

QUIVERS AND EQUATIONS A LA PLÜCKER FOR THE HILBERT SCHEME

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ABSTRACT:

Several moduli spaces parametrising linear subspaces of the projective space are cut out by linear and quadratic equations in their natural embedding: Grassmannians, Flag varieties, and Schubert varieties. The goal of this paper is to prove that a similar statement holds when one replaces linear subspaces with algebraic subschemes of the projective space. We exhibit equations of degree 1 and 2 that define schematically the Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^n}^p$ (for all, possibly non-constant, Hilbert polynomial p) in its standard embedding $\mathbf{Hilb}_{\mathbb{P}^n}^p \hookrightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ with R any degree larger than or equal to the Gotzmann number r of p . For every $R < r$ these linear and quadratic equations constructed, and suitable linear inequalities define the locally closed embedding $\mathbf{Hilb}_{\mathbb{P}^n}^{p,[R]} \hookrightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ of the Hilbert scheme parametrising subschemes with regularity upper bounded by R .

The equations are reminiscent of the Plücker relations for Grassmannians: they are explicit and built formally with permutations on indexes on the Plücker coordinates. Our method relies on a new description of the Hilbert scheme as a quotient of a scheme of quiver representations.

1. INTRODUCTION

The Plücker coordinates on a Grassmannian satisfy the well known Plücker relations [25]. Similarly, the flag varieties are defined by quadratic equations and Schubert varieties are defined by quadratic and linear equations [28, 13]. The goal of this paper is to prove that, in a similar way, the Hilbert schemes parametrising closed subschemes of a projective space are defined by simple explicit linear and quadratic equations in their natural embeddings.

The Hilbert schemes carry in general a natural non reduced structure inherited from their functorial construction. Our equations take into account the non reduced structure and define the Hilbert schemes schematically.

Let $\mathbf{Hilb}_{\mathbb{P}^n}^p$ be the Hilbert scheme parametrising closed subschemes of \mathbb{P}^n with Hilbert polynomial p . It can be embedded in the Grassmannian $\mathbf{Gr}_{S_R}^{p(R)}$, where R is any integer larger or equal to the Castelnuovo-Mumford-Gotzmann number r of p and $S_R = H^0\mathcal{O}_{\mathbb{P}^n}(R)$. Composing with the Plücker embedding $\mathbf{Gr}_{S_R}^{p(R)} \subset \mathbb{P}^{D(R)}$, $D(R) := \binom{\dim S_R}{p(R)} - 1$, we consider the problem of finding equations for the Hilbert scheme in $\mathbb{P}^{D(R)}$.

The question of finding equations for the Hilbert scheme as a subscheme of a Grassmannian has been addressed many times after its introduction by Grothendieck.

The equations that arise depend much on the way the Hilbert scheme is constructed. The initial construction by Grothendieck involved flattening stratifications [20, Lemme

3.4]. Techniques were developed to compute local equations for the flat stratum corresponding to the Hilbert scheme [14][17, Proposition 0.5].

The work by Gotzmann [16] leads to a description of the Hilbert scheme as a determinantal locus in a product of Grassmannians. A new description for the Hilbert scheme as a subscheme of a single Grassmannian given by local determinantal conditions was conjectured by Bayer in his PhD thesis [4] and proved by Iarrobino and Kleiman in [24, Appendix C] also exploiting an argument of Grothendieck. Haiman and Sturmfels obtain Bayer’s description as a special case of their own construction of the multigraded Hilbert scheme [23]. In [7] and [26], Brachat, Lella, Mourrain and Roggero define the Hilbert scheme using functors which involve symmetries of the Hilbert scheme given by the action of GL_n . Commuting matrices of multiplication by variables and border bases have been applied to define equations for Hilbert schemes of points by Alonso, Brachat and Mourrain [1].

The various approaches lead to equations of different degrees: degree $n + 1$, only depending on the “ambient” space \mathbb{P}^n , for those by Bayer, Iarrobino-Kleiman and Haiman-Sturmfels, degree $\deg(p) + 2$, only depending on the Hilbert polynomial, for those by Brachat-Lella-Mourrain-Roggero.

We will see that it is possible to find equations of degree 1 and 2 that cut out the Hilbert scheme for every, possibly nonconstant p . These are obviously the smallest possible degrees since in general Hilbert schemes are not linear spaces, not even linear sections of a Grassmannian [7, Section 7.2].

It was remarked by Haiman and Sturmfels [23] that the framework of a quite theoretical construction of the Hilbert scheme provides access to equations hardly accessible by direct computation. In cryptography, systems built with rich structures are possibly fragile because attackers may extract information from the structure. The above list of examples suggest that similarly each new description of the Hilbert scheme could expose a structure providing access to some new sets of equations.

Starting from these remarks the natural question is: how to produce a new description for the Hilbert scheme?

We considered the construction by Nakajima of $\mathbf{Hilb}_{\mathbb{A}^2}^p$, for constant p [27]. It is at a crossroads of several approaches. It is related to the framed moduli space of torsion free sheaves on \mathbb{P}^2 , monads and adhm-structures, quivers of commuting matrices.

Our project was to provide a description in the same vein for $\mathbf{Hilb}_{\mathbb{P}^n}^p$, i.e. we wanted to replace the constant p by any Hilbert polynomial p and the affine plane \mathbb{A}^2 by a projective space \mathbb{P}^n of any dimension.

An extension of Nakajima’s construction has been realized by Bartocci, Bruzzo, Lanza and Rava [3]. They replace the affine plane \mathbb{A}^2 with the total space of $\mathcal{O}_{\mathbb{P}^1}(-n)$ and use a description of the moduli space parametrising isomorphism classes of framed sheaves on the Hirzebruch surface Σ_n . The computations of the paper show that it is not possible to extend the initial construction directly. In the sheaf context, the trivialization at infinity of the sheaf is responsible for the loss of projectivity. Replacing the surface by a higher dimensional variety or considering a nonconstant Hilbert polynomial weakens the link between sheaves and Hilbert schemes.

We may reformulate the above obstructions to extend Nakajima’s construction in matrix terms. Recall that a zero-dimensional subscheme $Z \subset \mathbb{A}^2$ is represented by a pair of commuting matrices X, Y corresponding to the multiplication by the variables x, y on the vector space $O_Z \simeq k^{\text{length}(Z)}$, together with a cyclic vector $v \in k^{\text{length}(Z)}$ for the pair

(X, Y) . The matrices are determined up to the choice of the base of O_Z , and the cyclic vector is the algebraic counterpart of the constant function $1 \in O_Z$ generating O_Z as a $k[x, y]$ -module. Equivalently, the Hilbert scheme is constructed as a quotient of an open set of a commuting variety, where the commuting variety is a moduli space parametrising pairs (X, Y) of commuting matrices.

When one tries to extend the construction with commuting matrices from the case of zero-dimensional schemes in \mathbb{A}^2 to the case of projective schemes $Z \subset \mathbb{P}^n$ with any Hilbert polynomial, the first challenge is that of finding suitable vector spaces of finite dimension, as for instance $H^0(O_Z(t))$ (while the dimension of $H_*(O_Z)$ is infinite).

The multiplication by variables x_i yield morphisms $\mu_i : H^0(O_Z(t)) \rightarrow H^0(O_Z(t+1))$ and, if t is chosen larger than or equal to the Castelnuovo-Mumford regularity of Z , these maps contain much information on Z . However, the source space and the target space are different and the commutativity $\mu_i \mu_j = \mu_j \mu_i$ does not make sense. Indeed, when p is nonconstant, the underlying matrices \mathcal{M}_i are not square matrices and their sizes are incompatible; when p is constant, the matrix sizes are compatible but we miss a trivialization at infinity to identify $H^0(O_Z(t))$ with $H^0(O_Z(t+1))$. Indeed, in the projective case, there is no privileged element in $H^0(O_Z(t))$ and no natural cyclic vector notion.

The above analysis shows that for a construction of $\mathbf{Hilb}_{\mathbb{P}^n}^p$ based on the multiplicative action of the variables, we require a framework where we can formulate substitutes for the commutativity and the cyclic conditions.

In the first part of the paper, we introduce a quiver and we formulate these substitutes as technical conditions on the representations of the quiver that we consider. We proceed as follows.

We choose any integer number R larger than or equal to the Gotzmann number r of p and we consider the quiver Q_p with 4 vertices, $2n+3$ arrows, dimension vector $((\binom{R-1+n}{R-1}, \binom{R+n}{R}, p(R), p(R+1))$ and corresponding vector spaces $S_{R-1}, S_R, k^{p(R)}, k^{p(R+1)}$, where $S := k[x_0, \dots, x_n]$.

$$\begin{array}{ccccccc}
 S_{R-1} & \xrightarrow{\mu_0} & S_R & \xrightarrow{\rho} & k^{p(R)} & \xrightarrow{M_0} & k^{p(R+1)} \\
 \bullet & \xrightarrow{\mu_n} & \bullet & & \bullet & \xrightarrow{M_n} & \bullet
 \end{array}$$

Then we consider the representations $\mu_0, \dots, \mu_n, \rho, M_0, \dots, M_n$ of the quiver such that:

- The map μ_i is the multiplication by the variable x_i .
- The map ρ is surjective
- The images of the M_i satisfy the condition $Im(M_0) + \dots + Im(M_n) = k^{p(R+1)}$.
- $M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i$ for every $i, j \in \{0, \dots, n\}$.

There is a functor associated to these representations, which is the functor of points \underline{C}^p of a scheme C^p . There is an action of $GL_{p(R)} \times GL_{p(R+1)}$ on C^p corresponding to the base changes on the last two vertices of the quiver. Our description of the Hilbert scheme is summarized in the following theorem.

Theorem A. *C^p is a $GL_{p(R)} \times GL_{p(R+1)}$ principal bundle over the Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^n}^p$.*

The theorem provides a new universal property for the Hilbert scheme: it is possible to describe locally a family of subschemes of \mathbb{P}^n using families of matrices from the quiver description, up to action of a group. Describing schemes in terms of linear algebra up to

action may be more convenient than the usual description in terms of polynomial ideals (see [9, Prop. 3.14] for an explicit example).

Recall that Grassmannians are quotients of Stiefel varieties, and that Plücker coordinates are computable from Stiefel coordinates [15]. In our context, the “Stiefel” coordinates on C^p are the entries of the matrices ρ, M_0, \dots, M_n . In section 4 we describe the Plücker coordinates of the Hilbert scheme in terms of these Stiefel coordinates of C_p (Proposition 4.4):

- the maximal minors of ρ give Plücker coordinates for the embedding $\mathbf{Hilb}_{\mathbb{P}^n}^p \hookrightarrow \mathbf{Gr}_{S_R}^{p(R)}$;
- the maximal minors of $\sum_{i=0}^n (M_i \circ \rho): S_R^{n+1} \rightarrow k^{p(R+1)}$ give Plücker coordinates for the embedding $\mathbf{Hilb}_{\mathbb{P}^n}^p \hookrightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$.

