ON COMBINATORIAL TYPES OF CYCLES UNDER THE MULTIPLICATION BY k MAP OF THE CIRCLE.

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In memory of Tan Lei

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Let \( m_k : T \to T := \mathbb{R}/\mathbb{Z} \) denote the multiplication by \( k \geq 2 \) map of the circle

\[
m_k(x) = kx \pmod{\mathbb{Z}}.
\]

The central question of this work is whether a given combinatoric \( \sigma \in C_q \) and or combinatorial type \( \tau \) in \( C_q \) has a realization under \( m_k \) and if it does, how many such realizations there are.
There is a natural way to associate to each $q$-periodic point $z$ for $m_k$ belonging to a $q$-cycle $0 < z_1 < \ldots < z_q < 1$, say $z = z_j$, a pair of $q$-periodic points $(x_j, y_j)$ characterized as follows:

- $y_j - x_j = \frac{(k + 1)^{q - 1}}{(k + 1)^q - 1}$,
- Their cycles are interlaced
  
  $0 < x_1 < y_1 < x_2 < y_2 < \ldots x_q < y_q < 1$

- There is a monotone projection $P : \mathbb{T} \to \mathbb{T}$ with $P(0) = 0$, $P([x_j, y_j]) = z_j$ and semi-conjugating $m_{k+1}$ to $m_k$ on $\mathbb{T} \setminus [x_j, y_j]$. 
Connectedness locus for $\lambda z^2 + z^3$

e.g. $z = \frac{3}{5}$ with $m_2$-orbit

$$\left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\}$$

gives $(x_3, y_3) = (\frac{29}{80}, \frac{56}{80})$

with $m_3$ orbits

$$\left\{ \frac{7}{80}, \frac{21}{80}, \frac{29}{80}, \frac{63}{80} \right\}$$

and

$$\left\{ \frac{8}{80}, \frac{24}{80}, \frac{56}{80}, \frac{72}{80} \right\}.$$
I view the above as saying that for every periodic point \( z \) for \( m_k \) there is a pair of neighbouring periodic orbits for \( m_{k+1} \) with the same combinatorics and with critical interval corresponding to \( z \).

This motivates the following questions:

- Which combinatorics exists for \( m_{k+1} \), but does not exist for \( m_k \)?
- How does the number of orbits with a given combinatorics grow with the degree \( k \)?
- For rotation orbits with rational rotation number the answers to these questions are known.
- In fact for each irreducible rotation number \( p/q \), \( m_2 \) has a unique such orbit and Goldberg showed that in the general case, the number of such orbits is given by

\[
\binom{q + k - 2}{q}
\]
Cyclic Permutations

- We shall use cyclic permutations to represent combinatorics of periodic orbits on the circle $\mathbb{T}$.
- Denote by $S_q$ the group of permutations of $q$ symbols, which we take to be the representatives $\{1, \ldots, q\}$ of the cyclically ordered set $\mathbb{Z}/q\mathbb{Z}$.
- Denote by $C_q \subset S_q$ the set of $q$-cycles $\sigma$ in $S_q$:
  \[
  \sigma = (1 \sigma(1) \sigma^2(1) \ldots \sigma^{q-1}(1))
  \]
- And denote by $R_q \subset S_q$ the rotation group, that is the group generated by the $q$-cycle
  \[
  \rho = (1 \ 2 \ \ldots \ q)
  \]
  with rotation number $1/q$. 
What is a combinatorics?

Consider again the "Cocapeli"-orbit \( \{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \} \) under \( m_2 \).

\[
x_2 = \frac{2}{5} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
We shall use $\sigma = (1 2 4 3)$ as a synonym for the combinatorics of the orbit $\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \}$ under $m_2$.

More generally if $0 < x_1 < x_2 < \ldots < x_q < 1$ and

$$f : \{x_1, \ldots, x_q\} \longrightarrow \{x_1, \ldots, x_q\}$$

is a cyclic dynamics we shall say that the orbit $\{x_1, \ldots, x_q\}$ has combinatorics $\sigma \in \mathcal{C}_q$ iff

$$\forall i : f(x_i) = x_{\sigma(i)}.$$

And we shall call any $\sigma \in \mathcal{C}_q$ a $q$-combinatorics.
A few numbers

- For each $q$ the number of $q$-combinatorics is $(q - 1)!$.
- For each $k \geq 2$ and $q$ there are at most $k^q \frac{k^q}{q}$ periodic orbits for $m_k$ of period $q$.
- So for each fixed $k$ and sufficiently large $q$ the majority of the $q$-combinatorics are not realized by $m_k$.
- The next slide shows as examples the four possible non-rotational 4-combinatorics
\[ \sigma_1 = (1 \ 2 \ 4 \ 3) \]
\[ \sigma_2 = (1 \ 4 \ 2 \ 3) \]
\[ \sigma_3 = (1 \ 3 \ 4 \ 2) \]
\[ \sigma_4 = (1 \ 3 \ 2 \ 4) \]
Only $\sigma_1$ is realized by $m_2$, uniquely by our "Cocapeli"-orbit \{ $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ \}.

