via the Euler class of an obstruction bundle. The orbifold quantum product is defined by the Gromov-Witten potential. So, as for manifolds, the orbifold quantum cohomology is naturally endowed with a Frobenius structure.

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On the other side, A. Douai and C. Sabbah (*cf.* [DS03]) explained how to build a canonical Frobenius manifold on the base space of a universal unfolding for any Laurent polynomial which is convenient and nondegenerate with respect to its Newton polyhedron. In particular, in [DS04], the authors described this construction explicitly for the polynomial $w_0u_0 + \cdots + w_nu_n$ restricted to $U := \{(u_0, \dots, u_n) \in \mathbb{C}^{n+1} \mid \prod_i u_i^{w_i} = 1\}$ where $w_0, \dots, w_n$ are positive integers that are relatively prime.

In this article, we compare the Frobenius structures, whose existence is provided by the general results recalled above, on the orbifold quantum cohomology of the weighted projective space $\mathbb{P}(w_0, \dots, w_n)$ (A side) and the one attached to the Laurent polynomial $f(u_0, \dots, u_n) := u_0 + \cdots + u_n$ restricted to U (B side).

First we show a correspondence between “classical limits”. To state this result, we need to introduce some notation. For the A side, we denote by $H_{\text{orb}}^{2*}(\mathbb{P}(w_0, \dots, w_n), \mathbb{C})$ the orbifold cohomology of $\mathbb{P}(w_0, \dots, w_n)$, $\cup$ the orbifold cup product and $\langle \cdot, \cdot \rangle$ the orbifold Poincaré duality. For the B side, we consider the vector space $\Omega^n(U)/df \wedge \Omega^{n-1}(U)$ where $\Omega^n(U)$ is the space of algebraic n -forms on U . It is naturally endowed with an increasing filtration, called the Newton filtration and denoted by $\mathcal{N}_\bullet$, and a nondegenerate bilinear form. The choice of a volume form on U gives us an identification of this vector space with the Jacobian ring of f . Hence, we get a product on this vector space. As the product and the nondegenerate bilinear form respect the filtration $\mathcal{N}_\bullet$, we have a product, denoted by $\cup$, and a nondegenerate bilinear form, denoted by $\llbracket g \rrbracket(\cdot, \cdot)$, on the graded space of $\Omega^n(U)/df \wedge \Omega^{n-1}(U)$ with respect to the Newton filtration. The following theorem is shown in Section 6.b.

Theorem 1.1 (Classical correspondence). *We have an isomorphism of graded Frobenius algebras between*

$$(H_{\text{orb}}^{2*}(\mathbb{P}(w), \mathbb{C}), \cup, \langle \cdot, \cdot \rangle) \quad \text{and} \quad (\text{gr}_*^{\mathcal{N}}(\Omega^n(U)/df \wedge \Omega^{n-1}(U)), \cup, \llbracket g \rrbracket(\cdot, \cdot)).$$

Note that, in a more general and algebraic context, Borisov, Chen and Smith [BCS05] computed the orbifold cohomology ring for toric Deligne-Mumford stacks. We will not use these results because, firstly we will use the techniques developed by Chen and Ruan, and secondly the author did not find in the literature a complete and explicit description of weighted projective spaces as toric Deligne-Mumford stacks.

Afterward, using [CCLT06], we prove two propositions¹ (*cf.* Propositions 4.14 and 4.17) on the value of some orbifold Gromov-Witten invariants with 3 marked points, and we show in Section 6.c that these propositions imply an isomorphism between the Frobenius manifolds coming from the A side and

¹In a previous version of this article, these propositions were conjectures.

from the B side. Let us note that Theorem 5.13 shows that we can reconstruct, in an algorithmic way, the full genus 0 Gromov-Witten invariants from the 3-point invariants. This result is similar to the first reconstruction theorem of M. Kontsevich and Y. Manin in [KM94, Theorem 3.1].

The article is organized as follows. The first section is devoted to Frobenius manifolds. In the second section, we compute the orbifold cohomology ring of weighted projective spaces. In the third section, we compute the value of some specific Gromov-Witten invariants. In the fourth part, we briefly recall the results for the Laurent polynomial $f : U \rightarrow \mathbb{C}$. In the last section, we give the proofs of the two correspondences: the “classical correspondence” and the isomorphism between the two Frobenius manifolds.

2. Background on Frobenius manifolds

Let M be a complex manifold endowed with

- a perfect pairing $g : TM \times TM \rightarrow \mathbb{C}$,
- an associative and commutative product $\star$ on the complex tangent bundle TM with unit e ,
- a vector field $\mathfrak{E}$, called the Euler vector field.

These data $(M, \star, e, g, \mathfrak{E})$ define a Frobenius structure on M if they satisfy some compatibility conditions. We will not write them because we will not use them explicitly. The reader can find these conditions in Lecture 1 of [Dub96] (see also [Man99, p.19], [Her02, p.146], [Sab02, p.240]). Assume that M is simply connected. Let $(t_0, \dots, t_n)$ be a system of flat coordinates on M . According to Lemma 1.2 in Lecture 1 of [Dub96] (see also Section VII.2.b in [Sab02]), there exists a holomorphic function, called a potential, $F : M \rightarrow \mathbb{C}$ such that for any i, j, k in $\{1, \dots, n\}$, we have

$$\frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} = g(\partial_{t_i} \star \partial_{t_j}, \partial_{t_k}).$$

The potential is determined up to a polynomial of degree 2. As the product $\star$ is associative, the potential satisfies the WDVV equations. We have the following theorem.

Theorem 2.1 ([Dub96, lecture 3]; see also [Sab02, p. 250] or more generally Theorem 4.5 in [HM04]). *Let $g^\circ : \mathbb{C}^\mu \times \mathbb{C}^\mu \rightarrow \mathbb{C}$ be a perfect pairing. Let A_0° be a semi-simple and regular matrix of size $\mu \times \mu$ such that $(A_0^\circ)^* = A_0^\circ$. Let A_∞ be a matrix of size $\mu \times \mu$ such that $A_\infty + A_\infty^* = k \cdot \text{id}$ with $k \in \mathbb{Z}$. Let e° be an eigenvector of A_∞ for the eigenvalue q such that $(e^\circ, A_0^\circ e^\circ, \dots, A_0^{\circ\mu-1} e^\circ)$ is a basis of $\mathbb{C}^\mu$. The data $(A_0^\circ, A_\infty, g^\circ, e^\circ)$ determined a unique germ of a*

Frobenius manifold $((M, 0), \star, e, g, \mathfrak{E})$ such that via the isomorphism between T_0M and $\mathbb{C}^\mu$ we have $g^\circ = g(0)$, $A_0^\circ = \mathfrak{E}\star$, $A_\infty = (q+1)\text{id} - \nabla\mathfrak{E}$ and $e^\circ = e(0)$.

In order to show an isomorphism between the Frobenius manifold coming from $\mathbb{P}(w)$ and the one associated to the Laurent polynomial f , we will show that their initial conditions satisfy the hypotheses of the theorem above and that they are equal.

2.a. The A side. We construct the Frobenius manifold on the complex vector space $H_{\text{orb}}^*(\mathbb{P}(w_0, \dots, w_n), \mathbb{C})$ of dimension $\mu := w_0 + \dots + w_n$. The perfect pairing is the orbifold Poincaré duality, denoted by $\langle \cdot, \cdot \rangle$. In Section 3.b, we will define a basis $(\eta_0, \dots, \eta_{\mu-1})$ of the vector space $H_{\text{orb}}^*(\mathbb{P}(w_0, \dots, w_n), \mathbb{C})$. Denote by $(t_0, \dots, t_{\mu-1})$ the coordinates $H_{\text{orb}}^*(\mathbb{P}(w_0, \dots, w_n), \mathbb{C})$ in this basis. The Euler field is defined by the following formula:

$$\mathfrak{E} := \mu \partial_{t_1} + \sum_{i=0}^{\mu-1} (1 - \deg(\eta_i)/2) t_i \partial_{t_i}.$$

The big quantum product, denoted by $\star$, is defined with the full Gromov-Witten potential of genus 0, denoted by F^{GW} , by the following formula:

$$\frac{\partial^3 F^{GW}(t_0, \dots, t_{\mu-1})}{\partial t_i \partial t_j \partial t_k} = \langle \partial t_i \star \partial t_j, \partial t_k \rangle.$$

The initial conditions of the Frobenius manifold are the data $(A_0^\circ, A_\infty, \langle \cdot, \cdot \rangle, \eta_0)$ where $A_0^\circ := \mathfrak{E}\star|_{t=0}$ and $A_\infty := \text{id} - \nabla\mathfrak{E}$.

The matrix A_∞ is easy to compute (see Proposition 3.25), but in order to compute the matrix A_0° , we have to compute the orbifold cup product (cf. Section 3.c) and some specific Gromov-Witten invariants with 3 marked points (cf. Section 4.b). Via the correspondence, Theorem 5.13 implies that we can reconstruct the big quantum cohomology from the small one. In particular, the proof of Theorem 5.13 gives an algorithm to do so.

2.b. The B side. Let $U := \{(u_0, \dots, u_n) \in \mathbb{C}^{n+1} \mid \prod_i u_i^{w_i} = 1\}$. In the article [DS04], the polynomial is $w_0 u_0 + \dots + w_n u_n$ restricted to U and the weights are relatively prime. In our case, we consider general weights and the polynomial f is $u_0 + \dots + u_n$ restricted to U . Nevertheless, we will use the same techniques to show the following theorem.

Theorem 2.2 (see Theorem 5.3). *There exists a canonical Frobenius structure on any germ of universal unfolding of the Laurent polynomial $f(u_0, \dots, u_n) = u_0 + \dots + u_n$ restricted to U .*

3. Orbifold cohomology ring of weighted projective spaces

In this section, we will describe explicitly the orbifold cohomology ring of weighted projective spaces.

In the first part, we define the orbifold structure that we will consider on weighted projective spaces. In the second part, we give a natural $\mathbb{C}$ -basis of the orbifold cohomology. Then we compute the orbifold Poincaré duality in this basis. In the last part, we compute the orbifold cup product and we express it in the basis defined in the second part. The obstruction bundle is computed in Theorem 3.17.

In this article, we will use the following notation. Let n and $w_0, \dots, w_n$ be some integers greater than or equal to one.

3.a. Orbifold structure on weighted projective spaces. In this part, we describe the weighted projective spaces as Deligne-Mumford stacks in Section 3.a.1 and as orbifolds in Section 3.a.2.

3.a.1. Weighted projective spaces as Deligne-Mumford stacks.

We define the action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^{n+1} - \{0\}$ by $\lambda \cdot (y_0, \dots, y_n) := (\lambda^{w_0} y_0, \dots, \lambda^{w_n} y_n)$. We denote by $\mathbb{P}(w)$ the quotient stack $[\mathbb{C}^{n+1} - \{0\} / \mathbb{C}^*]$. This stack is a smooth proper Deligne-Mumford stack.

For any subset $I := \{i_1, \dots, i_k\} \subset \{0, \dots, n\}$, we denote $w_I := (w_{i_1}, \dots, w_{i_k})$. We have a closed embedding $\iota_I : \mathbb{P}(w_I) := \mathbb{P}(w_{i_1}, \dots, w_{i_k}) \rightarrow \mathbb{P}(w)$. We denote by $\mathbb{P}(w)_I$ the image of this stack morphism. In the following, we will identify $\mathbb{P}(w_I)$ with $\mathbb{P}(w)_I$.

Let us define the invertible sheaf $\mathcal{O}_{\mathbb{P}(w)}(1)$ on $\mathbb{P}(w)$. For any scheme X and for any stack morphism $X \rightarrow \mathbb{P}(w)$ given by a principal $\mathbb{C}^*$ -bundle $P \rightarrow X$ and a $\mathbb{C}^*$ -equivariant morphism $P \rightarrow \mathbb{C}^{n+1} - \{0\}$, we denote by $\mathcal{O}_{\mathbb{P}(w)}(1)_X$ the sheaf of sections of the associated line bundle of P .