The notations to formulate the main results about the equations for the Hilbert scheme are the following. We consider exterior products of the type

$$(1.1) \quad \ell z_1 \wedge \cdots \wedge \ell z_b \wedge v_{b+1} \wedge \cdots \wedge v_{p(R+1)}$$

where $z_i \in S_R, v_j \in S_{R+1}$ are monomials and $b \leq p(R+1) + 1$; note that (1.1) makes sense only if $b \leq p(R+1)$, but by convention we set that they are identically zero for $b = p(R+1) + 1$, so that the case $b = p(R) + 1$ makes sense also for a constant Hilbert polynomial.

If we chose as ℓ a variable x_i , (1.1) corresponds to a Plücker coordinate on the Grassmannian $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$. If ℓ is a linear form in S_1 , the multilinear expansion of (1.1) gives a linear combination of Plücker coordinates. If ℓ is the “generic” linear form $L = y_0 x_0 + \cdots + y_n x_n$ with indeterminate coefficients y_i , the multilinear expansion of (1.1) gives a homogeneous polynomial of degree b in the variables y_i and linear combinations of Plücker coordinates on $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ as coefficients. Let $m = y_{i_1} \cdots y_{i_b}$ be any such monomial, \underline{x} be the tuple $(x_{i_1}, \dots, x_{i_b})$ and $\underline{z}, \underline{v}$ be the tuples of monomials z_i, v_j . We denote the linear combination of Plücker coordinates which is the coefficient of m by $F(\underline{x}, \underline{z}, \underline{v})$ when $b = p(R)$ and by $E(\underline{x}, \underline{z}, \underline{v})$ when $b = p(R) + 1$.

Since, up to a sign, permutations on the lists $\underline{x}, \underline{z}, \underline{v}$ do not modify $E(\underline{x}, \underline{z}, \underline{v})$, we assume each list ordered in increasing order (for instance w.r.t. the lexicographic order).

Theorem B. *Let p be any Hilbert polynomial of subschemes of \mathbb{P}^n and let r be the Gotzmann number of p . For any positive integer R , consider the Plücker embedding $\mathbf{Gr}_{S_{R+1}}^{p(R+1)} \hookrightarrow \mathbb{P}^{D(R+1)}$ and the following three sets of equations on \mathbb{P}^D :*

- 1) *the quadratic Plücker relations of the Grassmannian,*
- 2) *the linear equations $E(\underline{x}, \underline{z}, \underline{v}) = 0$ (non trivial only for a non-constant p)*
- 3) *the quadratic equations $F(\underline{x}, \underline{z}, \underline{v})F(\underline{x}', \underline{z}', \underline{v}') - F(\underline{x}, \underline{z}', \underline{v})F(\underline{x}', \underline{z}, \underline{v}') = 0$*

If $R \geq r$, then the image of the embedding $j_{R+1}: \mathbf{Hilb}_{\mathbb{P}^n}^p \hookrightarrow \mathbb{P}^{D(R+1)}$ is defined by the linear and quadratic equations above.

We underline that our result does not apply to the minimal standard embedding j_r . In fact, we prove that (except for few very trivial cases) these equations considered in the case $R = r - 1$ define a subscheme of $\mathbb{P}^{D(r)}$ that properly contains the Hilbert scheme (Proposition 8.2). Furthermore, we explicitly present a Hilbert scheme whose image under the minimal embedding j_r cannot be cut out by any set of equations of degree ≤ 2 (Example 8.1).

Even though this could appear as a striking reason to motivate the non-applicability of Theorem B to the embedding j_r , in the final section we present a different, deeper motivation for this apparent failure, showing that our equations have an interesting meaning also for every $R \leq r$. We show that the problems are concentrated in points with high regularity and that our equations properly define the Hilbert scheme on the locus of points with adequate regularity.

More precisely, let us denote by $\mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']}$ the open subscheme of $\mathbf{Hilb}_{\mathbb{P}^n}^p$ parametrising subschemes with Hilbert polynomial p and regularity at most r' . It is proved in [5] that for every $s \geq r'$ there is a closed embedding $j_{r',s}: \mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']} \hookrightarrow \mathbb{P}^{D(s)} \setminus L_p^{r',s}$, where $L_p^{r',s}$ is a suitable linear subspace of $\mathbb{P}^{D(s)}$.

In Theorem 8.3 we complete Theorem B proving that:

If r' is any positive integer lower than or equal to r , and $s \geq r' + 1$, the image of $j_{r',s}: \mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']} \hookrightarrow \mathbb{P}^{D(s)} \setminus L_p^{r',s}$ is given by the linear and quadratic equations 1),2),3) of Theorem B.

Note that the embedding $j_{r',s}$ is defined for every $s \geq r'$, but our equations define its image only if $s \geq r' + 1$.

Overview of the proof of Theorem B. The standard way to find equations for $\mathbf{Hilb}_{\mathbb{P}^n}^p$ as a subscheme of a Grassmannian that we can find in literature is the following. One chooses a degree R larger or equal to the Gotzmann number of p , a subspace $V \subset S_R$ of codimension $p(R)$ and looks at its “expansion” S_1V in the next degree $R + 1$. By Gotzmann’s persistence (Theorem 2.2 (1)) V corresponds to a point of $\mathbf{Hilb}_{\mathbb{P}^n}^p$ if and only if the dimension of S_1V is the minimum allowed by Macaulay’s growth (Theorem 2.2 (2)).

In this paper we follow a different approach, that in some sense goes in the opposite direction. We consider a subspace W in S_{R+1} of codimension $p(R + 1)$ and look at the previous degree R . For subspaces W corresponding to points of the Hilbert scheme, $(W: S_1)$ has codimension $p(R)$ in S_R and, according to our construction, its Plücker coordinates are maximal minors of a map $(M_0 \circ \rho, \dots, M_n \circ \rho)$ which is a composition (Proposition 4.4).

If the dimension of the space F in the middle of a composition $E \rightarrow F \rightarrow G$ is too small, the maximal minors vanish. In our context, this happens if in (1.1) we choose $b = p(R) + 1$. After some algebraic manipulation this leads to the linear equations $E(\underline{x}, \underline{z}, \underline{v}) = 0$.

These linear equations define a subscheme \mathbf{E} of the Grassmannian that contains the Hilbert scheme, but in general does not coincide with it. In §6 we give an intrinsic description of \mathbf{E} as the locus of points $W \in \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ such that for ℓ general in S_1 the codimension of $(W: \ell)$ is $p(R)$, the maximum allowed by Green’s hyperplane restriction theorem (Theorem 2.2 (3) and Theorem 6.2).

Finally, in §7 we prove that $\mathbf{Hilb}_{\mathbb{P}^n}^p$ is the locus of points $W \in \mathbf{E}$ such that for ℓ general $(W: \ell)$ does not depend on ℓ , hence it coincides with $(W: S_1)$ (Theorem 7). We conclude the proof of Theorem B showing that the quadratic equations $F(\underline{x}, \underline{z}, \underline{v})F(\underline{x}', \underline{z}', \underline{v}') = F(\underline{x}, \underline{z}', \underline{v})F(\underline{x}', \underline{z}, \underline{v}')$ are fulfilled at a point W in \mathbf{E} exactly when for ℓ general the Plücker coordinates of $(W: \ell)$ in $\mathbf{Gr}_{S_R}^{p(R)}$ do not depend on ℓ .

The proof we just outlined is developed in the course of the paper in a more general framework, not only for k -points but for families, so that the equations we obtain define schematically the Hilbert scheme.

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2. GENERALITIES AND EMBEDDINGS OF THE HILBERT SCHEME

In this section, after some general notation, we recall some of the classical material used to embed Hilbert schemes into Grassmannians.

We work over a field k of any characteristic; in sections from 2 to 7 we assume that it is algebraically closed; in the final section we will prove that our equations are valid on any field.

Let $S = k[x_0, \dots, x_n]$ and $S_A = A[x_0, \dots, x_n]$ for any k -algebra A . We denote by S_d and $S_{A,d}$ the free submodules of homogeneous polynomials of degree d and by $N(d)$ their dimension. We denote by the same letter $\mu_i : S \rightarrow S$ and $\mu_i : S_A \rightarrow S_A$ the multiplication by the variable x_i .

When A is a field, we often consider the k -vector spaces $S_1 \simeq k^{n+1}$ and $S_{A,1} \simeq A^{n+1}$ as topological spaces endowed with the Zariski topology. We say that a property holds for a general linear form ℓ in S_1 (resp. $S_{A,1}$) if it holds for every ℓ in a non-empty Zariski open subset of S_1 (resp. $S_{A,1}$). We will use the following well known facts.

Lemma 2.1. *Let A be any k -algebra (k algebraically closed, but infinite is sufficient).*

- (1) *If $F \in A[y_1, \dots, y_m]$ vanishes on a dense subset of k^m , then it is the null polynomial.*
- (2) *If A is a field, the Zariski topology of k^m is the subspace topology induced by the Zariski topology of A^m .*

Proof. We prove that the set of zeros Z in k^m of a polynomial $F \in A[y_1, \dots, y_m]$ is also the set of common zeros of suitable polynomials $G_i \in k[y_1, \dots, y_m]$. If $a_1, \dots, a_v \in A$ are the coefficients of F , it is sufficient to prove the result assuming that A is the finitely generated k -algebra $k[a_1, \dots, a_v]$. Then, by Noether’s Normalization Lemma, $A = C[b_1, \dots, b_s]$ with $C := k[T_1, \dots, T_r]$ polynomial ring in the indeterminates T_i and b_i integral over C .

Then, it is sufficient to prove the result for polynomials with coefficients in $B[t]$ assuming that the result holds for polynomials with coefficients in a k -algebra B and that t is either integral or transcendent over B . In both cases, the coefficients in $B[t]$ of a polynomial $F \in B[t][y_1, \dots, y_m]$ are contained in some free B -module of finite rank d with basis given by powers of t : $B[t]$ itself, if t is integral, the B module generated by the powers of t up to the maximum appearing in F , if t is transcendent. Then $F = \sum_{i=0}^{d-1} G_i t^i$ with $G_i \in B[y_1, \dots, y_n]$ and Z is the set of the common zeros of the polynomials G_i .

We have proved the second point. The first one is now an easy induction on the number m of variables. \square

For any k -algebra A and A -module W in $S_{A,d}$, we will denote by $S_1 W$ the A -submodule of $S_{A,d+1}$ generated by the images of the multiplication maps. Moreover, for every linear form $\ell \in S_1$ we will denote by $(W : \ell)$ the A -module $\{f \in S_{A,d-1} \mid \ell f \in W\}$.