The others however are each uniquely realized by $m_3$:

\[
\begin{align*}
\sigma_2 &= (1 \ 4 \ 2 \ 3) : \left\{ \frac{23}{80}, \frac{47}{80}, \frac{61}{80}, \frac{69}{80} \right\} \\
\sigma_3 &= (1 \ 3 \ 4 \ 2) : \left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\} \\
\sigma_4 &= (1 \ 3 \ 2 \ 4) : \left\{ \frac{11}{80}, \frac{19}{80}, \frac{33}{80}, \frac{57}{80} \right\}
\end{align*}
\]
A 5-cycle example

\[ \sigma = (1 \ 2 \ 4 \ 5 \ 3) \]

This combinatorics is not realized by \( m_2 \) either.

It is however uniquely realized by \( m_3 \):

\[ \sigma = (1 \ 2 \ 4 \ 5 \ 3) : \left\{ \frac{8}{121}, \frac{24}{121}, \frac{43}{121}, \frac{72}{121}, \frac{95}{121} \right\} \]
Intervals in $\mathbb{Z}/q\mathbb{Z}$ and "lengths"

**Definition**

For $1 \leq i, j \leq q$ define the closed interval $[i, j]$ in $\mathbb{Z}/q\mathbb{Z}$ as:

$$[i, j] = \begin{cases} 
\{i, i + 1, \ldots, j\} & \text{if } i < j, \\
\{i, (i + 1), \ldots, (j + q)\} & \text{if } j < i.
\end{cases}$$

And the length $|[i, j]| := \#[i, j] - 1$ so that

$$|[i, j]| = j - i \text{ if } i \leq j \text{ and } |[i, j]| = j + q - i \text{ if } j < i.$$

All subsets of $\mathbb{Z}/q\mathbb{Z}$ are closed but we shall use the notion $[i, j)$ to indicate the "open interval $[i, j]$ minus the right end point.
The degree of a cycle.

**Definition**

For $\sigma \in \mathcal{C}_q$ define $\text{deg}(\sigma)$ as the integer:

$$\text{deg}(\sigma) = \frac{1}{q} \sum_{j=1}^{q} |[\sigma(j), \sigma(j+1)]|$$

- The degree of $\sigma$ is equal to the descent number $\text{des}(\sigma)$ of the permutation $\sigma$ as defined in combinatorial analysis.
- $\text{deg}(1243) = \text{deg}(1423) = \text{deg}(1342) = \text{deg}(1324) = 2$
- $\text{deg}(12453) = 3$
- $\text{deg}(\sigma) = 1$ if and only if $\sigma$ is a rotation cycle.
Example $\sigma = (1 \ 2 \ 4 \ 5 \ 3)$
Topological realization

Definition

A (topological) realization of the cycle $\sigma \in C_q$ is a pair $(f, \mathcal{O})$, where $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a positively oriented covering map, $\mathcal{O} = \{x_1, \ldots, x_q\}$, $0 < x_1 < \ldots, x_q < 1$ is a period $q$ orbit of $f$, and $f(x_i) = x_{\sigma(i)}$ for all $i$.

The degree of the realization $(f, \mathcal{O})$ is the mapping degree of $f$.

A realization of $\sigma$ is minimal if it has the smallest possible degree among all realizations.

- For any $x \neq y \in \mathbb{T}$ let $[x, y]$ denote the closed interval in $\mathbb{T}$ with end points $x, y$ such that for any $z$ in the corresponding open interval $]x, y[$ the triple $(x, z, y)$ is positively oriented.
- Equivalently let $\Pi : \mathbb{R} \longrightarrow \mathbb{T}$ denote the natural projection. Then $[x, y] = \Pi([\hat{x}, \hat{y}])$, where $\Pi(\hat{x}) = x$ and $\Pi(\hat{y}) = y$ and $\hat{x} < \hat{y} < \hat{x} + 1$. 