Let us consider the following map:

$$\begin{aligned} \tilde{f}_w : \mathbb{C}^{n+1} - \{0\} &\rightarrow \mathbb{C}^{n+1} - \{0\} \\ (z_0, \dots, z_n) &\mapsto (z_0^{w_0}, \dots, z_n^{w_n}). \end{aligned}$$

If we consider the standard action (*i.e.* with weights 1) on the source and the action with weights on the target, then the map $\tilde{f}_w$ is $\mathbb{C}^*$ -equivariant. This induces a stack morphism $f_w : \mathbb{P}^n \rightarrow \mathbb{P}(w)$. By remark (12.5.1) of [LMB00], the invertible sheaf $f_w^* \mathcal{O}_{\mathbb{P}(w)}(1)$ is the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$.

3.a.2. Weighted projective spaces as orbifolds. In this part, we are using the language of orbifold used by Satake [Sat56] and by Chen and Ruan [CR02]. In this setting, the author didn't find in the literature a complete reference for the orbifold structure on weighted projective spaces. The purpose

of this part is to fix it in this language: namely, we use the notion of good map, which is defined in [CR04].

First, we recall some general definitions about orbifold charts. Let U be a connected topological space. A *chart* of U is a triple $(\tilde{U}, G, \pi)$ where $\tilde{U}$ is a connected open set of $\mathbb{C}^n$, G is a *finite* commutative² group which acts holomorphically on $\tilde{U}$ and π is a map from $\tilde{U}$ on U such that π induces a homeomorphism between $\tilde{U}/G$ and U . We denote by $\text{Ker}(G)$ the subgroup of G that acts trivially on $\tilde{U}$. When we will not need to specify the group or the projection, we will write $\tilde{U}$ for a chart of U .

Let U be a connected open set of U' . Let $(\tilde{U}', G', \pi')$ be a chart of U' . A chart $(\tilde{U}, G, \pi)$ of U is *induced* by $(\tilde{U}', G', \pi')$ if there exists a monomorphism of groups $\kappa : G \rightarrow G'$ and an open κ -equivariant embedding α from $\tilde{U}$ to $\tilde{U}'$ such that κ induces an isomorphism between $\text{Ker}(G)$ and $\text{Ker}(G')$ and $\pi' = \alpha \circ \pi$. In [Sat57], Satake calls such a pair $(\alpha, \kappa) : (\tilde{U}, G, \pi) \hookrightarrow (\tilde{U}', G', \pi')$ an *injection* of charts.

We define the action of the multiplicative group $\mathbb{C}^\star$ on $\mathbb{C}^{n+1} - \{0\}$ by $\lambda \cdot (y_0, \dots, y_n) := (\lambda^{w_0} y_0, \dots, \lambda^{w_n} y_n)$. The *weighted projective space* is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by this action. Denote by $|\mathbb{P}(w)|$ this topological space and $\pi_w : \mathbb{C}^{n+1} - \{0\} \rightarrow |\mathbb{P}(w)|$ the quotient map. Denote by $[y_0 : \dots : y_n]$ the class of $\pi_w(y_0, \dots, y_n)$ in $|\mathbb{P}(w)|$. We have the following commutative diagram:

$$\begin{array}{ccccc} (z_0, \dots, z_n) & \mathbb{C}^{n+1} - \{0\} & \xrightarrow{\pi} & \mathbb{P}^n & [z_0 : \dots : z_n] \\ \downarrow & \downarrow \tilde{f}_w & & \downarrow f_w & \downarrow \\ (z_0^{w_0}, \dots, z_n^{w_n}) & \mathbb{C}^{n+1} - \{0\} & \xrightarrow{\pi_w} & |\mathbb{P}(w)| & [z_0^{w_0} : \dots : z_n^{w_n}] \end{array}$$

where π is the standard quotient map for complex projective spaces. Denote by μ_k the group of k -th roots of unity. We can endow $|\mathbb{P}(w)|$ with two different orbifold structures. In the algebraic settings, we say that the Deligne-Mumford stacks $\mathbb{P}(w)$ and $[\mathbb{P}^n / \mu_{w_0} \times \dots \times \mu_{w_n}]$ have the same coarse moduli space $|\mathbb{P}(w)|$.

- (i) The group $\mu_{w_0} \times \dots \times \mu_{w_n}$ acts on $\mathbb{P}^n$ in the following way:

$$\begin{aligned} \mu_{w_0} \times \dots \times \mu_{w_n} \times \mathbb{P}^n &\longrightarrow \mathbb{P}^n \\ ((\lambda_0, \dots, \lambda_n), [z_0 : \dots : z_n]) &\longmapsto [\lambda_0 z_0 : \dots : \lambda_n z_n]. \end{aligned}$$

The map $f_w : \mathbb{P}^n \rightarrow |\mathbb{P}(w)|$ induces a homeomorphism between $\mathbb{P}^n / \mu_{w_0} \times \dots \times \mu_{w_n}$ and $|\mathbb{P}(w)|$. So, the topological space $|\mathbb{P}(w)|$ is endowed with an orbifold structure.

²In the general case, one doesn't suppose that the groups are commutative (see [CR02]). Nevertheless, here we consider only examples where the groups are commutative.

- (ii) The topological space $|\mathbb{P}(w)|$ can also be endowed with an orbifold structure, which is not global, via the map π_w . The orbifold atlas which defines this structure is described below.

In this article, we will study only the orbifold structure that comes from (ii).³ For $i \in \{0, \dots, n\}$, let $U_i := \{[y_0 : \dots : y_n] \mid y_i \neq 0\} \subset |\mathbb{P}(w)|$. Let $\tilde{U}_i$ be the set of points of $\mathbb{C}^{n+1} - \{0\}$ such that $y_i = 1$. The subgroup of $\mathbb{C}^\star$ that stabilizes $\tilde{U}_i$ is μ_{w_i} . The map $\pi_i := \pi_w|_{\tilde{U}_i} : \tilde{U}_i \longrightarrow U_i$ induces a homeomorphism between $\tilde{U}_i/\mu_{w_i}$ and U_i .

Let U be a connected open set of $|\mathbb{P}(w)|$. A chart $(\tilde{U}, G_{\tilde{U}}, \pi_{\tilde{U}})$ of U is called *admissible* if there exists $i \in \{0, \dots, n\}$ such that $\tilde{U} \subset \tilde{U}_i$ is a connected component of $\pi_i^{-1}(U)$, $G_{\tilde{U}}$ is the subgroup of μ_{w_i} that stabilizes $\tilde{U}$ and $\pi_{\tilde{U}} = \pi_i|_{\tilde{U}}$. In particular, the charts $(\tilde{U}_i, \mu_{w_i}, \pi_i)$ of U_i are admissible charts. Denote by $\mathcal{A}(|\mathbb{P}(w)|)$ the set of all admissible charts. The set of charts of $\mathcal{A}(|\mathbb{P}(w)|)$ induces a cover, denoted by $\mathcal{U}_w$, of $|\mathbb{P}(w)|$.

Proposition 3.1. *The set $\mathcal{A}(|\mathbb{P}(w)|)$ is an orbifold atlas.*

We will denote by $\mathbb{P}(w)$ the orbifold $(|\mathbb{P}(w)|, \mathcal{A}(|\mathbb{P}(w)|))$.

Proof of Proposition 3.1. According to [MP97], we have to prove that the cover $\mathcal{U}_w$ satisfies the following conditions:

- (1) each open set U of the cover $\mathcal{U}_w$ has a chart $(\tilde{U}, G_U, \pi_U)$,
- (2) for any p in $U \cap V$, there exists $W \subset U \cap V$ which contains p and two injections of charts $\tilde{W} \hookrightarrow \tilde{U}, \tilde{W} \hookrightarrow \tilde{V}$.

The first point is clear. Let $(\tilde{U}, G_U, \pi_U)$ be a chart of U and $(\tilde{V}, G_V, \pi_V)$ be a chart of V in $\mathcal{A}(|\mathbb{P}(w)|)$. Let p be a point in $U \cap V$. By definition of $\mathcal{A}(|\mathbb{P}(w)|)$ there exists a unique pair $(i, j) \in \{0, \dots, n\}$ such that $\tilde{U} \subset \tilde{U}_i$ and $\tilde{V} \subset \tilde{U}_j$. We can find a chart $(\tilde{U}_p, G_p, \pi_p)$ of a small neighborhood U_p of p such that $\tilde{U}_p \subset \tilde{U} \subset \tilde{U}_i$ and the map

$$\begin{aligned} \psi_{ij} : \tilde{U}_p &\longrightarrow \tilde{V} \subset \tilde{U}_j \\ (y_0, \dots, 1_i, \dots, y_n) &\longmapsto (y_0/y_j^{w_0/w_j}, \dots, 1_j, \dots, y_n/y_j^{w_n/w_j}) \end{aligned}$$

where y_j^{1/w_j} is a w_j -th root of y_j , is an injection of charts. For more details about the existence of such a chart, see Proposition IV.1.10 of [Man05]. $\square$

Remark 3.2. On an orbifold, one can define a group which acts globally and trivially (cf. Part 4.1 of [CR02] or Lemma 3.1.10 of [Man05]). For $\mathbb{P}(w)$, it is easy to see that this group is $\mu_{\gcd(w)}$.

³The author has found in the literature a mixing between these two orbifold structures on the same topological space $|\mathbb{P}(w)|$. This is one of the motivations for making explicit the orbifold structure that comes from (ii).

Between two orbifolds, one can define orbifold maps (see Paragraph 4.1 in [CR02]). But then one has some problems when you want to pull back bundles. So, one defines a more restrictive map, which is called *good* (see Section 4.4 of [CR02]), that allows one to pull back bundles.

Proposition 3.3. *Let $I := \{i_1, \dots, i_k\} \subset \{0, \dots, n\}$. The inclusion map*

$$\begin{aligned} \iota_I : \mathbb{P}(w_I) &\longrightarrow \mathbb{P}(w) \\ [z_1 : \dots : z_\delta] &\longmapsto [0 : \dots : 0 : z_{i_1} : 0 : \dots : 0 : z_{i_\delta} : 0 : \dots : 0] \end{aligned}$$

is a good orbifold map.

Proof. We will prove this proposition for the set $I = \{0, \dots, \delta\}$. First we use Section 4.1 of [CR02] to construct a *compatible cover* (cf. Section 4.1 of [CR02]), denoted by $\mathcal{U}_I$, associated to the atlas $\mathcal{A}(|\mathbb{P}(w_I)|)$. To have a good map, we need a correspondence between open sets and injections of charts which satisfies some conditions. We denote this correspondence by $\mathfrak{F}$. For any open set U_I of $\mathcal{U}_I$, we put $\mathfrak{F}(U_I) := \{[y_0 : \dots : y_n] \mid [y_0 : \dots : y_n] \in U_I\}$. For any injection $(\alpha, \kappa) : \tilde{U}_I \hookrightarrow \tilde{V}_I$, we put $\mathfrak{F}(\alpha, \kappa) := (\alpha, \text{id}) : \mathfrak{F}(U_I) \hookrightarrow \mathfrak{F}(V_I)$. It is straightforward to check that these data satisfy the conditions for being a good map (see Proposition IV.1.15 of [Man05]). $\square$

For any subset I of $\{0, \dots, n\}$, we define the topological space $|\mathbb{P}(w)_I| := \iota_I(|\mathbb{P}(w_I)|)$. The orbifold atlas of $|\mathbb{P}(w)|$ induces a natural orbifold atlas (cf. Remark IV.1.11.(5) of [Man05]) which endows $|\mathbb{P}(w)_I|$ with an orbifold structure denoted by $\mathbb{P}(w)_I$. The orbifold map $\iota_I : \mathbb{P}(w_I) \rightarrow \mathbb{P}(w)$ induces an isomorphism between $\mathbb{P}(w_I)$ and $\mathbb{P}(w)_I$. In the following, we will identify the orbifolds $\mathbb{P}(w_I) \hookrightarrow \mathbb{P}(w)$ and $\mathbb{P}(w)_I \subset \mathbb{P}(w)$.

Proposition 3.4. *The map*

$$\begin{aligned} f_w : \mathbb{P}^n &\longrightarrow \mathbb{P}(w) \\ [z_0 : \dots : z_n] &\longmapsto [z_0^{w_0} : \dots : z_n^{w_n}] \end{aligned}$$

is a good orbifold map.

Remark 3.5. The degree of f_w , regarded as a map between topological spaces, is $\prod w_i / \gcd(w_0, \dots, w_n)$.