By simplicity for every tuple $\underline{w} = (w_1, \dots, w_b)$ of elements of an A -module M we shortly write $\wedge \underline{w}$ to denote the element $w_1 \wedge \dots \wedge w_b$ in $\wedge^b M$.

The vector spaces S_R and S_{R+1} are considered with their natural bases of monomials ordered in some way (for instance lexicographically); we say that tuple of monomials (w_1, \dots, w_b) is ordered increasingly if $w_1 \leq w_2 \leq \dots \leq w_b$.

If p is the Hilbert polynomial of a subscheme $Z \subset \mathbb{P}^n$, the Gotzmann number of p is the Castelnuovo-Mumford regularity of p , i.e. the smallest integer m such that every $Z \subset \mathbb{P}^n$ with Hilbert polynomial p is m -regular [24, Proposition C.24]. In particular the Hilbert function H_Z of Z satisfies $H_Z(d) = p(d)$ for every $d \geq r$. Note that r depends on p , but neither on the ground field nor on n .

From now on, p will denote a Hilbert polynomial for subschemes of \mathbb{P}^n , r its Gotzmann number and R any number $\geq r$. Moreover, for every integer t we will denote by $q(t)$ the number $N(t) - p(t)$.

Theorem 2.2. *Assume that the k -algebra A is a field. Let W be a vector space in $S_{A,d}$ with $\text{codim}_A(W, S_{A,d}) = p(d)$ and $d \geq r$. Then,*

- (1) (Macaulay) $\text{codim}_A(W S_{A,1}, S_{A,d+1}) \leq p(d+1)$.
- (2) (Gotzmann) *The equality $\text{codim}_A(W S_{A,1}, S_{A,d+1}) = p(d+1)$ holds if and only if the Hilbert polynomial of the ideal generated by W is p .*
- (3) (Green) *If $d \geq r+1$ and ℓ is general in S_1 , then $\text{codim}_A((W : \ell), S_{A,d-1}) \geq p(d-1)$.*

Proof. (1) is a consequence of Macaulay's theorem on the growth of the Hilbert functions and (2) is Gotzmann's persistence theorem [16]. These results can be found in several research papers and books; a version very close to ours for notation and intent is that of [23, Proposition 4.2].

(3) follows by Green's hyperplane restriction theorem proved in [18, Theorem 1]; in fact, if $c = p(d)$ and d is larger than the Gotzmann number of p , then the number that in [18] is denoted as $c_{\langle d \rangle}$ coincides with $p(d) - p(d-1)$. We also refer to [8, Theorem 4.2.12]) where it is clearly stated that this result only needs that the ground field is infinite. Note that in the quoted paper the result is proved for a general ℓ in $S_{A,1}$, hence it holds for a general ℓ in S_1 (Lemma 2.1 (2)). \square

Exploiting Theorem 2.2 the following result realizes $\mathbf{Hilb}_{\mathbb{P}^n}^p$ as a closed subscheme of the product of Grassmannians $\mathbf{Gr}_{S_R}^{p(R)} \times \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ [16, Bemerkung 3.2], [24, Proposition C.28, Theorem C.29], [23], [12, Exercise VI-3].

Following [23], We will denote by \underline{Y} the functor of points of a scheme Y from k -algebras to sets.

Theorem 2.3. *The Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^n}^p$ is the subscheme of $\mathbf{Gr}_{S_R}^{p(R)} \times \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ whose functor of points from k -algebras to sets is given by*

$$\mathbf{Hilb}_{\mathbb{P}^n}^p(A) = \{(I_{A,R}, I_{A,R+1}) \subset (S_{A,R}, S_{A,R+1}) \text{ that satisfy the following conditions } \}$$

- $S_{A,R}/I_{A,R}$ is locally free of rank $p(R)$
- $S_{A,R+1}/I_{A,R+1}$ is locally free of rank $p(R+1)$
- $x_i I_{A,R} \subset I_{A,R+1}$ for each variable x_i .

Moreover, the first and second projections give the embeddings $j_R: \mathbf{Hilb}_{\mathbb{P}^n}^p \hookrightarrow \mathbf{Gr}_{S_R}^{p(R)}$ and $j_{R+1}: \mathbf{Hilb}_{\mathbb{P}^n}^p \hookrightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$.

3. A NEW DESCRIPTION OF THE HILBERT SCHEME

Notation 3.1. If $\phi_j : E \rightarrow F$, for $j = 0, \dots, n$, are morphisms of A -modules and B is an A -algebra, we will use the following notations

- $\phi_j \otimes_A B : E \otimes_A B \rightarrow F \otimes_A B$ is the morphism of modules with $(\phi_j \otimes_A B)(e \otimes b) = \phi_j(e) \otimes b$,
- ϕ is the list (ϕ_0, \dots, ϕ_n) ,
- $\oplus \phi$ is the morphism $E \oplus \dots \oplus E \rightarrow F \oplus \dots \oplus F$ given by $\oplus \phi(e_0, \dots, e_n) = (\phi_0(e_0), \dots, \phi_i(e_n))$,
- $\Sigma \phi$ is the morphism $E \oplus \dots \oplus E \rightarrow F$ given by $\Sigma \phi(e_0, \dots, e_n) = \sum_{j=0}^n \phi_j(e_j)$.

Remark 3.2. By the functorial description of the Hilbert scheme given in Theorem 2.3, every map $\text{Spec}(A) \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p$ corresponds to a commutative diagram with exact rows.

$$(3.1) \quad \begin{array}{ccccccc} & & & (S_{A,R-1})^{n+1} & & & \\ & & & \downarrow (\oplus \mu)_{R-1} & & & \\ 0 & \rightarrow & (I_{A,R})^{n+1} & \hookrightarrow & (S_{A,R})^{n+1} & \xrightarrow{\pi_R \oplus \dots \oplus \pi_R} & (S_{A,R}/I_{A,R})^{n+1} \rightarrow 0 \\ & & \downarrow (\Sigma \mu)_{R,I} & & \downarrow (\Sigma \mu)_R & & \downarrow (\Sigma \bar{\mu})_R \\ 0 & \rightarrow & I_{A,R+1} & \hookrightarrow & S_{A,R+1} & \xrightarrow{\pi_{R+1}} & S_{A,R+1}/I_{A,R+1} \rightarrow 0 \end{array}$$

where μ is the list (μ_0, \dots, μ_n) with $\mu_i : S_A \rightarrow S_A$ the multiplication by the variable x_i , π_R, π_{R+1} are the projections on the quotients, $\bar{\mu} = (\bar{\mu}_0, \dots, \bar{\mu}_n)$ is the list of quotient maps, $\Sigma \mu$ and $\Sigma \bar{\mu}$ are defined by notation 3.1, and $(\Sigma \mu)_R, (\Sigma \mu)_{R,I}, (\Sigma \bar{\mu})_R$ are defined in the natural way by restrictions of $\Sigma \mu$ and $\Sigma \bar{\mu}$, the indices keeping track of the domain and codomain.

To build the variety C^p above the Hilbert scheme we elaborate on the above diagram and construct a functor of representations of the quiver Q_p of the introduction

$$\begin{array}{ccccc} S_{R-1} & \xrightarrow{\mu_0} & S_R & \xrightarrow{\rho} & k^{p(R)} & \xrightarrow{M_0} & k^{p(R+1)} \\ \bullet & \xrightarrow{\mu_n} & \bullet & & \bullet & \xrightarrow{M_n} & \bullet \end{array}$$

Definition 3.3. Let A be a k -algebra. Let $\mathfrak{C}^p(A) = \{(\mu, \rho, M)\}$ where:

- $\mu = (\mu_0, \dots, \mu_n)$ and $\mu_i : S_{A,R-1} \rightarrow S_{A,R}$ is the multiplication by the variable x_i ,
- $M = (M_0, \dots, M_n)$ and $M_i : A^{p(R)} \rightarrow A^{p(R+1)}$ is a morphism of A -modules,
- $\Sigma M : (A^{p(R)})^{n+1} \rightarrow A^{p(R+1)}$ is surjective
- $\rho : S_{A,R} \rightarrow A^{p(R)}$ is a surjective morphism of A -modules,
- for every pair $(i, j) \in \{0, \dots, n\}$, $M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i$

$$S_{A,R-1} \xrightarrow{\mu_i} S_{A,R} \xrightarrow{\rho} A^{p(R)} \xrightarrow{M_j} A^{p(R+1)}.$$

Since the tensorisation preserves the surjectivity, for any map of k -algebras $A \rightarrow B$, we have a morphism $\mathfrak{C}^p(A) \rightarrow \mathfrak{C}^p(B)$ which sends (μ, ρ, M) to $(\mu \otimes_A B, \rho \otimes_A B, M \otimes_A B)$. This makes \mathfrak{C}^p a functor from the category of k -algebras to the category of sets.

Remark 3.4. The set $\mathfrak{C}^p(A)$ and the map μ depend on R , but for brevity, R is not included in our notation. Similarly, we will use the notation $\mathfrak{C}^p(A) = \{(\rho, M)\}$ as a shortcut for $\mathfrak{C}^p(A) = \{(\mu, \rho, M)\}$ since there is only one possible choice for μ .

Proposition 3.5. *There exists a scheme C^p such that:*

- $\mathfrak{C}^p(A) = \underline{C}^p$, namely $\mathfrak{C}^p(A) = \text{Hom}(\text{Spec}(A), C^p)$ for every k -algebra A ;

- the k -points of \mathfrak{C}^p are representations of the quiver Q_p .

Proof. The non-trivial fact is the first item. It follows immediately that the k -points are representations of Q_p .

Let $\widetilde{\mathfrak{C}}^p$ be the extension of \mathfrak{C}^p to the category of k -schemes, i.e. $\widetilde{\mathfrak{C}}^p(Z) = \{(\mu, \rho, M)\}$ where:

- $\mu = (\mu_0, \dots, \mu_n)$ and $\mu_i : S_{R-1} \otimes \mathcal{O}_Z \rightarrow S_R \otimes \mathcal{O}_Z$ is the multiplication by the variable x_i ,
- $M = (M_0, \dots, M_n)$ and $M_i : \mathcal{O}_Z^{p(R)} \rightarrow \mathcal{O}_Z^{p(R+1)}$ is a morphism of \mathcal{O}_Z -modules,
- $\Sigma M : (\mathcal{O}_Z^{p(R)})^{n+1} \rightarrow \mathcal{O}_Z^{p(R+1)}$ is surjective
- $\rho : S_R \otimes \mathcal{O}_Z \rightarrow \mathcal{O}_Z^{p(R)}$ is a surjective morphism of \mathcal{O}_Z -modules,
- for every pair $(i, j) \in \{0, \dots, n\}$, $M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i$.