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Combinatorics of cycles
In memory of Tan Lei
For \((f, O)\) a topological realization of \(\sigma\) and any \(j \in \mathbb{Z}/q\mathbb{Z}\) the restriction of \(f\) to \(I_j := [x_j, x_{j+1}]\) lifts into \(\Pi\) as a homeomorphism \(\hat{f}_j : [x_j, x_{j+1}] \rightarrow [\hat{x}_\sigma(j), \hat{x}_\sigma(j+1)]\), where \(\hat{x}_\sigma(j) < \hat{x}_\sigma(j+1)\), \(\Pi(\hat{x}_\sigma(j)) = x_\sigma(j)\) and \(\Pi(\hat{x}_\sigma(j+1)) = x_\sigma(j+1)\).

It follows that \((f, O)\) is minimal iff \(\hat{x}_\sigma(j) < \hat{x}_\sigma(j+1) < \hat{x}_\sigma(j) + 1\) for each \(j\).

Or in other words \((f, O)\) is minimal only if for each \(j\)

\[
f(I_j) = [f(x_j), f(x_{j+1})] = [x_\sigma(j), x_\sigma(j+1)] = \bigcup_{i \in [\sigma(j), \sigma(j+1))} l_i.
\]

Thus \((f, O)\) is minimal if and only if \(\text{deg}(f) = \text{deg}(\sigma)\)

McMullen observed that a minimal realization of \(\sigma\) always exists:

Take any \(q\) points with \(0 < x_1 < \ldots < x_q < 1\) as \(O\) and let \(f\) be any map which for each \(j\) maps \([x_j, x_{j+1}]\) homeomorphically onto \([x_\sigma(j), x_\sigma(j+1)]\).
Minimal realization of $\sigma = (1 \ 2 \ 4 \ 5 \ 3)$
We see immediately why \( \sigma = (1 \ 2 \ 4 \ 5 \ 3) \) is not realized by \( m_2 \). It has degree 3 and thus any realizing map must have topological degree at least 3.

The four non-rotational period 4 combinatorics \( \sigma_1 = (1 \ 2 \ 4 \ 3) \), \( \sigma_2 = (1 \ 4 \ 2 \ 3) \), \( \sigma_3 = (1 \ 3 \ 4 \ 2) \) and \( \sigma_4 = (1 \ 3 \ 2 \ 4) \) are mutually conjugate by powers of the rotation \( \rho = (1 \ 2 \ 3 \ 4) \) and have degree 2. But only \( \sigma_1 \) is realized by \( m_2 \). Why is this?

Notice that 0 is a fixed point for any \( m_k \). Thus in general \( I_q \) must be mapped over itself, and in fact onto a larger interval in order for \( \sigma \in C_q \) to be realized by \( m_k \).

This means for a \( \sigma \in C_q \) of degree \( d \) to be realized by \( m_d \) we must have \( I_q = [x_q, x_1] \subset [x_{\sigma(q)}, x_{\sigma(1)}] \) or equivalently

\[
\sigma(1) < \sigma(q).
\]
We have thus arrived at

**Proposition**

*A necessary condition for a combinatoric* $\sigma \in \mathcal{C}_q$ *to be realized by* $m_k$ *is that*

$$\deg(\sigma) \leq k \quad \text{and} \quad \sigma(1) < \sigma(q).$$

The following theorem shows that these conditions are also sufficient.
Realization under $m_d$. I

**Theorem (Zakeri and P.)**

Let $\sigma \in \mathbb{C}_q$ be a $q$-cycle with $\deg(\sigma) = d \geq 2$.

- If $\sigma(1) < \sigma(q)$ then $\sigma$ has a realization under $m_d$ and
- if $\sigma(1) > \sigma(q)$ then $\sigma$ has a realization under $m_{d+1}$.

In both cases the realisation is unique.
Realization under $m_d$. II

**Theorem (Zakeri and P.)**

Let $\sigma \in \mathcal{C}_q$ be a $q$-cycle with $\deg(\sigma) = d \geq 2$ and let $k \geq d$. Then the number of realizations of $\sigma$ under $m_k$ is given by the binomial coefficient:

$$\begin{align*}
\binom{q+k-d}{q} &\text{ if } \sigma(1) < \sigma(q) \\
\binom{q+k-d-1}{q} &\text{ if } \sigma(1) > \sigma(q).
\end{align*}$$

- Note that for $d = 1$ (rotation cycles) and $k \geq 2$ this agrees with Goldbergs formula.
- I shall focus on the proof that a $q$-cycle $\sigma \in \mathcal{C}_q$ with $\deg(\sigma) = d \geq 2$ and $\sigma(1) < \sigma(q)$ is realised under $m_d$. 
The transition matrix of $\sigma$.