Proof of Proposition 3.4. We will just prove this proposition when the weights are relatively prime. We refer to Proposition IV.1.18 of [Man05] for the general case. Recall that the map

$$\begin{aligned} \tilde{f}_w : \mathbb{C}^{n+1} - \{0\} &\longrightarrow \mathbb{C}^{n+1} - \{0\} \\ (z_0, \dots, z_n) &\longmapsto (z_0^{w_0}, \dots, z_n^{w_n}) \end{aligned}$$

is $\mathbb{C}^*$ -equivariant and it lifts the map $|f_w| : |\mathbb{P}^n| \rightarrow |\mathbb{P}(w)|$. The map f_w is surjective and open. So, the map f_w is a regular orbifold map; that is,

$f_w^{-1}(|\mathbb{P}(w)_{\text{reg}}|)$ is open, connected and dense, where $|\mathbb{P}(w)_{\text{reg}}|$ is $\{p \in |\mathbb{P}(w)| \mid G_p = \mu_{\text{gcd}(w)}\}$. Then, Lemma 4.4.11 of [CR02] shows that the map f_w is good. $\square$

Proposition 3.6. *There exists an orbibundle of rank 1, denoted by $\mathcal{O}_{\mathbb{P}(w)}(1)$, over $\mathbb{P}(w)$ such that $f_w^* \mathcal{O}_{\mathbb{P}(w)}(1)$ is isomorphic to the bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ over $\mathbb{P}^n$.*

Remark 3.7. For any subset I of $\{0, \dots, n\}$, the orbibundle $\iota_I^* \mathcal{O}_{\mathbb{P}(w)}(1)$ is isomorphic to the orbibundle $\mathcal{O}_{\mathbb{P}(w_I)}(1)$.

Proof of Proposition 3.6. We will define the orbibundle $\mathcal{O}_{\mathbb{P}(w)}(1)$ on $\mathbb{P}(w)$ by its transition maps. Let $\alpha : \tilde{U} \hookrightarrow \tilde{V}$ be an injection between two charts of $\mathcal{A}(|\mathbb{P}(w)|)$. By definition of $\mathcal{A}(|\mathbb{P}(w)|)$, there exists a unique pair $(i, j) \in \{0, \dots, n\}$ such that $\tilde{U} \subset \tilde{U}_i$ and $\tilde{V} \subset \tilde{U}_j$. For any $y \in \tilde{U}$ and any $t \in \mathbb{C}$, we put

$$(3.8) \quad \psi_{\alpha}^{\mathcal{O}_{\mathbb{P}(w)}(1)}(y)(t) = \begin{cases} \zeta t & \text{if } i = j, \\ t/y_j^{1/w_j} & \text{if } i \neq j, \end{cases}$$

where $\zeta \in \mu_{w_i}$. The cocycle condition is easy to check. So, we have defined an orbibundle of rank 1, denoted by $\mathcal{O}_{\mathbb{P}(w)}(1)$, on $\mathbb{P}(w)$.

The map f_w is a good map (see Proposition 3.4), so we can define the pull-back bundle $f_w^* \mathcal{O}_{\mathbb{P}(w)}(1)$ over $\mathbb{P}^n$. A careful computation shows that the bundles $f_w^* \mathcal{O}_{\mathbb{P}(w)}(1) \rightarrow \mathbb{P}^n$ and $\mathcal{O}_{\mathbb{P}^n}(1) \rightarrow \mathbb{P}^n$ have the same transition maps. $\square$

3.b. Orbifold cohomology of weighted projective spaces as $\mathbb{C}$ -vector space. We refer to Definition 3.2.3 of [CR04] for the definition of orbifold cohomology.

For any $g \in \bigcup \mu_{w_i}$, there exists a unique $\gamma(g)$ in $[0, 1[$ such that $g = \exp(2i\pi\gamma(g))$. When there will be no confusion, we will simply write γ instead of $\gamma(g)$.

Proposition 3.9. *For any $g \in \bigcup \mu_{w_i}$, we put $\text{age}(g) := \{\gamma w_0\} + \dots + \{\gamma w_n\}$ and $I(g) := \{i \mid g \in \mu_{w_i}\}$ where $\{\cdot\}$ means the fractional part. The graded $\mathbb{C}$ -vector space structure of the orbifold cohomology of $\mathbb{P}(w)$ is given by the following:*

$$\begin{aligned} H_{\text{orb}}^{2\star}(\mathbb{P}(w), \mathbb{C}) &= \bigoplus_{g \in \bigcup \mu_{w_i}} H^{2(\star - \text{age}(g))}(|\mathbb{P}(w)_{(g)}|, \mathbb{C}) \\ &\simeq \bigoplus_{g \in \bigcup \mu_{w_i}} H^{2(\star - \text{age}(g))}(|\mathbb{P}(w_{I(g)})|, \mathbb{C}). \end{aligned}$$

Remark 3.10. In [Kaw73], T. Kawasaki shows the following results:

$$H^{2i}(|\mathbb{P}(w)|, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } i \in \{0, \dots, n\}; \\ 0 & \text{otherwise.} \end{cases}$$

Then, according to the proposition above, we have an explicit description of the $\mathbb{C}$ -vector space $H_{\text{orb}}^{2\star}(\mathbb{P}(w), \mathbb{C})$.

Proof of Proposition 3.9. Let p be in $|\mathbb{P}(w)|$. Let $(\tilde{U}_p, G_p, \pi_p)$ be a chart of a neighborhood of U_p of p . Denote by $\tilde{p}$ the lift of p in $\tilde{U}_p$. The action of G_p on the tangent space $T_{\tilde{p}}\tilde{U}_p$ induces the following representation of the group G_p :

$$\begin{aligned} G_p &\longrightarrow GL(n, \mathbb{C}) \\ g = \exp(2i\pi\gamma) &\longmapsto \text{diag}(e^{2i\pi\gamma w_0}, \dots, e^{2i\pi\gamma w_n}). \end{aligned}$$

According to Part 3.2 of [CR04], the *age* of g is $\{\gamma w_0\} + \dots + \{\gamma w_n\}$. We deduce the equality in the proposition.

For the second part, we note that the twisted sector $|\mathbb{P}(w)_{(g)}|$ is $|\mathbb{P}(w)_{I(g)}|$, which is identified with $|\mathbb{P}(w_{I(g)})|$. $\square$

For any g in $\bigcup \mu_{w_i}$, put

$$\eta_g^d := c_1(\mathcal{O}_{\mathbb{P}(w)_{(g)}}(1))^d \in H^{2d}(|\mathbb{P}(w)_{(g)}|, \mathbb{C}).$$

Remark that η_g^d vanishes for $d > \dim_{\mathbb{C}} \mathbb{P}(w)_{(g)}$. We deduce the following corollary.

Corollary 3.11.

- (1) *The dimension of the $\mathbb{C}$ -vector space $H_{\text{orb}}^{2\star}(\mathbb{P}(w), \mathbb{C})$ is $\mu := w_0 + \dots + w_n$.*
- (2) *The set $\boldsymbol{\eta} := \{\eta_g^d \mid g \in \bigcup \mu_{w_i}, d \in \{0, \dots, \dim_{\mathbb{C}} \mathbb{P}(w)_{(g)}\}\}$ is a basis of the $\mathbb{C}$ -vector space $H_{\text{orb}}^{\star}(\mathbb{P}(w), \mathbb{C})$. The orbifold degree of η_g^d is $2(d + \text{age}(g))$.*

We refer to Formula (2.4) of [CR04] for the definition of the orbifold integral (see also Formula (III.3.3) of [Man05]).

Proposition 3.12. *We have the following equality:*

$$\int_{\mathbb{P}(w)}^{\text{orb}} \eta_1^n = \prod_{i=0}^n w_i^{-1}.$$

Proof. We denote $|\mathbb{P}(w)_{\text{reg}}| := \{p \in |\mathbb{P}(w)| \mid G_p = \mu_{\text{gcd}(w)}\}$. Let us note that $|\mathbb{P}(w)_{\text{reg}}|$ is open and dense in $|\mathbb{P}(w)|$. By definition of the orbifold integral

and Proposition 3.6, we have

$$\int_{\mathbb{P}(w)}^{\text{orb}} \eta_1^n = \frac{1}{\gcd(w)} \int_{|\mathbb{P}(w)_{\text{reg}}|} c_1(\mathcal{O}_{\mathbb{P}(w)}(1))^n = \frac{1}{\gcd(w) \deg(f_w)} \int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^n.$$

Then Remark 3.5 implies the proposition. $\square$

According to Section 3.3 of [CR04], we can define an orbifold Poincaré duality, denoted by $\langle \cdot, \cdot \rangle$, on orbifold cohomology. The proposition below is a straightforward consequence of the definition and of Proposition 3.12.

Proposition 3.13. *Let η_g^d and $\eta_{g'}^{d'}$ be two elements of the basis $\boldsymbol{\eta}$.*

- (1) *If $g' \neq g^{-1}$, then we have $\langle \eta_g^d, \eta_{g'}^{d'} \rangle = 0$.*
- (2) *If $g' = g^{-1}$, then we have $I(g) = I(g')$ and*

$$\langle \eta_g^d, \eta_{g^{-1}}^{d'} \rangle = \begin{cases} \prod_{i \in I(g)} w_i^{-1} & \text{if } \deg(\eta_g^d) + \deg(\eta_{g^{-1}}^{d'}) = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

3.c. Orbifold cohomology ring of weighted projective spaces. Before computing the orbifold cup product in the basis $\boldsymbol{\eta}$, we will state a lemma about the orbifold tangent bundle of $\mathbb{P}(w)$.

According to Proposition 3.3, an orbundle on $\mathbb{P}(w)$ can be restricted to $\mathbb{P}(w_I)$.

Lemma 3.14. *For any subset I of $\{0, \dots, n\}$, we have the following decomposition:*

$$T\mathbb{P}(w)|_{\mathbb{P}(w_I)} \simeq \left(\bigoplus_{i \in I^c} \mathcal{O}_{\mathbb{P}(w_I)}(w_i) \right) \oplus T\mathbb{P}(w_I)$$

where I^c is the complement of I in $\{0, \dots, n\}$.

Proof. A straightforward computation shows that these two orbundles have the same transition functions. $\square$

In order to compute the orbifold cup product on $H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C})$, we will compute the trilinear form $(\cdot, \cdot, \cdot)$ introduced in [CR04] and get the cup product through the Poincaré pairing by the formula

$$(3.15) \quad (\alpha_1, \alpha_2, \alpha_3) = \langle \alpha_1 \cup \alpha_2, \alpha_3 \rangle \quad \forall \alpha_1, \alpha_2, \alpha_3 \in H_{\text{orb}}^*(\mathbb{P}(w), \mathbb{C}).$$

Let us fix g_0, g_1, g_∞ in $\bigcup \boldsymbol{\mu}_{w_i}$ satisfying $g_0 g_1 g_\infty = 1$. Let us fix a presentation of the fundamental group $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \star)$ as $\langle \lambda_0, \lambda_1, \lambda_\infty \mid \lambda_0 \lambda_1 \lambda_\infty = 1 \rangle$. The group homomorphism $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \star) \rightarrow \mathbb{C}^\star$ sending λ_j to g_j for any j in $\{0, 1, \infty\}$, whose image is denoted by H , defines a covering Σ° of $\mathbb{P}^1 - \{0, 1, \infty\}$ having H as its automorphism group. This covering extends as a ramified covering $\pi : \Sigma \rightarrow \mathbb{P}^1$, where Σ is a compact Riemann surface, and the action of H also extends to Σ in such a way that $\mathbb{P}^1 = \Sigma/H$.