It suffices to prove that $\widetilde{\mathfrak{C}}^p$ is representable to obtain the first item of the proposition.

Consider the functor \mathcal{V} from schemes to sets defined as follows. If Z is a k -scheme, an element of $\mathcal{V}(Z)$ is a couple (ρ, M) where:

- $\Sigma M : (\mathcal{O}_Z^{p(R)})^{n+1} \rightarrow \mathcal{O}_Z^{p(R+1)}$ is a morphism of \mathcal{O}_Z -modules,
- $\rho : S_R \otimes \mathcal{O}_Z \rightarrow \mathcal{O}_Z^{p(R)}$ is a (possibly not surjective) morphism of \mathcal{O}_Z -modules.

For any map of k -schemes $\phi : Z_2 \rightarrow Z_1$, we have a morphism $\mathcal{V}(Z_1) \rightarrow \mathcal{V}(Z_2)$ which sends (ρ, M) to $(\phi^* \rho, \phi^* M)$.

For any finite dimensional k -vector space V , let us denote by $t(V)$ the scheme $\text{Spec}(\text{Sym}(V^*))$ and consider the functor $\underline{t(V)}$ given by

$$\underline{t(V)}(Z) = \text{Hom}(Z, \text{Spec}(\text{Sym}(V^*))) \simeq \text{Hom}(\text{Sym}(V^*), H^0(\mathcal{O}_Z)) \simeq H^0(\mathcal{O}_Z) \otimes_k V$$

and, for any map of k -schemes $Z_2 \rightarrow Z_1$, the map $\underline{t(V)}(Z_1) \rightarrow \underline{t(V)}(Z_2)$ sends $H^0(\mathcal{O}_{Z_1}) \otimes_k V$ to $H^0(\mathcal{O}_{Z_2}) \otimes_k V$ by pullback. The above functor \mathcal{V} is a special case of the functor $\underline{t(V)}$ with $V = \text{Hom}((k^{p(R)})^{n+1}, k^{p(R+1)}) \oplus \text{Hom}(S_R, k^{p(R)})$.

We recall the notion of relative representability from [19]. Let F, G be functors from the category of k -schemes to sets. Suppose that F is a subfunctor of G . The inclusion $F \subset G$ is relatively representable if, for every k -scheme Z and every morphism of functors $\underline{Z} \rightarrow G$, the fiber product $\underline{Z} \times_G F$ is representable. Grothendieck, [19, Lemme 3.6] proves that if G is representable and if $F \subset G$ is relatively representable, then F is representable.

In our case, $\widetilde{\mathfrak{C}}^p$ is a subfunctor of the representable functor \mathcal{V} and it is defined by the surjectivity of ΣM and ρ , and by the equality $M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i$. Thus it suffices to prove that a subfunctor defined by the surjectivity of a morphism of locally free sheaves is relatively representable, and that a subfunctor defined by the equality of morphisms of locally free sheaves is relatively representable.

The locus in $\text{Spec}(A)$ where two matrices $M, N \in \text{Hom}(\text{Spec}(A), k^{pq})$ of size $p \times q$ with coefficients m_{ij}, n_{ij} in A coincide is closed. More precisely, if $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is a morphism, then the pullback matrices $M_B, N_B \in \text{Hom}(\text{Spec}(B), k^{pq})$ satisfy $M_B = N_B$ if and only if the morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ factorizes through the closed subscheme $Z = \text{Spec}(A/J)$ where the ideal J is generated by the elements $(m_{ij} - n_{ij})$. It follows that if \mathcal{F}, \mathcal{G} are locally free sheaves on Z , and if $f, g \in \text{Hom}_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{G})$ are two morphisms of sheaves, there exists a closed subscheme $i_W : W \rightarrow Z$, such that for all $\phi : Y \rightarrow Z$, $\phi^* f = \phi^* g$ iff ϕ factorizes through W .

Let G be a functor such that $G(Z) = \{(f, g, \dots)\}$, i.e. $G(Z)$ is a tuple, and two components f, g of this tuple correspond to a functorial morphism of locally free sheaves $\mathcal{F}_Z \rightarrow \mathcal{G}_Z$ above Z . Let F be the subfunctor of G defined by the condition $f = g$. By Yoneda, a morphism $\underline{Z} \rightarrow G$ is defined by an element in $G(Z)$. By the above, $\underline{Z}(Y) \times_{G(Y)} F(Y)$ can be identified with $\underline{W}(Y)$, where W is the closed subscheme of Z defined by the condition $f = g$. Thus $\underline{Z} \times_G F \simeq \underline{W}$ and $F \subset G$ is a relatively representable functor. It follows that the condition $M_i \circ \rho \circ \mu_j = M_j \circ \rho \circ \mu_i$ defines a relatively representable (closed) subfunctor of \mathcal{V} .

The fact that the surjectivity condition on a morphism of sheaves defines an open subfunctor is a classical argument used in the construction of the Grassmannians [21, Lemme 9.7.4.6]. Thus $\widetilde{\mathcal{C}}^p$ is representable as it is a locally closed subfunctor of the representable functor \mathcal{V} . \square

Our next goal is to prove that the Hilbert scheme is a quotient of C^p . We first explain how C^p is related to the description of $\mathbf{Hilb}_{\mathbb{P}^n}^p$ given in Theorem 2.3. We always refer to Notation 3.1.

Proposition 3.6. *For every k -algebra A and $(\rho, M) \in \underline{C}^p(A)$, let $I_{A,R} := \text{Ker}(\rho)$, $I_{A,R+1} := \Sigma\mu(I_{A,R}^{n+1})$ and $\Sigma\mu_{R,I}$ the restriction of $\Sigma\mu$ to $(I_{A,R})^{n+1} \rightarrow I_{A,R+1}$.*

Then, there is a morphism $\beta: S_{A,R+1} \rightarrow A^{p(R+1)}$ such that $I_{A,R+1} = \text{Ker}(\beta)$ and the following diagram is commutative with exact rows:

$$(3.2) \quad \begin{array}{ccccccc} & & & (S_{A,R-1})^{n+1} & & & \\ & & & \downarrow \oplus \mu & & & \\ 0 & \rightarrow & (I_{A,R})^{n+1} & \hookrightarrow & (S_{A,R})^{n+1} & \xrightarrow{\rho \oplus \dots \oplus \rho} & (A^{p(R)})^{n+1} \rightarrow 0 \\ & & \downarrow \Sigma\mu_{R,I} & & \downarrow \Sigma\mu_R & & \downarrow \Sigma M \\ 0 & \rightarrow & I_{A,R+1} & \hookrightarrow & S_{A,R+1} & \xrightarrow{\beta} & A^{p(R+1)} \rightarrow 0 \end{array}$$

Moreover, $(I_{A,R}, I_{A,R+1}) \in \mathbf{Hilb}_{\mathbb{P}^n}^p(A)$.

Proof. We observe that

- $\rho \oplus \dots \oplus \rho$, $\Sigma\mu_{R,I}$, $\Sigma\mu_R$ and ΣM are surjective by hypotheses and/or by construction;
- by construction the first row is exact and the square on the left commutes.

We use all these properties in order to define β so that also the last line is exact and the right square commutes.

We define β by diagram chasing in the following way: by the surjectivity of $\Sigma\mu$ every element of $S_{A,R+1}$ can be written (not uniquely) as $\Sigma x_i f_i$ where $F := (f_0, \dots, f_n) \in (S_{A,R})^{n+1}$; then we set $\beta(\Sigma x_i f_i) = \Sigma M(\rho \oplus \dots \oplus \rho(f))$.

To verify that β is well defined it is sufficient to prove that when $\Sigma x_i f_i = 0$ we have $\Sigma M(\rho \oplus \dots \oplus \rho(f)) = 0$.

This is obvious if $F = (0, \dots, 0, f_n)$, since $\Sigma x_i f_i = 0$ implies $f_n = 0$. Then, we prove the assertion for $F = (0, \dots, 0, f_{j-1}, \dots, f_n)$ assuming it holds for elements of the form $(0, \dots, 0, f_j, \dots, f_n)$

For every $i = j, \dots, n$ we set $f_i = x_{j-1}f'_i + f''_i$ with $f'_i \in S_{A,R-1}$ and x_{j-1} not appearing in f''_i . The equality $\sum_{i=0}^n x_i f_i = 0$ implies $f_{j-1} + \sum_{i=j}^n x_i f'_i = 0$ and $\sum_{i=j}^n x_i f''_i = 0$. Then

$$\Sigma M(\rho \oplus \dots \oplus \rho(f)) = \sum_{i=j-1}^n M_i(\rho(f_i)) = M_{j-1}(\rho(f_{j-1})) + \sum_{i=j}^n M_i(\rho(\mu_{j-1}(f'_i))) + \sum_{i=j}^n M_i(\rho(f''_i)).$$

The last summand is equal to $\Sigma M(\rho \oplus \dots \oplus \rho((0, \dots, 0, f''_j, \dots, f''_n)))$, hence it vanishes by the inductive assumption. Moreover, by the compatibility conditions, we have

$$M_i(\rho(\mu_{j-1}(f'_i))) = M_{j-1}(\rho(\mu_i(f'_i))) = M_{j-1}(\rho(x_i f'_i)). \text{ Therefore } \Sigma M(\rho \oplus \dots \oplus \rho(f)) = M_{j-1}(\rho(f_{j-1} + \sum_{i=j}^n x_i f'_i)) = M_{j-1}(\rho(0)) = 0.$$

The commutativity of the right square holds by construction of β and the surjectivity of β follows from that of $\rho \oplus \dots \oplus \rho$, and ΣM .

To complete the construction of our diagram, we now prove that $\text{Ker}(\beta)$ is equal to $I_{A,R+1}$. By the commutativity of the two squares and the surjectivity of $\Sigma \mu_{R,I}$ it follows that $I_{A,R+1}$ is contained in $\text{Ker}(\beta)$. To prove the reverse inclusion we observe that $I_{A,R}, I_{A,R+1}, \text{Ker}(\beta)$ depend functorially on A in the sense that if $A \rightarrow B$ is a morphism of k -algebras, if $L_A \in \{I_{A,R}, \text{Ker}(\beta)\}$ is one of these two A -modules, then $L_B = L_A \otimes_A B$, and $I_{B,R+1}$ is the image of $I_{A,R+1}$ in $S_{A,R+1} \otimes B$. Then, we may check that for each maximal ideal \mathfrak{m} , $(\text{Ker}(\beta)/I_{A,R+1}) \otimes_A A_{\mathfrak{m}} = 0$. In other words, we may replace A with $A_{\mathfrak{m}}$ and suppose that A is local with maximal ideal \mathfrak{m} .