**Definition**

The *transition matrix* of $\sigma \in \mathcal{C}_q$ is the $q \times q$ matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 
1 & \text{if } j \in [\sigma(i), \sigma(i + 1)) \\
0 & \text{otherwise.}
\end{cases}$$

We may also view the transition matrix $A$ geometrically:

Let $(f, \mathcal{O})$ be a(ny) minimal realization of $\sigma$, where $\mathcal{O} = \{x_1, \ldots, x_q\}$ and as usual $0 < x_1 < \ldots < x_q < 1$. Then we saw above that

$$f(I_i) = \bigcup_{j \in [\sigma(i), \sigma(i + 1))} I_j$$

for all $i$, where $I_j = [x_j, x_{j+1}]$. 

It follows that the entries of the transition matrix $A = [a_{ij}]$ satisfy

$$a_{ij} = \begin{cases} 1 & \text{if } f(I_i) \supset I_j \\ 0 & \text{otherwise.} \end{cases}$$

Since $f$ is a covering map of degree $d$, every column of the transition matrix $A$ contains exactly $d$ entries of 1.

The column stochastic matrix $\frac{1}{d} \cdot A$ describes a Markov chain with states $I_1, \ldots, I_q$, with the probability of going from $I_j$ to $I_i$ equal to $1/d$ if $I_j \subset f(I_i)$ and equal to 0 otherwise.
The Transition matrix and iteration

- Let $A$ be the transition matrix of a $q$-cycle $\sigma$.
- Let $(f, \mathcal{O})$ be a minimal realization of $\sigma$ with the partition intervals $I_1, \ldots, I_q$ as above.
- A straightforward induction shows that the $ij$-entry $a_{ij}^{(n)}$ of the power $A^n$ is the number of times the $n$-th iterated image $f^{\circ n}(I_i)$ covers $I_j$ or, equivalently, the number of connected components of $f^{-n}(I_j)$ in $I_i$.

**Lemma**

Let $A$ be the transition matrix of $\sigma \in \mathcal{C}_q$ with $\deg(\sigma) \geq 2$. Then the power $A^q$ has positive entries.

- This is shows that the transition matrix is irreducible.
A Perron – Frobenius Theorem

Theorem (Perron – Frobenius)

Let $S$ be a $q \times q$ column stochastic matrix with the property that some power of $S$ has positive entries. Then

(i) $S$ has a simple eigenvalue at $\lambda = 1$ and the remaining eigenvalues are in the open unit disk $\{\lambda : |\lambda| < 1\}$.

(ii) The eigenspace corresponding to $\lambda = 1$ is generated by a unique probability vector $\ell = (\ell_1, \ldots, \ell_q)$ with $\ell_i > 0$ for all $i$.

(iii) The powers $S^n$ converges to the matrix with identical columns $\ell$ as $n \to \infty$. 
We immediately have:

**Theorem**

Let $A$ be the transition matrix of $\sigma \in \mathcal{C}_q$ with $\deg(\sigma) = d \geq 2$. Then, there is a unique probability vector $\ell \in \mathbb{R}^q$ such that $A\ell = d\ell$. Moreover, $\ell$ has positive components and satisfies

$$\ell = \lim_{n \to \infty} \frac{1}{d^n} A^n \nu$$

for every probability vector $\nu \in \mathbb{R}^q$. 
We are now ready to prove the theorem:

**Theorem**

Let $\sigma \in C_q$ be any $q$-cycle with $\text{deg}(\sigma) = d \geq 2$ and with $\sigma(1) < \sigma(q)$. Then $\sigma$ has a unique realization under $m_d$.

**PROOF:**

- We are looking for a $q$-periodic orbit $O = \{x_1, \ldots, x_q\}$ for $m_d$, $0 < x_1 < \ldots < x_q < 1$ such that $m_d(x_i) = x_{\sigma(i)}$ for all $i$.
- Assume for a moment that such $O$ exists, let $I_i = [x_i, x_{i+1}]$, consider the lengths $\ell_i = |I_i|$, and form the probability vector $\ell = (\ell_1, \ldots, \ell_q) \in \mathbb{R}_+^q$.
- Since $m_d$ maps $I_i$ homeomorphically onto $\bigcup_{j \in [\sigma(i), \sigma(i+1)]} I_j$, we have
  \[
  \sum_{j \in [\sigma(i), \sigma(i+1)]} \ell_j = d \ell_i \quad \text{for all } i. \tag{1}
  \]
The $q$ relations (1) can be written as

$$A\ell = d\ell,$$  \hspace{1cm} (2)

where $A$ is the transition matrix of $\sigma$.