The group H acts in a natural way on $T\mathbb{P}(w) \mid_{\mathbb{P}(w)_{(g_0, g_1, g_\infty)}}$ where $\mathbb{P}(w)_{(g_0, g_1, g_\infty)}$ is the standard notation for the triple twisted sectors. For any $k \in \{0, 1, \infty\}$, we denote by ι_{g_k} the injection $\mathbb{P}(w)_{(g_k)} \hookrightarrow \mathbb{P}(w)_{(g_0, g_1, g_\infty)}$. We define the following orbibundle:

$$E_{(g_0, g_1, g_\infty)} := \left(T\mathbb{P}(w) \mid_{\mathbb{P}(w)_{(g_0, g_1, g_\infty)}} \otimes H^{0,1}(\Sigma, \mathbb{C}) \right)^H.$$

In the basis $\boldsymbol{\eta}$, we define the trilinear form $(\cdot, \cdot, \cdot)$:

$$(3.16) \quad (\eta_{g_0}^{d_0}, \eta_{g_1}^{d_1}, \eta_{g_\infty}^{d_\infty}) := \int_{\mathbb{P}(w)_{(g_0, g_1, g_\infty)}}^{\text{orb}} \iota_{g_0}^* \eta_{g_0}^{d_0} \wedge \iota_{g_1}^* \eta_{g_1}^{d_1} \wedge \iota_{g_\infty}^* \eta_{g_\infty}^{d_\infty} \wedge c_{\max}(E_{(g_0, g_1, g_\infty)}).$$

Theorem 3.17. *Let g_0, g_1 and g_∞ be in $\bigcup \mu_{w_i}$ such that $g_0 g_1 g_\infty = 1$. For $i \in \{0, 1, \infty\}$, we denote by γ_i the unique element in $[0, 1[$ such that $g_i = \exp(2i\pi\gamma_i)$. The orbibundle $E_{(g_0, g_1, g_\infty)}$ is isomorphic to*

$$\bigoplus_{j \in J(g_0, g_1, g_\infty)} \mathcal{O}_{\mathbb{P}(w)_{(g_0, g_1, g_\infty)}}(w_j)$$

where $J(g_0, g_1, g_\infty) := \{i \in \{0, \dots, n\} \mid \{\gamma_0 w_i\} + \{\gamma_1 w_i\} + \{\gamma_\infty w_i\} = 2\}$.

Proof. According to the decomposition of Lemma 3.14, the obstruction bundle $E_{(g_0, g_1, g_\infty)}$ is isomorphic to

$$\left(\bigoplus_{\substack{i \in \{0, \dots, n\} - \\ I(g_0) \cap I(g_1) \cap I(g_\infty)}} \mathcal{O}_{\mathbb{P}(w)_{(g_0, g_1, g_\infty)}}(w_i) \otimes H^{0,1}(\Sigma, \mathbb{C}) \right)^H \oplus \left(T\mathbb{P}(w)_{(g_0, g_1, g_\infty)} \otimes H^{0,1}(\Sigma, \mathbb{C}) \right)^H.$$

As H acts trivially on $T\mathbb{P}(w)_{(g_0, g_1, g_\infty)}$, we get

$$E_{(g_0, g_1, g_\infty)} = \bigoplus_{\substack{i \in \{0, \dots, n\} - \\ I(g_0) \cap I(g_1) \cap I(g_\infty)}} \left(\mathcal{O}_{\mathbb{P}(w)_{(g_0, g_1, g_\infty)}}(w_i) \otimes H^{0,1}(\Sigma, \mathbb{C}) \right)^H.$$

Note that the group H acts on the fiber of $\mathcal{O}_{\mathbb{P}(w)_{(g_0, g_1, g_\infty)}}(w_i)$ by multiplication by the character $\chi_i : H \rightarrow \mathbb{C}^*$ which sends h to h^{w_i} . Now we apply Proposition 6.3 of [BCS05] (see also Proposition 3.4 of [CH04] or Theorem IV.5.13 of [Man05]) and we get the theorem. $\square$

Theorem 3.17 and Formula (3.16) of the trilinear form $(\cdot, \cdot, \cdot)$ imply that we can compute this trilinear form in the basis $\boldsymbol{\eta}$. Then, the definition of the cup product via Formula (3.15) gives us the following corollary.

Corollary 3.18. *Let $\eta_{g_0}^{d_0}$ and $\eta_{g_1}^{d_1}$ be two elements of the basis η . We have*

$$\eta_{g_0}^{d_0} \cup \eta_{g_1}^{d_1} = \left(\prod_{i \in K(g_0, g_1)} w_i \right) \eta_{g_0 g_1}^d$$

where $K(g_0, g_1) := J(g_0, g_1, (g_0 g_1)^{-1}) \sqcup I(g_0 g_1) - I(g_0) \cap I(g_1)$ and $d := \frac{\deg(\eta_{g_0}^{d_0})}{2} + \frac{\deg(\eta_{g_1}^{d_1})}{2} - \text{age}(g_0 g_1) = d_0 + d_1 + \text{age}(g_0) + \text{age}(g_1) - \text{age}(g_0 g_1)$.

Example 3.19. Let us consider the case where the weights are $w = (1, 2, 2 : 3 : 3, 3)$ (this example was considered in [Jia03]). The orbifold cup product is computed in the table below in basis η . We put $j := \exp(2i\pi/3)$.

	η_1^0	η_1^1	η_1^2	η_1^3	η_1^4	η_1^5	η_j^0	η_j^1	η_j^2	η_{-1}^0	η_{-1}^1	$\eta_{j^2}^0$	$\eta_{j^2}^1$	$\eta_{j^2}^2$
η_1^0	η_1^0	η_1^1	η_1^2	η_1^3	η_1^4	η_1^5	η_j^0	η_j^1	η_j^2	η_{-1}^0	η_{-1}^1	$\eta_{j^2}^0$	$\eta_{j^2}^1$	$\eta_{j^2}^2$
η_1^1		η_1^2	η_1^3	η_1^4	η_1^5	0	η_j^1	η_j^2	0	η_{-1}^1	0	$\eta_{j^2}^1$	$\eta_{j^2}^2$	0
η_1^2			η_1^4	η_1^5	0	0	η_j^2	0	0	0	0	$\eta_{j^2}^2$	0	0
η_1^3				0	0	0	0	0	0	0	0	0	0	0
η_1^4					0	0	0	0	0	0	0	0	0	0
η_1^5						0	0	0	0	0	0	0	0	0
η_j^0							$4\eta_{j^2}^2$	0	0	0	0	$4\eta_1^3$	$4\eta_1^4$	$4\eta_1^5$
η_j^1								0	0	0	0	$4\eta_1^4$	$4\eta_1^5$	0
η_j^2									0	0	0	$4\eta_1^5$	0	0
η_{-1}^0										$3^3\eta_1^4$	$3^3\eta_1^5$	0	0	0
η_{-1}^1											0	0	0	0
$\eta_{j^2}^0$												$1\eta_j^1$	$1\eta_j^2$	0
$\eta_{j^2}^1$													0	0
$\eta_{j^2}^2$														0

The upper left corner is the standard cup product on $H^*(|\mathbb{P}(1, 2, 2 : 3 : 3, 3)|)$.

Example 3.20. Let $\mathbb{P}(w_0, w_1)$ be a weighted projective line. Let us denote by d the greatest common divisor of w_0 and w_1 . Choose integers m and n such that $mw_0 + nw_1 = d$. Denote $g_{w_0} := \exp(2i\pi n/w_0)$, $g_{w_1} := \exp(2i\pi m/w_1)$ and $g_d := \exp(2i\pi/d)$. If we put $x := \eta_{g_{w_0}}^0$, $y := \eta_{g_{w_1}}^0$ and $\xi := \eta_{g_d}^0$, we have that

$$H_{\text{orb}}^*(\mathbb{P}(w_0, w_1), \mathbb{C}) = \mathbb{C}[x, y, \xi] / \langle xy, w_0 x^{w_0/d} - w_1 y^{w_1/d} \xi^{n-m}, \xi^d - 1 \rangle.$$

This agrees completely with the computation of [AGV06, Section 9].

3.d. Some initial conditions for the Frobenius manifold. In Sections 3.b and 3.c we have computed two initial conditions for the Frobenius manifold, namely the orbifold Poincaré duality $\langle \cdot, \cdot \rangle$ and the unit η_0^0 . In this section, we will compute a third one, which is $\text{id} - \nabla \mathfrak{E}$, where ∇ is the torsion free connection associated to the nondegenerate pairing $\langle \cdot, \cdot \rangle$ and $\mathfrak{E}$ is the Euler field defined in (3.23) below. We start with the following lemma.

Lemma 3.21. *We have the exact sequence of orbifold bundles over $\mathbb{P}(w)$*

$$0 \longrightarrow \underline{\mathbb{C}} \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}(w)}(w_0) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}(w)}(w_n) \xrightarrow{\varphi} T\mathbb{P}(w) \longrightarrow 0$$

where $\underline{\mathbb{C}}$ is the orbifold trivial bundle of rank 1 over $\mathbb{P}(w)$.

The proof is a straightforward generalization of the proof in Section 3 of Chapter 3 of [GH94] (all the details are explained in Lemma V.2.1 of [Man05]).

Recall that $\eta_1^1 = c_1(\mathcal{O}_{\mathbb{P}(w)}(1))$. Lemma 3.21 and Section 4.3 of [CR02] (see also Proposition III.4.13 in [Man05]) imply that

$$c(T\mathbb{P}(w)) = c(\mathcal{O}_{\mathbb{P}(w)}(w_0) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}(w)}(w_n)) = \prod_{i=0}^n (1 + w_i \eta_1^1).$$

We deduce that

$$(3.22) \quad c_1(T\mathbb{P}(w)) = \mu \eta_1^1$$

where $\mu := w_0 + \dots + w_n$.

For any $g \in \bigcup \mu_{w_i}$ and any $0 \leq d \leq \dim_{\mathbb{C}} \mathbb{P}(w)_{(g)}$, we denote by $t_{g,d}$ the coordinates of $H_{\text{orb}}^{2*}(\mathbb{P}(w), \mathbb{C})$ with respect to the element of the basis η_g^d . As in the book of Y. Manin [Man99, p.37], we define the Euler field by

$$(3.23) \quad \mathfrak{E} := \sum_{\substack{g \in \bigcup \mu_{w_i} \\ 0 \leq d \leq \dim_{\mathbb{C}} \mathbb{P}(w)_{(g)}}} (1 - \deg(\eta_g^d)/2) t_{g,d} \partial_{t_{g,d}} + \mu \partial_{t_{1,1}}.$$

Notation 3.24. For the following, it will be useful to have an order on the basis $\boldsymbol{\eta}$ of $H_{\text{orb}}^{2*}(\mathbb{P}(w), \mathbb{C})$. Choose any determination of the argument in $\mathbb{C}$. Hence, for any $g \in \bigcup \mu_{w_i}$, there exists a unique $\gamma(g) \in [0, 1[$ such that $g = \exp(2i\pi\gamma(g))$. We say that $\eta_g^d \leq \eta_{g'}^{d'}$ if $\gamma(g) < \gamma(g')$ or $\gamma(g) = \gamma(g')$ and $d \leq d'$. We denote by $g_{\max}$ the greatest element in $\bigcup \mu_{w_i}$.

Proposition 3.25. *Denote by $d_{\max}$ the complex dimension of the twisted sector $\mathbb{P}(w)_{(g_{\max})}$. The matrix $A_{\infty} := \text{id} - \nabla \mathfrak{E}$ in the basis $\boldsymbol{\eta}$ is $\text{diag}(\deg(\eta_0^0)/2, \dots, \deg(\eta_0^n)/2, \deg(\eta_{g_{\max}^{-1}}^0), \dots, \deg(\eta_{g_{\max}}^{d_{\max}}))$ where ∇ is the torsion free connection associated to the nondegenerate pairing $\langle \cdot, \cdot \rangle$. This matrix satisfies $A_{\infty} + A_{\infty}^* = n \text{id}$ where A_{∞}^* is the adjoint of A_{∞} with respect to the nondegenerate bilinear form $\langle \cdot, \cdot \rangle$.*

Proof. By definition, we have

$$A_{\infty} = \frac{1}{2} \text{diag} \left(\deg(\eta_0^0), \dots, \deg(\eta_0^n), \deg(\eta_{g_{\max}^{-1}}^0), \dots, \deg(\eta_{g_{\max}}^{d_{\max}}) \right).$$

The adjoint matrix of A_{∞} with respect to the nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ is

$$A_{\infty}^* = \frac{1}{2} \text{diag} \left(\deg(\eta_0^n), \dots, \deg(\eta_0^0), \deg(\eta_{g_{\max}}^{d_{\max}}), \dots, \deg(\eta_{g_{\max}^{-1}}^0) \right).$$

In order to end the proof, it is enough to check that

$$\deg(\eta_g^d) + \deg\left(\eta_{g^{-1}}^{\dim_{\mathbb{C}} \mathbb{P}(w)_{(g)} - d}\right) = 2n. \quad \square$$

4. Orbifold quantum cohomology of weighted projective spaces

The orbifold cohomology algebra, with its Poincaré pairing, of weighted projective spaces is now completely determined. We have computed three out of four initial conditions of the Frobenius manifold. In this section, we will study the last initial condition $\mathfrak{E}\star|_{t=0}$.

4.a. The orbifold Gromov-Witten invariants. First, we recall the definition of orbifold stable maps to $\mathbb{P}(w)$ (see Paragraph 2.3 of [CR02] for details).