The A -module $\text{Ker}(\beta)$ is finitely generated as a kernel of a map between finitely generated free modules ([2, Exercise 12, p.32]). Thus $\text{Ker}(\beta)/I_{A,R+1}$ is finitely generated and, by Nakayama, we may even suppose that A is a field. When A is a field, the inclusion $I_{A,R+1} \subset \text{Ker}(\beta)$ is an equality if $\dim I_{A,R+1} \geq \dim \text{Ker}(\beta)$ as vector spaces. Since $\text{codim}(I_{A,R}, S_R) = p(R)$, Macaulay's maximal growth theorem (Theorem 2.2 (3)) gives the inequality $\text{codim}(I_{A,R+1}, S_{A,R+1}) \leq p(R+1) = \text{codim}(\text{Ker}(\beta), S_{A,R+1})$.

The final assertion directly follows from Theorem 2.3. \square

Now we are ready to conclude the proof of the first main result of the paper.

Proof. of Theorem A. Our goal is to prove that the Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^n}^p$ is a quotient of C^p by a natural action of $GL_{p(R)} \times GL_{p(R+1)}$. We start with the construction of a morphism $\chi : C^p \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p$. We consider the description of $\mathbf{Hilb}_{\mathbb{P}^n}^p$ given in Theorem 2.3.

Claim 1 There exists a surjective morphism $\chi : C^p \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p$.

Making reference to Proposition 3.6, we define χ at the functorial level by setting $\chi((\rho, M)) = (I_{A,R} = \text{Ker}(\rho), I_{A,R+1} = \text{Ker}(\beta)) \in \mathbf{Hilb}_{\mathbb{P}^n}^p(A)$, for every k -algebra A and $(\rho, M) \in \underline{C}^p(A)$. By construction this association depends functorially on A : indeed the exactness and commutativity of (3.2) is preserved by tensorisation since the modules on the right are free.

We observe that the difference between the diagram (3.1) associated with the Hilbert scheme and the diagram (3.2) associated with C^p comes from identifications $\tilde{\rho} : S_{A,R}/I_{A,R} \rightarrow A^{p(R)}$ and $\tilde{\beta} : S_{A,R+1}/I_{A,R+1} \rightarrow A^{p(R+1)}$ such that $\rho = \tilde{\rho} \circ \pi_R$ and $\beta = \tilde{\beta} \circ \pi_{R+1}$. Starting from (3.2), the isomorphisms $\tilde{\rho}, \tilde{\beta}$ are obtained by factorization. Starting from (3.1) and the isomorphisms $\tilde{\rho}, \tilde{\beta}$, we will see that it is possible to form the following

diagram that includes both (3.1) and (3.2).

$$(3.3) \quad \begin{array}{ccccccc} & & (S_{A,R-1})^{n+1} & & & & \\ & & \downarrow \oplus \mu & & & & \\ 0 & \rightarrow & (I_{A,R})^{n+1} & \hookrightarrow & (S_{A,R})^{n+1} & \xrightarrow{\pi_R \oplus \dots \oplus \pi_R} & (S_{A,R}/I_{A,R})^{n+1} & \xrightarrow{\tilde{\rho} \oplus \dots \oplus \tilde{\rho}} & (A^{p(R)})^{n+1} & \rightarrow & 0 \\ & & \downarrow \Sigma \mu_{R,I} & & \downarrow \Sigma \mu_R & & \downarrow \Sigma \bar{\mu}_R & & \downarrow \Sigma M & & \\ 0 & \rightarrow & I_{A,R+1} & \hookrightarrow & S_{A,R+1} & \xrightarrow{\pi_{R+1}} & S_{A,R+1}/I_{A,R+1} & \xrightarrow{\tilde{\beta}} & A^{p(R+1)} & \rightarrow & 0 \end{array}$$

It remains to prove that χ is surjective. Let $\phi : U = \text{Spec}(A) \hookrightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p$ be an open embedding such that the quotients $S_{A,R}/I_{A,R}$ and $S_{A,R+1}/I_{A,R+1}$ are free of rank $p(R)$ and $p(R+1)$ respectively, and consider the restriction of $\chi|_U : \chi^{-1}(U) \rightarrow U$. We check the surjectivity of $\chi|_U$.

Let us choose isomorphisms $\tilde{\rho} : S_{A,R}/I_{A,R} \rightarrow A^{p(R)}$ and $\tilde{\beta} : S_{A,R+1}/I_{A,R+1} \rightarrow A^{p(R+1)}$ and their lifts ρ and β to $S_{A,R}$ and $S_{A,R+1}$. We obtain a diagram as in (3.3) provided we can define a vertical map ΣM that fulfills all the commutativity conditions required.

For this we let $M_i := \tilde{\beta} \circ \bar{\mu}_{i,R} \circ \tilde{\rho}^{-1} : A^{p(R)} \rightarrow A^{p(R+1)}$, for $i = 0, \dots, n$. By the commutativity of the second square and that of the multiplication by two variables x_i, x_j we get

$$M_j \circ \rho \circ \mu_i = \tilde{\beta} \circ \pi_{R+1} \circ \mu_{j,R} \circ \mu_i = \tilde{\beta} \circ \pi_{R+1} \circ \mu_{i,R} \circ \mu_j = M_i \circ \rho \circ \mu_j.$$

Therefore, $(\rho, M) \in \underline{C}^p(A)$, i.e. it defines a map $\alpha : \text{Spec}(A) \rightarrow C^p$ such that $\chi \circ \alpha = \phi$, as directly follows by the definition of χ .

As a consequence of the above, we observe that a pair $(\rho, M) \in \underline{C}^p(A)$, is completely determined by the pair (ρ, β) of (3.2), since M is given by $M_i := \tilde{\beta} \circ \bar{\mu}_{i,R} \circ \tilde{\rho}^{-1}$ for $i = 0, \dots, n$. Therefore, in the sequel of this proof we will denote an element of $\underline{C}^p(A)$ as a triple (ρ, M, β) and by $\tilde{\rho}$ and $\tilde{\beta}$ the corresponding isomorphisms as in diagram (3.3).

Claim 2 $\chi : C^p \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p$ is a principal bundle with fibers isomorphic to $GL_{p(R)} \times GL_{p(R+1)}$.

The claimed property is local on the Hilbert scheme. Then, we may consider any open subset $U = \text{Spec}(A)$ of $\mathbf{Hilb}_{\mathbb{P}^n}^p$ as above and prove that $\chi^{-1}(\text{Spec}(A))$ is isomorphic to $\text{Spec}(A) \times GL_{p(R)}(A) \times GL_{p(R+1)}(A)$. In this case, as proved in the Claim 1, $(\rho, M, \beta) \in \chi^{-1}(\text{Spec}(A))$ if and only if $\text{Ker}(\rho) = I_{A,R}$, $\text{Ker}(\beta) = I_{A,R+1}$.

We choose and fix an element $(\rho_*, M_*, \beta_*) \in \chi^{-1}(\text{Spec}(A))$.

Then we can associate to every $(\rho, M, \beta) \in \chi^{-1}(\text{Spec}(A))$ the pair $(\tilde{\rho} \circ \tilde{\rho}_*^{-1}, \tilde{\beta} \circ \tilde{\beta}_*^{-1})$ in $GL_{p(R)}(A) \times GL_{p(R+1)}(A)$.

On the other hand, we can associate to any pair of isomorphisms $(\gamma_R, \gamma_{R+1}) \in GL_{p(R)}(A) \times GL_{p(R+1)}(A)$, the triple (ρ, M, β) given by $\rho := \gamma_R \circ \rho_*$, $\beta := \gamma_{R+1} \circ \beta_*$ and $M_i := \gamma_{R+1} \circ M_{*,i} \circ \gamma_R^{-1}$ for every $i = 0, \dots, n$. This is again an element of $\chi^{-1}(\text{Spec}(A))$ and, obviously, for different pairs (γ_R, γ_{R+1}) we obtain different elements (ρ, M, β) of $\chi^{-1}(\text{Spec}(A))$. \square

4. PLUCKER COORDINATES

Recall that there are two conventions for the Plücker coordinates, which give different signs [15, eq. 1.6]. The next propositions recall the basics about Grassmannians. They

introduce the notations that we need and they precise our sign convention for the Plücker coordinates.

We denote by \mathbf{Gr}_V^q the Grassmannian of codimension q subspaces of a vector space V . If $E = \{e_1, \dots, e_v\}$ is an ordered basis of V , for every k -algebra A , we also denote by E the corresponding basis $\{e_i \otimes 1_A\}$ of the free A -module $V_A := V \otimes_k A$.

A morphism $\text{Spec}(A) \rightarrow \mathbf{Gr}_V^q$ is functorially defined by an inclusion of A -modules $W_A \subset V_A$ such that the quotient V_A/W_A is locally free of rank q . The Plücker coordinates of a morphism $f \in \text{Hom}(\text{Spec}(A), \mathbf{Gr}_V^q)$ are defined as follows.

Proposition 4.1. *Let $F: \text{Spec}(A) \rightarrow \mathbf{Gr}_V^q$ such that V_A/W_A is free of rank q with basis F . Let $N \in M_{q,v}(A)$ be the matrix with columns N_1, \dots, N_v corresponding to the canonical morphism $V_A \rightarrow V_A/W_A$ with respect to the bases E and F . Consider a multi-index (i_1, \dots, i_q) with $1 \leq i_1 < i_2 < \dots < i_q \leq v$. The Plücker coordinate $P_{i_1 \dots i_q}$ of F is by definition the determinant $\det(N_{i_1}, \dots, N_{i_q}) \in A$. It is well defined up to multiplication by an invertible constant depending on the basis F . Equivalently, it is $(e_{i_1} \otimes 1_A) \wedge \dots \wedge (e_{i_q} \otimes 1_A) \in \Lambda^q(V_A/W_A) \simeq A$.*

Let us consider for every multi-index (i_1, \dots, i_q) with $1 \leq i_1 < i_2 < \dots < i_q \leq v$ an indeterminate X_{i_1, \dots, i_q} and the projective space $\mathbb{P} = \text{Proj}(k[X_{i_1, \dots, i_q}])$ of dimension $\binom{v}{q} - 1$. The Plücker embedding we now define is compatible with our convention for the Plücker coordinates.

Definition 4.2. The Plücker embedding $P: \mathbf{Gr}_V^q \rightarrow \mathbb{P} = \text{Proj}(k[X_{i_1, \dots, i_q}])$ is the embedding characterized by the following: if $F \in \text{Hom}(\text{Spec}(A), \mathbf{Gr}_V^q)$ is such that V_A/W_A is free of rank q , then $P \circ F \in \text{Hom}(\text{Spec}(A), \mathbb{P})$ is described in coordinates by $X_{i_1, \dots, i_q} = P_{i_1, \dots, i_q}$.