By the Perron-Frobenius Theorem, this equation has a unique solution $\ell$ which determines the lengths of the partition intervals $\{l_i\}$, hence the orbit $\mathcal{O}$ once we find $x_1$.

To construct the orbit $\mathcal{O} = \{x_1, \ldots, x_q\}$, take the unique solution $\ell = (\ell_1, \ldots, \ell_q)$ of (2) and define

$$\begin{cases} 
    x_1 = \frac{1}{d-1} \sum_{j \in [1, \sigma(1))] \ell_j \\
    x_i = x_1 + \sum_{j \in [1, i)} \ell_j \quad \text{for } 2 \leq i \leq q.
\end{cases}$$  \hspace{1cm} (3)

A few tedious computations shows that (3) works.
The higher degree cases

- Let $\sigma \in \mathbb{C}$. In order to describe the higher degree case $k > d = \deg(\sigma)$, we need some further notation.
- As before let $A$ denote the transition matrix for $\sigma$.
- It can be shown that the diagonal of the $0-1$ matrix $A$ contains precisely $d - 1$ entries of 1.
- A diagonal entry say $a_{ii}$ with value 1 corresponds to a fixed point for realizations.
- That is for any minimal realization $(f, \mathcal{O})$ of $\sigma$, the interval $I_i$ contains a fixed point for $f$ iff $a_{ii} = 1$, that is iff $f(I_i) \supseteq I_i$.
- In particular the $q$-th diagonal entry $a_{qq} = 1$ if and only if $\sigma(1) < \sigma(q)$. 
The signature of $\sigma$.

We define

**Definition**

Let $A = [a_{ij}]$ be the transition matrix of $\sigma \in \mathcal{C}_q$ with $\text{deg}(\sigma) = d$. The *signature* of $\sigma$ is the integer vector formed by the main diagonal entries of $A$:

$$\text{sig}(\sigma) = (a_{11}, \ldots, a_{qq}).$$

If $(f, \mathcal{O})$ is any realization of $\sigma$ (minimal or not), and if $I_1, \ldots, I_q$ are the corresponding partition intervals, then $I_i$ is called a *marked interval* if $a_{ii} = 1$.

- Let $p = (p_1, \ldots, p_q) \in \mathbb{N}^q$ be a $q$-vector with non negative integer valued coordinates. And let $1$ denote the $q$-vector of ones $1 = (1, \ldots, 1)$.
- Let $P = p^T \cdot 1$ be the $q \times q$ matrix with identical columns equal to $p^T \ldots$
The transformation matrix for non minimal realizations.

- Then

\[ B = A + P \]

can be regarded as the transition matrix for realizations \((f, O)\) of \(\sigma\) with winding \(p_i\) on interval \(l_i\).
- That is lifts of \(f\) on \([x_i, x_{(i+1)}]\) to \(\Pi\) have homeomorphic images of the form \([\hat{x}_{\sigma(i)}, \hat{x}_{\sigma(i+1)}]\) where \(\hat{x}_{\sigma(i)} + p_i < \hat{x}_{\sigma(i+1)} < \hat{x}_{\sigma(i)} + p_i + 1, \, \Pi(\hat{x}_{\sigma(i)}) = x_{\sigma(i)}\) and \(\Pi(\hat{x}_{\sigma(i+1)}) = x_{\sigma(i+1)}\).
- Then \(b_{ij}\) is the number of connected components of \(f^{-1}(l_j)\) contained in \(l_i\).
- The total sum of the elements in each column is

\[ k = \deg(\sigma) + \sum_{j=1}^{q} p_j. \]
Thus $\frac{1}{k}B$ is a column stochastic matrix.

Applying the Perron-Frobenius theorem again we find that $B$ has a unique simple leading eigenvalue 1 and a unique corresponding positive probability eigen-vector.

With this in place the following theorem is easily proved.

**Theorem**

*If the diagonal element $b_{qq} = a_{qq} + p_q > 0$, then there are $b_{qq}$ orbits for $m_k$ realizing $\sigma$.***
Workshop on Holomorphic Dynamics
- Iterated Monodromy groups and
Henon maps with a semi-neutral fixed point -
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http://thiele.ruc.dk/~lunde/Monodromy/index.html