A good orbifold map f between the orbifolds X and Y is an orbifold map and a compatible structure. A compatible structure is a correspondence between open sets and injections of charts of X and open sets and injections of charts of Y which satisfies some conditions (see Section 4.4 of [CR02] for more details). In particular, the compatible structure induces a homomorphism between the local groups; that is, for any $x \in X$ we have a morphism of groups $G_x \rightarrow G_{f(x)}$. We will not be more precise because we will not use this notion explicitly.

Definition 4.1. *An orbifold stable map to $\mathbb{P}(w)$ consists of the following data:*

- a nodal orbicurve $(C, \mathbf{z}, \mathbf{m}, \mathbf{n})$ where $\mathbf{z} := (z_1, \dots, z_k)$ are k distinct marked points such that $G_{z_i} = \mu_{m_i}$ for $i \in \{1, \dots, k\}$ and the j -th nodal point has the action of μ_{n_j} ;
- a continuous map $f : C \rightarrow \mathbb{P}(w)$;
- an isomorphism class of compatible structure, denoted by ξ .

These data $(f, (C, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi)$ satisfy

- (1) for each i in I , the orbifold map $f_i := f \circ \varphi_i : C_i \rightarrow \mathbb{P}(w)$ is holomorphic;
- (2) for each marked or nodal point z_i , the morphism of groups induced by ξ from G_{z_i} to $G_{f(z_i)}$ is injective;
- (3) if the map $f_i : C_i \rightarrow \mathbb{P}(w)$ is constant, then the curve C_i has more than three singular points (i.e. nodal or marked).

We endow the set of orbifold stable maps with the standard equivalence relation. Denote by $[f, (C, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi]$ the equivalence class of the orbifold stable map $(f, (C, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi)$.

Let $(f, (C, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi)$ be a stable map. We can associate to this stable map a homology class in $H_2(|\mathbb{P}(w)|, \mathbb{Z})$ defined by $f_*([C]) := \sum_i (f \circ \varphi_i)_*[C_i]$ where $[C_i]$ is the fundamental class of the curve C_i . This homology class does not depend on the equivalence class of the stable map. For each marked point z_i , the class of compatible structure ξ induces a monomorphism of groups $\kappa_i : G_{z_i} \hookrightarrow G_{f(z_i)}$. This monomorphism depends only on the equivalence class of the stable map.

Let us define the inertia orbifold $\mathcal{IP}(w)$ by $\bigsqcup_{g \in \bigcup \mu_{w_i}} \mathbb{P}(w)_{(g)} \times \{g\}$. We have an evaluation map, denoted by ev , which maps a class $[f, (C, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi]$ of stable maps to

$$\left((f(z_1), \kappa_1(e^{2i\pi/m_1})), \dots, (f(z_k), \kappa_k(e^{2i\pi/m_k})) \right) \in \mathcal{IP}(w) \times \dots \times \mathcal{IP}(w).$$

An orbifold stable map $(f, (C, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi)$ is said to have *type* $(g_1, \dots, g_k) \in \bigcup \mu_{w_i}$ if for any $\ell \in \{1, \dots, k\}$, $(f(z_\ell), \kappa_\ell(e^{2i\pi/m_\ell}))$ belongs to $\mathbb{P}(w)_{(g_\ell)} \times \{g_\ell\}$. When there is no ambiguity in the notation, we will write $\underline{g}$ for the k -tuple $(g_1, \dots, g_k)$.

Definition 4.2. Let A be in $H_2(|\mathbb{P}(w)|, \mathbb{Z})$. We define $\overline{\mathcal{M}}_k(A, \underline{g})$ to be the moduli space of equivalence classes of orbifold stable maps with k marked points, of homology class A and of type $\underline{g}$, i.e.

$$\overline{\mathcal{M}}_k(A, \underline{g}) = \left\{ \begin{array}{l} [(f, (C, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi)] \mid \#\mathbf{z} = k, f_*[C] = A, \\ \text{ev}(f, (C, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi) \in \prod_{\ell=1}^k (\mathbb{P}(w)_{(g_\ell)} \times \{g_\ell\}) \end{array} \right\}.$$

According to the results of [CR02] (cf. Proposition 2.3.8), the moduli space $\overline{\mathcal{M}}_k(A, \underline{g})$ is compact and metrizable. Chen and Ruan also define a Kuranishi structure on this moduli space whose dimension is given by the following theorem.

Theorem 4.3 (cf. Theorem A of [CR02]). *The dimension of the Kuranishi structure described by W. Chen and Y. Ruan of $\overline{\mathcal{M}}_k(A, \underline{g})$ is*

$$2 \left(\int_A c_1(T\mathbb{P}(w)) + \dim_{\mathbb{C}} \mathbb{P}(w) - 3 + k - \sum_{\ell=1}^k \text{age}(g_\ell) \right).$$

This Kuranishi structure defines (cf. Theorem 6.12 and Section 17 of [FO99]) a homology class, called the fundamental class of the Kuranishi structure,

$$(4.4) \quad \text{ev}_*[\overline{\mathcal{M}}_k(A, \underline{g})] \in H_2\left(\int_A c_1(T\mathbb{P}(w)) + n - 3 + k - \sum_{\ell=1}^k \text{age}(g_\ell)\right)(\mathbb{P}(w)_{(g_1)} \times \dots \times \mathbb{P}(w)_{(g_k)}, \mathbb{C}).$$

Remark 4.5. This notation is a bit tendentious because the class $\text{ev}_*[\overline{\mathcal{M}}_k(A, \underline{g})]$ is not a push forward of a homology class of $\overline{\mathcal{M}}_k(A, \underline{g})$. However, in algebraic geometry, we can construct a virtual fundamental class in the Chow group of $\overline{\mathcal{M}}_k(A, \underline{g})$ with the same degree (see Section 5.3 in [AGV06, Section 4.5]).

For each $\ell \in \{1, \dots, k\}$, let α_ℓ be a class in $H^{2(\star - \text{age}(g_\ell))}(\mathbb{P}(w)_{(g_\ell)}, \mathbb{C}) \subset H_{\text{orb}}^{2\star}(\mathbb{P}(w), \mathbb{C})$. Formula (1.3) of [CR02] defines the orbifold Gromov-Witten invariants by

$$(4.6) \quad \Psi_{k, \underline{g}}^A : H^\star(\mathbb{P}(w)_{(g_1)}, \mathbb{C}) \otimes \cdots \otimes H^\star(\mathbb{P}(w)_{(g_k)}, \mathbb{C}) \longrightarrow \mathbb{C}$$

$$\alpha_1 \otimes \cdots \otimes \alpha_k \longmapsto \int_{\text{ev}_*[\overline{\mathcal{M}}_k(A, \underline{g})]} \alpha_1 \wedge \cdots \wedge \alpha_k.$$

Let $g_1, \dots, g_k$ be in $\bigcup \mu_{w_i}$. Recall that $\mu := w_0 + \cdots + w_n$. Theorem 4.3 and Formula (3.22) imply that

$$\deg \text{ev}_* [\overline{\mathcal{M}}_k(A, \underline{g})] = 2 \left(\mu \int_A \eta_1 + n - 3 + k - \sum_{\ell=1}^n \text{age}(g_\ell) \right).$$

We recall that for any $g \in \bigcup \mu_{w_i}$ and any $0 \leq d \leq \dim_{\mathbb{C}} \mathbb{P}(w)_{(g)}$, we denote by $t_{g,d}$ the coordinate of $H_{\text{orb}}^{2\star}(\mathbb{P}(w), \mathbb{C})$ with respect to the element of the basis $\eta_{g,d}$. Let us put

$$T := \sum_{\substack{g \in \bigcup \mu_{w_i} \\ 0 \leq d \leq \dim_{\mathbb{C}} \mathbb{P}(w)_{(g)}}} t_{g,d} \eta_g^d.$$

The full Gromov-Witten potential of genus 0 of the weighted projective space $\mathbb{P}(w)$, denoted by F^{GW} , is defined by

$$F^{GW} := \sum_{k \geq 0} \sum_{\substack{A \in H_2(\mathbb{P}(w), \mathbb{Z}), \\ \underline{g} \in \bigcup (\mu_{w_i})^k}} \frac{\Psi_{k, \underline{g}}^A(T, \dots, T)}{k!}.$$

We define the orbifold quantum product by the equation

$$(4.7) \quad \frac{\partial^3 F^{GW}(\mathbf{t})}{\partial t_{g,d} \partial t_{g',d'} \partial t_{g'',d''}} = \langle \partial t_{g,d} \star \partial t_{g',d'}, \partial t_{g'',d''} \rangle.$$

4.b. Computation of some orbifold Gromov-Witten invariants.

To compute the last initial condition of the Frobenius manifold, we should compute the matrix $A_0^\circ := \mathfrak{E} \star |_{\mathbf{t}=0}$ (cf. Equation (4.7) for the definition of the quantum orbifold product). As the Euler field restricted to $\mathbf{t} = 0$ is $\mu \partial t_{1,1}$, we have to compute the Gromov-Witten invariants $\Psi_3^A(\eta_1^1, \eta_g^d, \eta_{g'}^{d'})$ for any $g, g' \in \bigcup \mu_{w_i}$, for any $(d, d') \in \{0, \dots, \dim_{\mathbb{C}} \mathbb{P}(w)_{(g)}\} \times \{0, \dots, \dim_{\mathbb{C}} \mathbb{P}(w)_{(g')}\}$

and for any class $A \in H_2(\mathbb{P}(w), \mathbb{Z})$. By definition of the Gromov-Witten invariant, if the class $A \in H_2(|\mathbb{P}(w)|, \mathbb{Z})$ does not satisfy

$$(4.8) \quad \mu \int_A \eta_1^1 = 1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n,$$

then the Gromov-Witten invariant $\Psi_3^A(\eta_1^1, \eta_g^d, \eta_{g'}^{d'})$ is zero.

Notation 4.9. In the following we denote by $A(g, d, g', d')$ the unique class in $H_2(|\mathbb{P}(w)|, \mathbb{Z})$ that satisfies (4.8). When there will be no confusion, we denote this class by $\tilde{A}$.

Motivated by Corollary 5.7 for the B side, we decompose this set of Gromov-Witten invariants into the following three subsets:

$$(4.10)$$

$$\left\{ \begin{array}{l} \Psi_3^{\tilde{A}}(\eta_1^1, \eta_g^d, \eta_{g'}^{d'}) \text{ such that} \\ 1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n + \mu(\gamma(g^{-1}) + \gamma(g'^{-1})) \not\equiv 0 \pmod{\mu} \end{array} \right\},$$

$$(4.11)$$

$$\left\{ \begin{array}{l} \Psi_3^{\tilde{A}}(\eta_1^1, \eta_g^d, \eta_{g'}^{d'}) \text{ such that} \\ 1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n + \mu(\gamma(g^{-1}) + \gamma(g'^{-1})) \equiv 0 \pmod{\mu} \\ \text{and } 2 + \deg(\eta_g^d) + \deg \eta_{g'}^{d'} = 2n \end{array} \right\},$$

$$(4.12)$$

$$\left\{ \begin{array}{l} \Psi_3^{\tilde{A}}(\eta_1^1, \eta_g^d, \eta_{g'}^{d'}) \text{ such that} \\ 1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n + \mu(\gamma(g^{-1}) + \gamma(g'^{-1})) \equiv 0 \pmod{\mu} \\ \text{and } 2 + \deg(\eta_g^d) + \deg \eta_{g'}^{d'} \neq 2n \end{array} \right\}.$$

Remark 4.13. (1) The number $1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n + \mu(\gamma(g^{-1}) + \gamma(g'^{-1}))$ is equal to the integer

$$1 + d + d' + n - \dim \mathbb{P}(w)_{(g)} - \dim \mathbb{P}(w)_{(g')} + \sum_{i=0}^n [\gamma(g^{-1})w_i] + [\gamma(g'^{-1})w_i]$$

where $[\cdot]$ is the integer part.

(2) Conditions (4.11) are equivalent to the conditions $gg' = 1$ and $2 + \deg(\eta_g^d) + \deg(\eta_{g'}^{d'}) = 2n$.

First we study the set (4.10) of Gromov-Witten invariants. The following proposition is a straightforward consequence of Proposition 7.2.