Remark 4.3. Starting from $F: \text{Spec}(A) \rightarrow C^p$, we define $F_R: \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_R}^{p(R)}$ and $F_{R+1}: \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ by the following compositions:

$$(4.1) \quad F_R: \text{Spec}(A) \rightarrow C^p \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p \rightarrow \mathbf{Gr}_{S_R}^{p(R)}$$

$$(4.2) \quad F_{R+1}: \text{Spec}(A) \rightarrow C^p \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}.$$

We associate to each tuple $\underline{z} = (z_1, \dots, z_{p(s)})$ of monomials of degree s (where $s = R$ or $s = R + 1$) ordered increasingly, a Plücker coordinate on $\mathbf{Gr}_{S_s}^{p(s)}$ that we denote $P_{z_1, \dots, z_{p(s)}}$ or $P_{\underline{z}}$. To simplify statements and proofs, we also consider tuples of monomials \underline{z} possibly not ordered in increasing order and with possibly repeated monomials, and associate to them the symbol $P_{\underline{z}}$ (that, by abuse, we call Plücker coordinate). If two lists of monomials \underline{z} and \underline{z}' are equal up to a permutation, then $P_{\underline{z}} = \pm P_{\underline{z}'}$ with the sign given by the parity of the permutation. Then, if \underline{z} contains repeated monomials, $P_{\underline{z}}$ simply stands for 0, while if the monomials are all distinct, $P_{\underline{z}}$ is a true Plücker coordinate up to a sign.

The next proposition describe the Plücker coordinates of F_R and F_{R+1} in terms of the entries of the matrices ρ, M_0, \dots, M_n which are associated to F through the functorial description of C^p .

Proposition 4.4. *In the above notations, the Plücker coordinates $P_{z_1, \dots, z_{p(R)}}$ of F_R are the maximal minors of ρ . The Plücker coordinates $P_{v_1, \dots, v_{p(R+1)}}$ of F_{R+1} are the maximal minors of $\Sigma M \circ (\rho \oplus \dots \oplus \rho)$. More precisely, if for each monomial $v_i \in S_{R+1}$, we choose a monomial $z_{t(i)} \in S_R$ and a variable $x_{j(i)}$ such that $v_i = x_{j(i)} z_{t(i)}$ and set $\tilde{z}_i =$*

$(0, \dots, 0, z_{t(i)}, 0, \dots, 0) \in (S_{A,R})^{n+1}$, where $z_{t(i)} \in S_{A,R}$ is located at position $j(i)$ so that $\Sigma\mu(\tilde{z}_i) = \mu_{j(i)}(z_{t(i)}) = v_i$, then $P_{v_1, \dots, v_{p(R+1)}}$ is the determinant of the matrix whose i -th column is $C_i := \Sigma M \circ (\rho \oplus \dots \oplus \rho)(\tilde{z}_i)$.

Proof. From our constructions, we have the two following diagrams with exact rows and commutative squares.

$$\begin{array}{ccccccc} 0 & \rightarrow & I_{A,R} & \rightarrow & S_{A,R} & \xrightarrow{\rho} & A^{p(R)} \rightarrow 0 \\ & & & & & & \\ 0 & \rightarrow & (I_{A,R})^{n+1} & \rightarrow & (S_{A,R})^{n+1} & \xrightarrow{\rho \oplus \dots \oplus \rho} & (A^{p(R)})^{n+1} \rightarrow 0 \\ & & \downarrow \Sigma\mu_{I,R} & & \downarrow \Sigma\mu_R & & \downarrow \Sigma M \\ 0 & \rightarrow & I_{A,R+1} & \rightarrow & S_{A,R+1} & \xrightarrow{\beta} & A^{p(R+1)} \rightarrow 0 \end{array}$$

Using the functorial description of the Grassmannian, the morphism F_R is described by the inclusion $I_{A,R} \subset S_{A,R}$. The first line shows that the Plücker coordinates in degree R are given by the maximal minors of ρ .

The morphism F_{R+1} is described by the inclusion $I_{A,R+1} \subset S_{A,R+1}$. The last line shows that the Plücker coordinates in degree $R+1$ are given by the maximal minors of β . Since $\Sigma\mu_R$ is surjective and sends the monomial basis of $(S_{A,R})^{n+1}$ to the monomial basis of $S_{A,R+1}$, the maximal minors of β coincide with the maximal minors of $\beta \circ \Sigma\mu_R = \Sigma M \circ \rho \oplus \dots \oplus \rho$. More precisely, if for each monomial $v_i \in S_{A,R+1}$, we choose a monomial \tilde{z}_i , as described in the statement, then $\beta(v_i) = \beta(\Sigma\mu_R(\tilde{z}_i)) = (\Sigma M \circ \rho \oplus \dots \oplus \rho)(\tilde{z}_i)$. The Plücker coordinate $P_{v_1, \dots, v_{p(R+1)}}$ is the determinant built with the $\beta(v_i)$ as columns, so the second equality of the proposition follows. \square

From the description of the Plücker coordinates, we get for free the vanishing of some Plücker coordinates over the whole Hilbert scheme if the Hilbert polynomial p has a positive degree. Indeed, among the minors of $\Sigma M \circ (\rho \oplus \dots \oplus \rho)$ are the minors of $M_i \circ \rho$, and they vanish for a nonconstant p . The idea is as follows, in the case where the image $Im(M_i \circ \rho)$ is a free module. In the composition $S_{A,R} \xrightarrow{\rho} A^{p(R)} \xrightarrow{M_i} A^{p(R+1)}$, the rank of the image $Im(M_i \circ \rho)$ is at most $p(R)$, the rank of the space in the middle, so that all minors of $M_i \circ \rho$ of order $p(R+1) > p(R) \geq rank(M_i \circ \rho)$ vanish. Beyond the vanishing of these particular Plücker coordinates, it is possible to get more general linear equations using a similar trick and elaborating on the above observation.

5. A STRATIFICATION OF THE GRASSMANNIAN $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$

In this section we introduce for every integer $b \geq p(R)$ a subscheme $\mathbf{H}^{(b)} \subset \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ of the Grassmannian whose closed points are the vector spaces $I_{R+1} \subset S_{R+1}$ with $codim((I_{R+1} : \ell) \subset S_R) < b$. We will prove that $\mathbf{H}^{(b)}$ is cut out by a linear space, is empty for $b = p(R)$ and contains the Hilbert scheme for $b > p(R)$.

We recall that a map $\text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ is given in the functorial description of the Grassmannian by a submodule $I_{A,R+1} \subset S_{A,R+1}$ with locally free quotient of rank $p(R+1)$.

Definition 5.1. Let X be a noetherian scheme. For every integer $b \geq p(R)$, we say that a morphism $X \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ satisfies the property \mathcal{P}^b if for every noetherian k -algebra A and every morphism $\text{Spec}(A) \rightarrow X$, the composed map $\text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ satisfies $\wedge^b S_{A,R}/(I_{A,R+1} : \ell) = 0$ for every $\ell \in S_1$.

It is not obvious that this definition is local as in general the colon ideal does not commute with the change of scalars, namely for a morphism of k -algebras $A \rightarrow B$ the module $S_{A,R}/(I_{A,R+1} : \ell) \otimes_A B$ can be different from $S_{B,R}/(I_{A,R+1} \otimes_A B : \ell)$. However, this locality is true, as formulated in the next proposition. The proof is an easy application of Lemma 5.3.

Proposition 5.2. *The morphism $X \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ satisfies the property \mathcal{P}^b if for some (or any) open covering of X by affine subschemes $\mathrm{Spec}(A_i) \rightarrow X$, the equality $\wedge^b S_{A_i,R}/(I_{A_i,R+1} : \ell) = 0$ holds for all i and $l \in S_1$.*

Lemma 5.3. *If $\mathrm{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ is given by the submodule $I_{A,R+1} \subset S_{A,R+1}$, then*

- (1) $\wedge^b S_{A,R}/(I_{A,R+1} : \ell) = 0$ if and only if $\wedge^b S_{A_{\mathfrak{p}},R}/(I_{A_{\mathfrak{p}},R+1} : \ell) = 0$ for every prime (or maximal) ideal \mathfrak{p} of A .
- (2) If $A \rightarrow B$ is a morphism of k -algebras and $\wedge^b S_{A,R}/(I_{A,R+1} : \ell) = 0$, then also $\wedge^b S_{B,R}/(I_{B,R+1} : \ell) = 0$.

Proof. (1) Let us consider any prime (or maximal) ideal \mathfrak{p} of A and any linear form ℓ .

The localization commutes with the colon ideal over a finitely generated ideal [2, Corollary 3.15]. Then, we have the equalities

$$S_{A_{\mathfrak{p}},R}/(I_{A_{\mathfrak{p}},R+1} : \ell) = S_{A_{\mathfrak{p}},R}/((I_{A,R+1} : \ell) \otimes_A A_{\mathfrak{p}}) = (S_{A,R} \otimes_A A_{\mathfrak{p}})/((I_{A,R+1} : \ell) \otimes_A A_{\mathfrak{p}}).$$

Moreover, the localization commutes with the quotient [2, Corollary 3.4] and with the exterior powers [11, Proposition A2.2 b], so that

$$\wedge^b(S_{A_{\mathfrak{p}},R}/(I_{A_{\mathfrak{p}},R+1} : \ell)) = \wedge^b((S_{A,R}/(I_{A,R+1} : \ell)) \otimes_A A_{\mathfrak{p}}) = (\wedge^b S_{A,R}/(I_{A,R+1} : \ell)) \otimes_A A_{\mathfrak{p}}.$$

Therefore, if $\wedge^b S_{A,R}/(I_{A,R+1} : \ell) = 0$, then also $\wedge^b S_{A_{\mathfrak{p}},R}/(I_{A_{\mathfrak{p}},R+1} : \ell) = 0$ for every prime (or every maximal) ideal \mathfrak{p} of A . Moreover, also the converse is true, since the property of being the null module is local [2, Proposition 3.8].