Proposition 4.14. *Let g, g' be in $\bigcup \mu_{w_i}$ and let (d, d') be in $\{0, \dots, \dim \mathbb{P}(w)_{(g)}\} \times \{0, \dots, \dim \mathbb{P}(w)_{(g')}\}$ such that $1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n + \mu(\gamma(g^{-1}) + \gamma(g'^{-1})) \not\equiv 0 \pmod{\mu}$. We have $\Psi_3^{\tilde{A}}(\eta_1^1, \eta_g^d, \eta_{g'}^{d'}) = 0$ where $\tilde{A}$ is defined by $\mu \int_{\tilde{A}} \eta_1^1 = 1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n$.*

Remark 4.15. The proposition above is proved in [Man05, Theorem V.3.3, p. 98] when μ and $\text{lcm}(w_0, \dots, w_n)$ are coprime.

The following proposition computes the Gromov-Witten invariant for the subset defined by Conditions (4.11). According to Remark 4.13(2), these conditions are equivalent to the hypothesis of the theorem below.

Proposition 4.16. *Let g, g' be in $\bigcup \mu_{w_i}$ and let (d, d') be in $\{0, \dots, \dim \mathbb{P}(w)_{(g)}\} \times \{0, \dots, \dim \mathbb{P}(w)_{(g')}\}$ such that $g'g = \text{id}$ and $2 + \deg \eta_g^d + \deg \eta_{g'}^{d'} = 2n$. Let A be a class in $H_2(|\mathbb{P}(w)|, \mathbb{Z})$. We have*

$$\Psi_3^A(\eta_1^1, \eta_g^d, \eta_{g'}^{d'}) = \begin{cases} \prod_{i \in I(g)} w_i^{-1} & \text{if } A = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If this invariant is not zero, then the class A should satisfy the equality $2\mu \int_{A(g,d,g',d')} \eta_1^1 = 2 + \deg \eta_g^d + \deg \eta_{g'}^{d'} - 2n$. By hypothesis this implies that $A = 0$. Hence Theorem 3.17 and Formula (3.16) finish the proof. $\square$

In order to simplify Conditions (4.12), we will recall some combinatorics. Let us denote the elements of $\bigcup \mu_{w_i}$ by $1 = g_0 < g_1 < \dots < g_\delta$ where the order is defined by choosing the principal determination of the argument (cf. Notation 3.24). Let us fix $g_k \in \bigcup \mu_{w_i}$. There exists a unique triple (d, g', d') in $\{0, \dots, \dim \mathbb{P}(w)_{(g_k)}\} \times \bigcup \mu_{w_i} \times \{0, \dots, \dim \mathbb{P}(w)_{(g')}\}$ that satisfies

$$\begin{cases} 1 + \deg(\eta_{g_k}^d)/2 + \deg(\eta_{g'}^{d'})/2 - n + \mu(\gamma(g_k^{-1}) + \gamma(g'^{-1})) = 0 \pmod{\mu} \\ \text{and } 2 + \deg \eta_{g_k}^d + \deg \eta_{g'}^{d'} \neq 2n. \end{cases}$$

Such a triple is given by $(\dim \mathbb{P}(w)_{(g_k)}, g_{k-1}^{-1}, \dim \mathbb{P}(w)_{(g_{k-1})})$ where $g_{-1} := g_\delta$.

Proposition 4.17. *Let $g_k \in \bigcup \mu_{w_i}$. Let $\tilde{A}$ be the class in $H_2(|\mathbb{P}(w)|, \mathbb{Z})$ defined by $\mu \int_{\tilde{A}} \eta_1^1 = 1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n$. We have that*

$$\text{ev}_* \left[\overline{\mathcal{M}}_2(\tilde{A}, g_k, g_{k-1}^{-1}) \right] = (\gamma(g_k) - \gamma(g_{k-1}))^{-1} [\mathbb{P}(w)_{(g_k)} \times \mathbb{P}(w)_{(g_{k-1})}].$$

Remark 4.18. In [AGV06, Section 9], D. Abramovich, T. Graber and A. Vistoli have computed the small quantum cohomology of $\mathbb{P}(w_0, w_1)$. This result implies Proposition 7.2, hence Propositions 4.14 and 4.17 for weighted projective lines.

Proof of Proposition 4.17. The divisor axiom implies that this proposition is equivalent to

$$\Psi_3^{\tilde{A}} \left(\eta_1^1, \eta_{g_k}^{\dim \mathbb{P}(w)_{(g_k)}}, \eta_{g_{k-1}^{-1}}^{\dim \mathbb{P}(w)_{(g_{k-1})}} \right) = \prod_{i \in I(g_k) \sqcup I(g_{k-1})} w_i^{-1}.$$

Formula (4.7) and Proposition 7.2 imply the equality above. $\square$

Let us put

$$((\eta_1^1, \eta_g^d, \eta_{g'}^{d'})) := \frac{\partial^3 F^{GW}}{\partial t_{1,1} \partial t_{g,d} \partial t_{g',d'}} \Big|_{t=0}.$$

Propositions 4.14, 4.16 and 4.17 imply the following corollary.

Corollary 4.19. *Let g, g' be in $\bigcup \mu_{w_i}$ and let (d, d') be in $\{0, \dots, \dim_{\mathbb{C}} \mathbb{P}(w)_{(g)}\} \times \{0, \dots, \dim_{\mathbb{C}} \mathbb{P}(w)_{(g')}\}$.*

- (1) *If $1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n + \mu(\gamma(g^{-1}) + \gamma(g'^{-1})) \neq 0 \pmod{\mu}$, then $((\eta_1^1, \eta_g^d, \eta_{g'}^{d'})) = 0$.*
- (2) *If $1 + \deg(\eta_g^d)/2 + \deg(\eta_{g'}^{d'})/2 - n + \mu(\gamma(g^{-1}) + \gamma(g'^{-1})) = 0 \pmod{\mu}$, then we have*

$$((\eta_1^1, \eta_g^d, \eta_{g'}^{d'})) = \begin{cases} \left(\prod_{i \in I(g) \amalg I(g')} w_i \right)^{-1} & \text{if } 2 + \deg \eta_g^d + \deg \eta_{g'}^{d'} \neq 2n, \\ \left(\prod_{i \in I(g)} w_i \right)^{-1} & \text{if } 2 + \deg \eta_g^d + \deg \eta_{g'}^{d'} = 2n. \end{cases}$$

To determine the matrix $A_0^\circ = \mathfrak{E} \star|_{t=0}$, we use formula (4.7). This formula shows that the data $((\eta_1^1, \eta_g^d, \eta_{g'}^{d'}))$ and the orbifold Poincaré duality $\langle \cdot, \cdot \rangle$ enable us to compute the matrix A_0° .

5. Frobenius structure associated to the Laurent polynomial f

In this part, we will use the following notation.

Notation 5.1. Let n and $w_0, \dots, w_n$ be some integers greater than or equal to one. We put $\mu := w_0 + \dots + w_n$. Consider the set $\bigsqcup_{i=0}^n \{\ell/w_i \mid \ell \in \{0, \dots, w_i - 1\}\}$ where $\bigsqcup$ means the disjoint union. Choose a bijection $s : \{0, \dots, \mu - 1\} \rightarrow \bigsqcup_{i=0}^n \{\ell/w_i \mid \ell \in \{0, \dots, w_i - 1\}\}$ such that $\mathcal{V} \circ s$ is nondecreasing. Let us consider, as in [DS04], the rational numbers $\sigma(i) := i - \mu s(i)$ for $i \in \{0, \dots, \mu - 1\}$.

Let $U := \{(u_0, \dots, u_n) \in \mathbb{C}^{n+1} \mid u_0^{w_0} \dots u_n^{w_n} = 1\}$. Let $f : U \rightarrow \mathbb{C}$ be the function defined by $f(u_0, \dots, u_n) = u_0 + \dots + u_n$. The polynomial f is not exactly the one considered in [DS04], but we can apply the same methods.

An easy computation shows that the critical value of f is $\mu \zeta (\prod_{i=0}^n w_i^{w_i})^{-1/\mu}$ where ζ is a μ -th root of unity. In [DS03], there exists a Frobenius structure on the base space of any universal unfolding of f . In this example, we will see (*cf.* Theorem 5.3) that the Frobenius structure is semi-simple, *i.e.* we can reconstruct the Frobenius structure from some initial data.

Let A_0° be the matrix of size $\mu \times \mu$ defined by (recall that $\overline{a+b} = a+b \pmod{\mu}$)

$$(A_0^\circ)_{\overline{j+1},j} = \begin{cases} \mu & \text{if } s(\overline{j+1}) = s(j); \\ \mu \prod_{i \in I(s(j))} w_i^{-1} & \text{otherwise.} \end{cases}$$

The eigenvalues of A_0° are exactly the critical values of f . Hence, A_0° is a semi-simple regular matrix. In the canonical basis $(e_0, \dots, e_{\mu-1})$ of $\mathbb{C}^\mu$, we define the bilinear nondegenerate form g by

$$(5.2) \quad g(e_j, e_k) = \begin{cases} \prod_{i \in I(s(k))} w_i^{-1} & \text{if } \overline{j+k} = n; \\ 0 & \text{otherwise.} \end{cases}$$

Let A_∞ be the matrix of size $\mu \times \mu$ defined by $A_\infty = \text{diag}(\sigma(0), \dots, \sigma(\mu-1))$ (cf. Section 6.a for the definition of $\sigma(\cdot)$). This matrix satisfies $A_\infty + A_\infty^* = n \cdot \text{id}$ where A_∞^* is the adjoint of A_∞ with respect to g .

Theorem 5.3 (Theorem 2 of [DS04]). *The canonical Frobenius structure on any germ of a universal unfolding of the Laurent polynomial $f(u_0, \dots, u_n) = u_0 + \dots + u_n$ on U is isomorphic to the germ of a universal semi-simple Frobenius structure with initial data $(A_0^\circ, A_\infty, e_0, g)$ at the point*

$$\left(\mu \prod_{i=0}^n w_i^{-w_i}, \mu \zeta \prod_{i=0}^n w_i^{-w_i}, \dots, \mu \zeta^{\mu-1} \prod_{i=0}^n w_i^{-w_i} \right) \in \mathbb{C}^\mu.$$

5.a. The Gauss-Manin system and the Brieskorn lattice of the Laurent polynomial f . One can compute the initial data $(A_0^\circ, A_\infty, e_0, g)$ of the Frobenius structure from the Jacobian algebra of f . Namely, the product on the Frobenius manifold at the origin comes from the product on the Jacobian algebra of f via the isomorphism given by the primitive form

$$\omega_0 := \frac{\frac{du_0}{u_0} \wedge \dots \wedge \frac{du_n}{u_n}}{d(\prod_i u_i^{w_i})} \Big|_{\prod_i u_i^{w_i} = 1}.$$

Moreover, the multiplication by the Euler field is induced, via this isomorphism, by the multiplication by f . Finally, the nondegenerate form g is given by a residue formula. In this example, the form g can also be computed from a duality on the Brieskorn lattice of f . In this example, this way is easier. For this reason, we will use the Gauss-Manin system and the Brieskorn lattice of f to get the initial data.

In the following, we will not give details, we refer to [DS03], [DS04]. For a better exposition, we will suppose that the weights are relatively prime (see [Man05] for the general case). The Gauss-Manin system of f is defined by $G := \Omega^n(U)[\theta, \theta^{-1}]/(\theta d - df) \wedge \Omega^{n-1}(U)[\theta, \theta^{-1}]$. The Brieskorn lattice of f ,

defined by $G_0 := \text{Im}(\Omega^n(U)[\theta] \rightarrow G)$, is a free $\mathbb{C}[\theta]$ -module of rank μ . We define inductively the sequence $(\underline{a}(k), i(k)) \in \mathbb{N}^{n+1} \times \{0, \dots, n\}$ by

$$\begin{aligned} \underline{a}(0) &= (0, \dots, 0), & i(0) &= 0, \\ \underline{a}(k+1) &= \underline{a}(k) + 1_{i(k)}, & i(k+1) &= \min\{j \mid \underline{a}(k+1)_j / w_j\}. \end{aligned}$$

For $k \in \{0, \dots, \mu-1\}$, we put $\omega_k := w^{\underline{a}(k_{\min(s(k))) - \underline{a}(k)}} u^{\underline{a}(k)} \omega_0$ where $u^{\underline{a}(k)} := u_1^{\underline{a}(k)_1} \dots u_n^{\underline{a}(k)_n}$ and $w^{\underline{a}(k)} := w_1^{\underline{a}(k)_1} \dots w_n^{\underline{a}(k)_n}$. The classes of $\omega_0, \dots, \omega_{\mu-1}$ form a $\mathbb{C}[\theta]$ -basis of G_0 , denoted by ω . This basis induces a basis, denoted by $[\omega]$, of the vector space $G_0/\theta G_0$. The product structure on the Jacobian quotient $\mathcal{O}(U)/(\partial f)$ is carried to $G_0/\theta G_0$ through the isomorphism $\varphi \mapsto \varphi \omega_0$.

Proposition 5.4. *In the basis $[\omega]$ of $G_0/\theta G_0$, the product is given by*

$$[\omega_i] \star [\omega_j] = w^{\underline{a}(k_{\min(s(i))) + \underline{a}(k_{\min(s(j)))} - \underline{a}(k_{\min(s(\overline{i+j}))}) + \underline{a}(\overline{i+j}) - \underline{a}(i+j)} [\omega_{\overline{i+j}}]$$

where $\overline{i+j}$ is the sum modulo μ .

Remark 5.5. This proposition will be used in Section 6.b in order to prove the classical correspondence. In fact, we will define a product on the graded ring $\text{gr}_\star^\mathcal{N}(G_0/\theta G_0)$ where $\mathcal{N}_\bullet$ is the Newton filtration of f .

According to Section 4 of [DS04], the metric g on $G_0/\theta G_0$ in the basis $[\omega]$ is given by Formula (5.2).

We define

$$(5.6) \quad ([a], [b], [c]) := g([a] \star [b], [c])$$

for any $[a], [b], [c]$ in $G_0/\theta G_0$. Proposition 5.4 and Formula (5.2) imply

Corollary 5.7. *Let j, k be in $\{0, \dots, \mu-1\}$.*

- (1) *If $\overline{1+j+k} \neq n$, then $(([\omega_1], [\omega_j], [\omega_k])) = 0$.*
- (2) *If $\overline{1+j+k} = n$, then*

$$(([\omega_1], [\omega_j], [\omega_k])) = \begin{cases} \prod_{i \in I(j,k)} w_i^{-1} & \text{if } \sigma(1) + \sigma(j) + \sigma(k) \neq n, \\ \prod_{i \in I(s(j))} w_i^{-1} & \text{if } \sigma(1) + \sigma(j) + \sigma(k) = n, \end{cases}$$

where $I(j, k) := I(s(j)) \sqcup I(s(k))$.

Remark 5.8. (1) This corollary will be useful for proving the quantum correspondence in Section 6.c. Because the bilinear form g is nondegenerate, one can reconstruct the multiplication by $[\omega_1]$ from this corollary and the bilinear form g .

- (2) Let us remark that the numbers $A_{1jk}(\mathbf{0})$ in Theorem 5.13 are exactly $(([\omega_1], [\omega_j], [\omega_k]))$.

5.b. Potential of the Frobenius structure. In this section we study the potential of the Frobenius structure, and we show that the potential is determined by some numbers (see Theorem 5.13).

Let X be the base space of a universal unfolding of f . Let $t_0, \dots, t_{\mu-1}$ be the flat coordinates in a neighborhood of 0 in X . We define the Euler field by

$$(5.9) \quad \mathfrak{E} = \sum_{k=0}^{\mu-1} (1 - \sigma(k)) t_k \partial_{t_k} + \mu \partial_{t_1}.$$

We develop the potential of the Frobenius structure in series, and we denote it by

$$F^{sing}(\mathbf{t}) = \sum_{\alpha_0, \dots, \alpha_{\mu-1} \geq 0} A(\boldsymbol{\alpha}) \frac{t^\alpha}{\boldsymbol{\alpha}!},$$

where $\boldsymbol{\alpha} := (\alpha_0, \dots, \alpha_{\mu-1})$ and $\frac{t^\alpha}{\boldsymbol{\alpha}!} := \frac{t_0^{\alpha_0}}{\alpha_0!} \cdots \frac{t_{\mu-1}^{\alpha_{\mu-1}}}{\alpha_{\mu-1}!}$. We define $|\alpha| := \alpha_0 + \dots + \alpha_{\mu-1}$ and we call it the *length* of $\boldsymbol{\alpha}$. We denote by (g^{ab}) the inverse matrix of the nondegenerate pairing g in the coordinates $\mathbf{t}$. For any $i, j, k, \ell \in \{0, \dots, \mu-1\}$, the potential satisfies the following conditions:

(5.10)

$$(i, j, k, \ell) : \sum_{a=0}^{\mu-1} F_{ija}^{sing} g^{aa^*} F_{a^*k\ell}^{sing} = \sum_{a=0}^{\mu-1} F_{jka}^{sing} g^{aa^*} F_{a^*i\ell}^{sing} \quad (\text{WDVV equations}),$$

$$(5.11) \quad \mathfrak{E} \cdot F^{sing} = (3 - n) F^{sing} \quad \text{up to quadratic terms},$$

$$(5.12) \quad F_{ijk}^{sing}(\mathbf{0}) = g|_{\mathbf{t}=\mathbf{0}} (\partial_{t_i} \star \partial_{t_j}, \partial_{t_k}) \text{ where } F_{ijk}^{sing} := \frac{\partial^3 F^{sing}}{\partial t_i \partial t_j \partial t_k}.$$

Note that $g|_{\mathbf{t}=\mathbf{0}} (\partial_{t_i} \star \partial_{t_j}, \partial_{t_k}) = ((\tilde{\omega}_i, \tilde{\omega}_j, \tilde{\omega}_k))$. Denote by $A_{ijk}(\boldsymbol{\alpha})$ the number $A(\alpha_0, \dots, \alpha_i + 1, \dots, \alpha_j + 1, \dots, \alpha_k + 1, \dots, \alpha_{\mu-1})$.

Theorem 5.13. *The potential F^{sing} is determined by the numbers $A_{1jk}(\mathbf{0})$ with $j, k \in \{0, \dots, \mu-1\}$ such that $\overline{1+j+k} = n$.*

Remark 5.14. (1) If $\overline{1+j+k} \neq n$, then Condition (5.12) implies that $A_{1jk}(\mathbf{0}) = 0$. This condition is exactly Proposition 4.14 on the A side.

(2) If we interpret this theorem on the A side, this means that we have an algorithm to reconstruct the full quantum cohomology from the small one. Note that recently, M. Rose has proved in [Ros06] a general reconstruction theorem for smooth Deligne-Mumford stacks. As the small quantum cohomology of weighted projective spaces is generated by $H^2(|\mathbb{P}(w)|, \mathbb{Q})$ (cf. Corollary 1.2 of [CCLT06]), Theorem 0.3 of [Ros06] implies that all genus zero Gromov-Witten invariants can be reconstructed from the 3-point invariants.

Proof. First, we will show that the potential is determined by the numbers $A_{ijk}(\mathbf{0})$ for any $i, j, k \in \{0, \dots, \mu - 1\}$.

We will show this by induction on the length of the numbers $A(\alpha)$. For any $i, j, k, \ell \in \{0, \dots, \mu - 1\}$, the term of $F_{ija}^{sing} g^{aa^*} F_{a^*k\ell}^{sing}$ between $\frac{t^\alpha}{\alpha!}$ is

$$g^{aa^*} \sum_{\beta+\gamma=\alpha} \binom{\beta_0}{\alpha_0} \dots \binom{\beta_{\mu-1}}{\alpha_{\mu-1}} A_{ija}(\beta) A_{a^*k\ell}(\gamma).$$

Hence, the terms of the greatest length, that is of length $|\alpha| + 3$, in the above sum are $g^{aa^*} A_{ija}(\alpha) A_{a^*k\ell}(\mathbf{0})$ and $g^{aa^*} A_{ija}(\mathbf{0}) A_{a^*k\ell}(\alpha)$. As the potential satisfies Conditions (5.12), we deduce that $A_{ija}(\mathbf{0}) \neq 0$ if and only if $a = \overline{i+j}^*$.

In the WDVV equation $(1, j, k, \ell)$, the terms of length $|\alpha| + 3$ in front of $\frac{t^\alpha}{\alpha!}$ are

$$\begin{aligned} g^{\overline{1+j}, \overline{1+j}^*} A_{1j\overline{1+j}^*}(\mathbf{0}) A_{\overline{1+j}k\ell}(\alpha), & \quad g^{\overline{k+\ell}, \overline{k+\ell}^*} A_{1j\overline{k+\ell}}(\alpha) A_{\overline{k+\ell}^*k\ell}(\mathbf{0}), \\ g^{\overline{j+k}, \overline{j+k}^*} A_{jk\overline{j+k}^*}(\mathbf{0}) A_{\overline{j+k}1\ell}(\alpha), & \quad g^{\overline{1+\ell}, \overline{1+\ell}^*} A_{jk\overline{1+\ell}}(\alpha) A_{\overline{1+\ell}^*1\ell}(\mathbf{0}). \end{aligned}$$

The terms of the form $A_{1??}(\mathbf{0})$ are computed by (5.12), and the homogeneity condition (5.11) implies that

$$A(\alpha_0, \alpha_1 + 1, \alpha_2, \dots, \alpha_{\mu-1}) = \frac{1}{\mu} A(\alpha) d(\alpha) \text{ for } |\alpha| \geq 3$$

where $d(\alpha) = 3 - n + \sum_{k=0}^{\mu-1} \alpha_k(\sigma(k) - 1)$. Hence, we can express the numbers $A_{1??}(\alpha)$ with numbers of strictly smaller length. The WDVV equation $(1, j, k, \ell)$ gives a relation between $A_{\overline{1+j}k\ell}(\alpha)$ and $A_{jk\overline{1+\ell}}(\alpha)$. We conclude that after a finite number of steps, we can express any number $A_{ijk}(\alpha)$ with terms of strictly smaller length. By induction we have that the potential is determined by the numbers $A_{ijk}(\mathbf{0})$ for any $i, j, k \in \{0, \dots, \mu - 1\}$.

In order to finish the proof, it is enough to show that we can compute any numbers $A_{ijk}(\mathbf{0})$ from the numbers of the form $A_{1??}(\mathbf{0})$. The numbers of length 3 in the equation $(1, j, k, \ell)$ are nonzero if and only if $\overline{1+j} + \overline{k} + \overline{\ell} = n$. Under this condition, we have $\overline{1+j} = \overline{k+\ell}^*$ and $\overline{j+k}^* = \overline{1+\ell}$. Hence, the terms of length 3 in the equation $(1, j, k, \ell)$ are

$$A_{1j\overline{1+j}^*}(\mathbf{0}) A_{\overline{1+j}k\ell}(\mathbf{0}) \quad \text{and} \quad A_{jk\overline{1+\ell}}(\mathbf{0}) A_{\overline{1+\ell}^*1\ell}(\mathbf{0}).$$

Considering successively the WDVV equations $(1, j-1, k, \ell+1)$, $(1, j-2, k, \ell+2), \dots$, we can express $A_{\overline{1+j}k\ell}(\mathbf{0})$ in terms of the numbers of the form $A_{1??}(\mathbf{0})$. $\square$

6. Correspondences

6.a. Combinatorics of numbers σ . We define an order on the circle S^1 by choosing the principal determination of the argument. Choose a non-decreasing bijection $\tilde{s} : \{0, \dots, \mu - 1\} \rightarrow \bigsqcup \mu_{w_i}$. For any $g \in \bigsqcup \mu_{w_i}$, we put

$$k_{\min}(g) := \min\{i \in \{0, \dots, \mu - 1\} \mid \tilde{s}(i) = g\}.$$

For any $g \in \bigsqcup \mu_{w_i}$, we define $\gamma(g)$ to be the unique element in $[0, 1[$ such that $\exp(2i\pi\gamma(g)) = g$. Recall that we have defined $\sigma(\cdot)$ in Notation 5.1. We have that $\sigma(i) = i - \mu\gamma(\tilde{s}(i))$.

The following proposition is straightforward.

Proposition 6.1. (1) For any $g \in \bigsqcup \mu_{w_i}$, we have

$$k_{\min}(g) = \text{codim } \mathbb{P}(w)_{(g)} + \sum [\gamma(g)w_i]$$

where $[\cdot]$ means the integer part.