(2) Recall that by the functorial definition of Grassmannian, $I_{B,R+1} = I_{A,R+1} \otimes_A B$. Tensorizing the sequence

$$(5.1) \quad 0 \rightarrow (I_{A,R+1} : \ell) \rightarrow S_{A,R} \rightarrow S_{A,R}/(I_{A,R+1} : \ell) \rightarrow 0$$

we get

$$(5.2) \quad (I_{A,R+1} : \ell) \otimes_A B \xrightarrow{f} S_{B,R} \rightarrow (S_{A,R}/(I_{A,R+1} : \ell)) \otimes_A B \rightarrow 0.$$

We observe that $f((I_{A,R+1} : \ell) \otimes_A B)$ is contained in $(I_{B,R+1} : \ell)$; indeed, if m is any element in $S_{A,R}$ such that $\ell m \in I_{A,R+1}$, then $m \otimes_A 1_B$ belongs to $(I_{B,R+1} : \ell)$ since $\ell(m \otimes_A 1_B) = \ell m \otimes_A 1_B$ belongs to $I_{A,R+1} \otimes_A B$. Therefore, there is a surjective map $S_{B,R}/f((I_{A,R+1} : \ell) \otimes_A B) \rightarrow S_{B,R}/(I_{B,R+1} : \ell)$; by [11, Prop. A.2.2,d] also the following is surjective

$$(5.3) \quad \wedge^b S_{B,R}/f((I_{A,R+1} : \ell) \otimes_A B) \rightarrow \wedge^b S_{B,R}/(I_{B,R+1} : \ell) \rightarrow 0.$$

By (5.2), we see that $\mathrm{coker}(f) = S_{B,R}/f((I_{A,R+1} : \ell) \otimes_A B) \simeq (S_{A,R}/(I_{A,R+1} : \ell)) \otimes_A B$. Then, applying [11, Proposition A.2.2,b] we get $\wedge^b((S_{A,R}/(I_{A,R+1} : \ell)) \otimes_A B) = (\wedge^b(S_{A,R}/(I_{A,R+1} : \ell))) \otimes_A B = 0$. We conclude by (5.3). \square

Remark 5.4. We can easily see that a morphism $\text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ which satisfies \mathcal{P}^b , also satisfies $\mathcal{P}^{b'}$ for $b < b'$. Moreover, all the morphisms $\text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ satisfy the property $\mathcal{P}^{p(R+1)+1}$, as can be proved by a variation on proposition 5.5.

Therefore, in the following we will study only the properties \mathcal{P}^b with $b \leq p(R+1)$.

Furthermore, for a constant Hilbert polynomial $p(t) = d$, then $p(R) = p(R+1) = d$ so that we may reduce to study \mathcal{P}^b with $b \leq d = p(R)$.

Proposition 5.5. *Let $b \leq p(R+1)$. The following are equivalent for a map $\alpha: \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$*

- i) α satisfies \mathcal{P}^b ;*
- ii) for every tuples \underline{z} of b monomials in S_R and \underline{v} of $p(R+1) - b$ monomials in S_{R+1} , the image of $e^{(b)}(\ell, \underline{z}, \underline{v}) := \wedge(\ell \underline{z}, \underline{v})$ in $\wedge^{p(R+1)}(S_{A,R+1}/I_{A,R+1})$ vanishes for every $\ell \in S_{A,1}$ (or for every ℓ in a dense subset of S_1 , or for every ℓ in S_1).*

Proof. We first prove the equivalence between *i)* and the alternative in *ii)* for all $l \in S_1$.

By proposition 5.2, the condition *i)* asserts that $\wedge^b(S_{A,R}/(I_{A,R+1} : l)) = 0$ or equivalently that

$$\wedge^b S_{A,R} \rightarrow \wedge^b(S_{A,R}/(I_{A,R+1} : l))$$

is the null morphism for every l . This is also equivalent to the vanishing of the morphism

$$\begin{aligned} \wedge^b S_{A,R} &\xrightarrow{\phi_l} \wedge^b(S_{A,R+1}/I_{A,R+1}) \\ z_1 \wedge \cdots \wedge z_b &\mapsto lz_1 \wedge \cdots \wedge lz_b \end{aligned}$$

since both morphisms have the same kernel $(I_{A,R+1} : l) \wedge \wedge^{b-1} S_{A,R}$ according to [11, Prop. A.2.2,d]. The vanishing of ϕ_l is equivalent to the vanishing of $\wedge(\ell \underline{z})$ in $\wedge^b(S_{A,R+1}/I_{A,R+1})$ for all monomials z_i , or to the vanishing of $e^{(b)}(\ell, \underline{z}, \underline{v}) := \wedge(\ell \underline{z}, \underline{v})$ for all monomials z_i, v_i .

Finally, we prove that the alternatives in *ii)* are equivalent, namely that the vanishing of $e^{(b)}(\ell, \underline{z}, \underline{v})$ for ℓ in a dense subset of S_1 implies its vanishing for all $\ell \in S_{A,1}$.

Let us consider the linear form $L := y_0 x_0 + \cdots + y_n x_n \in S_{B,1}$, where B is the polynomial ring $A[y_0, \dots, y_n]$ in the indeterminates y_0, \dots, y_n , and let \underline{z} and \underline{v} as in the statement. If we formally develop $\wedge(L \underline{z}, \underline{v})$ with respect to the indeterminates y_0, \dots, y_n and coefficients in $\wedge^{p(R+1)} S_{R+1}$, we obtain a homogeneous polynomial of degree b . If we now consider the image of the coefficients under the projection $\wedge^{p(R+1)} S_{R+1} \rightarrow \wedge^{p(R+1)} S_{A,R+1}/I_{A,R+1} \simeq A$, we obtain polynomials $Q_{\underline{z}, \underline{v}}^{(b)}$ in $A[y_0, \dots, y_n]$. When L specializes to any $\ell \in S_1$, the polynomial $Q_{\underline{z}, \underline{v}}^{(b)}$ specializes to $e^{(b)}(\ell, \underline{z}, \underline{v})$. We then conclude by Lemma 2.1 (1). \square

Definition 5.6. For every $b \leq p(R+1)$, let $\underline{z} = (z_1, \dots, z_b)$ be a tuple of monomials in S_R , $\underline{v} = (v_{b+1}, \dots, v_{p(R+1)})$ be a tuple of monomials in S_{R+1} , and $\underline{x} = (x_{i_1}, x_{i_2}, \dots, x_{i_b})$ be a tuple of variables, all of them ordered increasingly. We will denote by $P_{\underline{x}, \underline{z}, \underline{v}}$ be the Plücker coordinate on $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ associated to the tuple $(x_{i_1} z_1, \dots, x_{i_b} z_b, v_{b+1}, \dots, v_{p(R+1)})$ and by $\mathbf{H}_{\underline{x}, \underline{z}, \underline{v}}^{(b)}$ the hyperplane section of $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ given by the vanishing of the linear form $E^{(b)}(\underline{x}, \underline{z}, \underline{v}) := \sum P_{\sigma(\underline{x}), \underline{z}, \underline{v}}$ where the sum runs over all the possible distinct permutations $\sigma(\underline{x})$ of the tuple \underline{x} (two permutations $\sigma(\underline{x})$ and $\sigma'(\underline{x})$ are distinct if $\sigma(\underline{x})$ and $\sigma'(\underline{x})$ are different as ordered tuples).

The scheme $\mathbf{H}^{(b)}$ is by definition the closed subscheme $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ cut out by the hyperplanes $\mathbf{H}_{\underline{x}, \underline{z}, \underline{v}}^{(b)}$ for every tuples \underline{x} , \underline{z} and \underline{v} .

Proposition 5.7. *A morphism $\alpha : X \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ satisfies the property \mathcal{P}^b if and only if it factorizes through $\mathbf{H}^{(b)}$.*

Proof. We can check the statement locally, namely considering a local k -algebra A and a map $\alpha : \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$, so that $S_{A,R+1}/I_{A,R+1}$ is free and α has Plücker coordinates. We exploit the argument presented in the proof of Proposition 5.5. The coefficient in $Q_{\underline{z},\underline{v}}^{(b)} \in A[y_0, \dots, y_n]$ of each monomial $y_{i_1} \cdots y_{i_b}$ is (up to a sign) the value at $I_{A,R+1}$ by the linear form $E^{(b)}(\underline{x}, \underline{z}, \underline{v})$ with $\underline{x} = (x_{i_1}, \dots, x_{i_b})$. Thus α factorizes through $\mathbf{H}^{(b)}$ iff all the polynomials $Q_{\underline{z},\underline{v}}^{(b)}$ vanish. By Proposition 5.5, this is equivalent to the property \mathcal{P}^b for α . \square

Remark 5.8. Reformulating Remark 5.4 with Proposition 5.7, we get that $\mathbf{H}^{(b)}$ is a subscheme of $\mathbf{H}^{(b')}$ if $b < b'$, and $\mathbf{H}^{(b)} = \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ for every $b \geq p(R+1) + 1$. Therefore, in the following we will study only the schemes $\mathbf{H}^{(b)}$ with $b \leq p(R+1)$.

6. THE SCHEMES \mathbf{E} , \mathbf{F} AND $\mathbf{\Delta}$

Notation 6.1. To bridge the notations of the previous section and the notations of Theorem B 2) and 3), we observe that by definition $E(\underline{x}, \underline{z}, \underline{v}) = E^{(p(R)+1)}(\underline{x}, \underline{z}, \underline{v})$ and $F(\underline{x}, \underline{z}, \underline{v}) = E^{(p(R))}(\underline{x}, \underline{z}, \underline{v})$.

We will denote by \mathbf{E} and \mathbf{F} the closed subschemes of $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ that are defined by the linear forms $E(\underline{x}, \underline{z}, \underline{v})$ and $F(\underline{x}, \underline{z}, \underline{v})$. In other words, $\mathbf{E} = \mathbf{H}^{(p(R)+1)}$ and $\mathbf{F} = \mathbf{H}^{(p(R))}$.

We will denote by $\mathbf{\Delta}$ the closed subscheme of $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ defined by the quadratic equations $F(\underline{x}, \underline{z}, \underline{v})F(\underline{x}', \underline{z}', \underline{v}') - F(\underline{x}, \underline{z}', \underline{v})F(\underline{x}', \underline{z}, \underline{v}')$ of Theorem B 3).

The present section is devoted to a further study of the schemes \mathbf{E} , \mathbf{F} and $\mathbf{\Delta}$. First we give an intrinsic description of \mathbf{E} that depends on, and in some sense generalizes, Green's Theorem (Theorem 2.2 (3)). In fact we can rephrase the following result saying that \mathbf{E} is the locus of the Grassmannian where Green's bound is sharp.

Theorem 6.2. *Let us consider a map $\alpha : \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ with (A, \mathfrak{m}, K) a local k -algebra. The following are equivalent*

- 1) α factorizes through \mathbf{E}
- 2) for every general $\ell \in S_1$ the quotient $J_\ell := S_{A,R}/(I_{A,R+1} : \ell)$ is free of rank $p(R)$, namely it corresponds to a map $\alpha_{R,\ell} : \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_R}^{p(R)}$.

Proof. 2) \Rightarrow 1). In fact for every tuple \underline{z} of $p(R) + 1$ monomials in S_R and a general ℓ in S_1 , $\wedge \underline{z}$ vanishes since it is an element of $\wedge^{p(R)+1} S_{A,R}/(I_{A,R+1} : \ell) = 0$. Hence also $\wedge \ell \underline{z}, \underline{v}$ vanishes in $\wedge^{p(R+1)}(S_{A,R+1}/I_{A,R+1})$. Thus α has property $\mathcal{P}^{p(R+1)}$ from Proposition 5.5 and we conclude by Proposition 5.7.