(2) For any $g \in \bigsqcup \mu_{w_i}$ and $d \in \{0, \dots, \dim \mathbb{P}(w)_{(g)}\}$, we have

$$2\sigma(k_{\min}(g^{-1}) + d) = \deg(\eta_g^d).$$

(3) For any $g, g' \in \bigsqcup \mu_{w_i}$ and $(d, d') \in \{0, \dots, \dim \mathbb{P}(w)_{(g)}\} \times \{0, \dots, \dim \mathbb{P}(w)_{(g')}\}$, we have the following equivalence:

$$\begin{aligned} k_{\min}(g^{-1}) + d + k_{\min}(g'^{-1}) + d' &= n \pmod{\mu} \\ \Leftrightarrow \begin{cases} g \cdot g' = \text{id}, \\ d + d' = \dim \mathbb{P}(w)_{(g)}. \end{cases} \end{aligned}$$

6.b. Proof of the classical correspondence. Denote by $\mathcal{N}_\bullet G$ the Newton filtration of the Gauss-Manin system G (see Paragraph 2.e of [DS03]). This filtration induces a filtration on $G_0/\theta G_0$, denoted by $\mathcal{N}_\bullet(G_0/\theta G_0)$ or just $\mathcal{N}_\bullet$ when there is no ambiguity. Let Ξ be the $\mathbb{C}$ -linear map defined by

$$\begin{aligned} \Xi : H_{\text{orb}}^{2\star}(\mathbb{P}(w), \mathbb{C}) &\longrightarrow \text{gr}_\star^\mathcal{N}(G_0/\theta G_0), \\ \eta_g^d &\longmapsto \llbracket \omega_{k_{\min}(g^{-1})+d} \rrbracket \end{aligned}$$

where $\llbracket \cdot \rrbracket$ means the class in $\text{gr}_\star^\mathcal{N}(G_0/\theta G_0)$. The nondegenerate bilinear form $g(\cdot, \cdot)$ on $G_0/\theta G_0$ induces a nondegenerate bilinear form, denoted by $\llbracket g \rrbracket(\cdot, \cdot)$, on $\text{gr}_\star^\mathcal{N}(G_0/\theta G_0)$. Because for any $\beta_1, \beta_2 \in \mathbb{Q}$, we have that $\mathcal{N}_{\beta_1}(G_0/\theta G_0) \star \mathcal{N}_{\beta_2}(G_0/\theta G_0)$ is included in $\mathcal{N}_{\beta_1+\beta_2}(G_0/\theta G_0)$ (see Proposition VI.3.1 of [Man05]), we can define a product, denoted by $\cup$, on $\text{gr}_\star^\mathcal{N}(G_0/\theta G_0)$.

Theorem 6.2. The map Ξ is a graded isomorphism between the graded Frobenius algebras $(H_{\text{orb}}^{2\star}(\mathbb{P}(w), \mathbb{C}), \cup, \langle \cdot, \cdot \rangle)$ and $(\text{gr}_\star^\mathcal{N}(G_0/\theta G_0), \cup, \llbracket g \rrbracket(\cdot, \cdot))$.

Proof. According to Corollary 3.11, we have $\deg(\eta_\gamma^d) = 2(d + \text{age}(g))$. Proposition 6.1(2) implies that $\Xi(\eta_\gamma^d)$ is in the graded $\text{gr}_{d+a(\gamma)}^{\mathcal{N}}(G_0/\theta G_0)$. We conclude that Ξ is a graded map. On the one hand, Proposition 3.13, Formula (5.2) and Proposition 6.1(3) imply that $\langle \eta_\gamma^d, \eta_{\gamma'}^{d'} \rangle = \llbracket g \rrbracket(\Xi(\eta_\gamma^d), \Xi(\eta_{\gamma'}^{d'}))$. On the other hand, Corollary 3.18 and Proposition 5.4 imply that $\Xi(\eta_{\gamma_0}^{d_0} \cup \eta_{\gamma_1}^{d_1}) = \Xi(\eta_{\gamma_0}^{d_0}) \cup \Xi(\eta_{\gamma_1}^{d_1})$. $\square$

6.c. Proof of the quantum correspondence. Let $\tilde{\Xi}$ be the $\mathbb{C}$ -linear map defined by

$$\begin{aligned} \tilde{\Xi} : QH_{\text{orb}}^{2\star}(\mathbb{P}(w), \mathbb{C}) &\longrightarrow G_0/\theta G_0, \\ \eta_g^d &\longmapsto [\omega_{k_{\min}(g^{-1})+d}]. \end{aligned}$$

This map is an isomorphism of vector spaces.

Corollary 6.3. *Let $w_0, \dots, w_n$ be integers. The Frobenius manifolds associated to the Laurent polynomial f and the orbifold $\mathbb{P}(w)$ are isomorphic.*

Proof. Since the Frobenius manifold associated to f is semi-simple (cf. Theorem 5.3) it is enough to show that both Frobenius manifolds carry the same initial conditions.

Theorem 5.3 gives us the initial condition $(A_0^\circ, A_\infty, e_0, g)$ for the Frobenius manifold associated to the Laurent polynomial f . Theorem 6.2 implies that we have the same matrix A_∞ , the same eigenvector e_0 for the eigenvalue $q = 0$ and the same bilinear nondegenerate form.

We have to compare the matrices A_0° which correspond to the multiplication by the Euler fields at the origin. Formulas (3.23) and (5.9) show that the Euler fields are the same. Corollary 4.19, via Propositions 4.14 and 4.17, and Corollary 5.7 imply that for any $g, g' \in \bigcup \mu_{w_i}$ and $(d, d') \in \{0, \dots, \dim \mathbb{P}(w)_g\} \times \{0, \dots, \dim \mathbb{P}(w)_{(g')}\}$, we have

$$\begin{aligned} \langle \eta_1^1 \star \eta_g^d, \eta_{g'}^{d'} \rangle &= (\langle \eta_1^1, \eta_g^d, \eta_{g'}^{d'} \rangle) \\ &= (\langle \tilde{\Xi}(\eta_1^1), \tilde{\Xi}(\eta_g^d), \tilde{\Xi}(\eta_{g'}^{d'}) \rangle) \\ &= g(\tilde{\Xi}(\eta_1^1) \star \tilde{\Xi}(\eta_g^d), \tilde{\Xi}(\eta_{g'}^{d'})). \end{aligned}$$

Moreover, Proposition 3.13 and Formula (5.2) imply that we have $\langle \eta_g^d, \eta_{g'}^{d'} \rangle = g(\tilde{\Xi}(\eta_g^d), \tilde{\Xi}(\eta_{g'}^{d'}))$. Hence, we deduce that the multiplications by the Euler field at the origin are the same. $\square$

7. Appendix : Small quantum cohomology of weighted projective spaces

In [CCLT06], T. Coates, A. Corti, Y.-P. Lee and H.-H. Tseng have computed the small quantum cohomology of weighted projective spaces. We recall their results with our notation.

Denote the elements of $\bigcup \mu_{w_i}$ by $1 = g_0 < \dots < g_\delta$ where the order is defined by choosing the principal determination of the argument (*cf.* Notation 3.24). Recall that for any $g \in \bigcup \mu_{w_i}$, there exists a unique $\gamma(g) \in [0, 1[$ such that $g = \exp(2i\pi\gamma(g))$.

For any $k \in \{0, \dots, \delta\}$, put

$$s_k = \begin{cases} 1 & \text{if } k = 1, \\ \frac{\prod_{\gamma(g_m) < \gamma(g_k)} (\gamma(g_k) - \gamma(g_m))^{\dim \mathbb{P}(w)_{(g_m)} + 1}}{\prod_{i=0}^n (\gamma(g_j)w_i)^{\lceil \gamma(g_k)w_i \rceil}} & \text{otherwise} \end{cases}$$

where we put

$$x^{\underline{n}} = x(x-1)\dots(x-n+1).$$

According to Corollary 1.2 of [CCLT06], the small quantum cohomology of weighted projective spaces is generated by η_1^1 and η_g^0 for any $g \in \bigcup \mu_{w_i}$. The relations are

$$(7.1) \quad \begin{aligned} (\eta_1^1)^{k_{\min}(g_k)} &= Q^{\gamma(g_k)} s_k \eta_{g_k}^0, \\ \underbrace{\eta_1^1 \star \dots \star \eta_1^1}_{\sum w_i} &= Q \prod w_i^{-w_i} \end{aligned}$$

where $k_{\min}(g) := \sum [w_i \gamma(g)] + \text{codim } \mathbb{P}(w)_{(g)}$ (*cf.* Section 6.a for another interpretation of $k_{\min}$) and Q is a formal variable of degree μ . The careful reader will notice that Equation (7.1) is not exactly the one of Corollary 1.2 in [CCLT06]. Indeed, they define weighted projective spaces differently; that is, as the quotient stack of $[\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*]$ where $\mathbb{C}^*$ acts with weights $-w_0, \dots, -w_n$.

Proposition 7.2. *For any $k \in \{0, \dots, \delta\}$, we have that*

$$\eta_1^1 \star \eta_{g_{k-1}}^{\dim \mathbb{P}(w)_{(g_{k-1})} - 1} = \eta_{g_k}^0 Q^{\gamma(g_k) - \gamma(g_{k-1})} \prod_{i \in I(g_{k-1})} w_i^{-1}.$$

Proof. First, we will show that for any $k \in \{0, \dots, \delta\}$, we have

$$(7.3) \quad s_k = \prod_{i=0}^n w_i^{-\lceil \gamma(g_k)w_i \rceil}.$$

For any $k \in \{0, \dots, \delta\}$ and any $i \in \{0, \dots, n\}$, we have

$$\frac{\lceil \gamma(g_k)w_i \rceil - 1}{w_i} < \gamma(g_k) \leq \frac{\lceil \gamma(g_k)w_i \rceil}{w_i}.$$

We deduce that

$$\prod_{i=0}^n (\gamma(g_j)w_i)^{\lceil \gamma(g_k)w_i \rceil} = w_i^{\lceil \gamma(g_k)w_i \rceil} \prod_{\substack{\ell \mid w_i \\ \ell/w_i < \gamma(g_k)}} \left(\gamma(g_k) - \frac{\ell}{w_i} \right).$$

We deduce Formula (7.3).

Put $d(g_k) := \dim \mathbb{P}(w)_{(g_k)}$. We have

$$\eta_{g_{k-1}}^{d(g_{k-1})} = \eta_{g_{k-1}}^0 \star (\eta_1^1)^{d(g_{k-1})}.$$

Section 6.a and Proposition 6.1 imply that $k_{\min}(g_{k-1}) + d(g_{k-1}) + 1 = k_{\min}(g_k)$.

We deduce that

$$\begin{aligned} \eta_1^1 \star \eta_{g_{k-1}}^{d(g_{k-1})} &= (\eta_1^1)^{k_{\min}(g_k)} Q^{-\gamma(g_{k-1})} \prod_{i=0}^n w_i^{\lceil \gamma(g_{k-1})w_i \rceil} \\ &= \eta_{g_k}^0 Q^{\gamma(g_k) - \gamma(g_{k-1})} \prod_{i=0}^n w_i^{\lceil \gamma(g_{k-1})w_i \rceil - \lceil \gamma(g_k)w_i \rceil}. \end{aligned}$$

Then the following lemma finishes the proof. □

Lemma 7.4. *For any $i \in \{0, \dots, n\}$, we have*

$$\lceil w_i \gamma(g_{k-1}) \rceil - \lceil w_i \gamma(g_k) \rceil = \begin{cases} -1 & \text{if } i \in I(g_{k-1}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By choosing the principal determination of the argument, we order the elements in $\bigcup \mu_{w_i}$ by $1 = g_0 < g_1 < \dots < g_d$. Hence we have

$$0 < w_i \gamma(g_1) < \dots < 1 < \dots < 2 < \dots < w_i - 1 < \dots < w_i \gamma(g_d).$$

The above formula implies the following alternative:

- If $w_i \gamma(g_{k-1}) \in \mathbb{N}$ (i.e. $i \in I(g_{k-1})$), we have $\lceil w_i \gamma(g_{k-1}) \rceil - \lceil w_i \gamma(g_k) \rceil = -1$.
- If $w_i \gamma(g_k) \in \mathbb{N}$, we have $\lceil w_i \gamma(g_{k-1}) \rceil - \lceil w_i \gamma(g_k) \rceil = 0$.
- If $w_i \gamma(g_{k-1}), w_i \gamma(g_k) \notin \mathbb{N}$, we have $\lceil w_i \gamma(g_{k-1}) \rceil - \lceil w_i \gamma(g_k) \rceil = 0$.

□

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