1) \Rightarrow 2) As A is local, $S_{A,R+1}/I_{A,R+1}$ is free with rank $p(R+1)$ and $I_{A,R+1}$ is free with rank $q(R+1)$. Recall that $N(t)$ and $q(t)$ are $\dim_k S_t$ and respectively $N(t) - p(t)$.

We choose any ℓ (general) such that Green's Theorem holds for $I_{K,R+1} = I_{A,R+1} \otimes_A K$. Green's results holds for every ℓ in a suitable open subset U of K^n . According to lemma 2.1, we may choose ℓ general in $k^n = S_1$. We then perform a change of coordinates leading ℓ to x_0 .

In this way we have $d := \dim_K S_{K,R}/(I_{K,R+1} : x_0) \geq p(R)$. We can then choose a tuples \underline{z} of d monomials in S_R and \underline{v} of $p(R+1) - d$ monomials in S_{R+1} such that \underline{z} is a basis of $S_{K,R}/(I_{K,R+1} : x_0)$ and $x_0\underline{z}, \underline{v}$ is a basis of $S_{K,R+1}/I_{K,R+1}$. As $\wedge(x_0\underline{z}, \underline{v})$ is non zero when computed in $\wedge^{p(R+1)} S_{K,R+1}/I_{K,R+1} = \wedge^{p(R+1)} S_{A,R+1}/I_{A,R+1} \otimes_A A/\mathfrak{m}$, then it is also invertible in $\wedge^{p(R+1)} S_{A,R+1}/I_{A,R+1}$.

We now construct a special basis for $I_{A,R+1}$ starting from $x_0\underline{z}, \underline{v}$.

Let $B = \{x_0\underline{w}, \underline{u}\}$ be the set of $q(R+1)$ monomials in $S_{R+1} \setminus \{x_0\underline{z}, \underline{v}\}$, where \underline{u} is the tuple of those not divisible by the variable x_0 : note that by construction $\underline{w}, \underline{z}$ is the complete list of monomials in S_R ; hence \underline{w} contains $c := N(R) - d \leq q(R)$ monomials. By Nakayama, every monomial in B can be written modulo $I_{A,R+1}$ as a linear combination of monomials in $\{x_0\underline{z}, \underline{v}\}$, thus we can find in $I_{A,R+1}$ polynomials $T_1, \dots, T_{q(R+1)}$ such that the i -th monomial in B appears only in T_i and its coefficient is 1_k .

These polynomials T_i are in fact a free set of generators of $I_{A,R+1}$. Indeed, every polynomial $G \in I_{A,R+1}$ has a unique writing $G = \sum_{i=1}^{q(R+1)} a_i T_i$, where each a_i is the coefficient in G of the i -th monomial of B . Unicity is clear considering the coefficients on the monomials of B . On the other hand $G - \sum_{i=1}^{q(R+1)} a_i T_i$ lies in $I_{A,R+1}$, is a linear combination of the monomials $\{x_0\underline{z}, \underline{v}\}$ which form a base of $S_{K,R+1}/I_{K,R+1} = S_{A,R+1}/I_{A,R+1} \otimes K$, hence by Nakayama a base of $S_{A,R+1}/I_{A,R+1}$.

Hence, for every $D \in (I_{A,R+1} : x_0)$ we have $x_0 D = \sum_{i=1}^c a_i T_i$. Furthermore, if $T_i = x_0 T'_i + T''_i$ with $T''_i \in A[x_1, \dots, x_n]$, then $x_0 D = x_0 \sum_{i=1}^c a_i T'_i$ since the summand $\sum_{i=1}^c a_i T''_i$ belongs to $x_0 S_{A,R} \cap A[x_1, \dots, x_n] = \{0\}$. Therefore, $D = \sum_{i=1}^c a_i T'_i$, so that $(I_{A,R+1} : x_0)$ is contained in the A -submodule Q of $S_{A,R}$ generated by the polynomials T'_1, \dots, T'_c : note that by construction these polynomials are linearly independent on A because their matrix of coefficients corresponding to the monomials of \underline{w} is the identity. This matrix property also shows that $S_{A,R} = Q \oplus P$ with P the free submodule of rank $d = N(R) - c \geq p(R)$ generated by the monomials in $S_R \setminus \{\underline{w}\}$.

Then, in the standard exact sequence

$$(6.1) \quad 0 \rightarrow (I_{A,R+1} : x_0) \xrightarrow{i} S_{A,R} = Q \oplus P \rightarrow S_{A,R}/(I_{A,R+1} : x_0) \rightarrow 0$$

the image of the first map is contained in Q . Therefore, $S_{A,R}/(I_{A,R+1} : x_0)$ is isomorphic to $Q/\text{Im}(i) \oplus P$. By hypothesis, $\wedge^{p(R)+1} S_{A,R}/(I_{A,R+1} : x_0) = 0$, hence $\text{rk } P = p(R)$ and $Q/\text{Im}(i) = 0$, namely $S_{A,R}/(I_{A,R+1} : x_0) \simeq P$ is free of rank $p(R)$ and $(I_{A,R+1} : x_0) = Q$ is free of rank $q(R)$. \square

Now we prove that \mathbf{F} is empty and then use this fact to give an intrinsic description of the scheme theoretical intersection $\mathbf{E} \cap \mathbf{\Delta}$.

Lemma 6.3. *For every map $\text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ with A local, there exist tuples \underline{z}_0 and \underline{v}_0 of monomials such that the following elements are invertible at $I_{A,R+1}$*

- $e^{(p(R))}(\ell_0, \underline{z}_0, \underline{v}_0) = \wedge(\ell_0 \underline{z}_0, \underline{v}_0)$ for every general linear form ℓ_0 in S_1 .
- $F(\underline{x}, \underline{z}_0, \underline{v}_0)$ for a suitable tuple \underline{x} of $p(R)$ variables.

Therefore, the scheme \mathbf{F} is empty.

Proof. We first prove the result assuming that $A = K$ is a field. If b is the dimension of the K -vector space $S_{K,R}/(I_{K,R+1} : \ell)$ with ℓ a general linear form in $S_{1,K}$, by Green's Theorem 2.2 (3) we have $b \geq p(R)$. By Lemma 2.1 this is also true for a general $\ell_0 \in S_1$. Therefore, $\wedge^{p(R)} S_{K,R}/(I_{K,R+1} : \ell_0) \neq 0$. It follows by Proposition 5.5 that $e^{(p(R))}(\ell_0, \underline{z}_0, \underline{v}_0) = \wedge(\ell_0 \underline{z}_0, \underline{v}_0)$ is invertible in $\wedge^{p(R+1)} S_{K,R+1}/I_{K,R+1} \simeq K$.

We can extend the proof to the case of a local ring (A, \mathfrak{m}, K) applying the first part to $\text{Spec}(K) \rightarrow \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$. If $e^{(p(R))}(\ell_0, \underline{z}_0, \underline{v}_0)$ gives a non-zero element of $\wedge^{p(R+1)} S_{K,R+1}/I_{K,R+1} \simeq K$, then it gives an invertible element of $\wedge^{p(R+1)} S_{A,R+1}/I_{A,R+1} \simeq A$, since this element does not vanish modulo \mathfrak{m} .

The second item follows from the first one by Proposition 5.7. As a consequence, \mathbf{F} is empty since it has no closed points. \square

Lemma 6.4. *Let A be a local k -algebra and $\alpha: \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$. The following are equivalent:*

- (1) α factorizes through Δ
- (2) $\wedge(\ell \underline{z}, \underline{v}) \cdot \wedge(\ell' \underline{z}', \underline{v}') - \wedge(\ell' \underline{z}, \underline{v}') \cdot \wedge(\ell \underline{z}', \underline{v})$ vanishes in $\wedge^{p(R+1)} S_{A,R+1}/I_{A,R+1} \simeq A$ for every tuples $(\underline{v}, \underline{z})$ and $(\underline{v}', \underline{z}')$ and every (general) linear form ℓ, ℓ' in S_1 ;
- (3) $\wedge(\ell \underline{z}, \underline{v}) \cdot \wedge(\ell_0 \underline{z}_0, \underline{v}_0) - \wedge(\ell_0 \underline{z}, \underline{v}_0) \cdot \wedge(\ell \underline{z}_0, \underline{v})$ vanishes in $\wedge^{p(R+1)} S_{A,R+1}/I_{A,R+1} \simeq A$ for every tuples $(\underline{v}, \underline{z})$, every general ℓ in S_1 and $(\ell_0, \underline{z}_0, \underline{v}_0)$ such that $\wedge(\ell_0 \underline{z}_0, \underline{v}_0)$ is invertible.

Proof. The equivalence of the the first two conditions is obtained by a direct computation, similarly to Proposition 5.7: consider two linear forms $L = x_0 y_0 + \dots, x_n y_n$ and $L' = x_0 y'_0 + \dots, x_n y'_n$ with y_i, y'_j indeterminates and formally expand $M(y_i, y'_i, \underline{z}, \underline{v}, \underline{z}', \underline{v}') := \wedge(L \underline{z}, \underline{v}) \cdot \wedge(L' \underline{z}', \underline{v}') - \wedge(L' \underline{z}, \underline{v}') \cdot \wedge(L \underline{z}', \underline{v})$ with respect to the indeterminates y_i, y'_j .

By definition of Δ , the condition (1) is equivalent to the vanishing of all the coefficients in these expansions, hence is equivalent to the fact that all the $M(y_i, y'_i, \underline{z}, \underline{v}, \underline{z}', \underline{v}')$ are identically zero. On the other hands, (2) is equivalent to their vanishing after the specialization $L \mapsto \ell, L' \mapsto \ell'$ for every general $\ell, \ell' \in S_1$. Again this is equivalent to the vanishing of all the $M(y_i, y'_i, \underline{z}, \underline{v}, \underline{z}', \underline{v}')$ (Lemma 2.1).

It remains to prove that (3) implies (2). Note that the existence of a triple $(\ell_0 \underline{z}_0, \underline{v}_0)$ such that $\wedge(\ell_0 \underline{z}_0, \underline{v}_0)$ is proved in Lemma 6.3. Let a be the invertible element of A that we obtain computing $\wedge(\ell_0 \underline{z}_0, \underline{v}_0)$ in $\wedge^{p(R+1)} S_{A,R+1}/I_{A,R+1}$. Let us consider any element

$$(6.2) \quad \wedge(\ell_1 \underline{z}_1, \underline{v}_1) \cdot \wedge(\ell_2 \underline{z}_2, \underline{v}_2) - \wedge(\ell_2 \underline{z}_1, \underline{v}_2) \cdot \wedge(\ell_1 \underline{z}_2, \underline{v}_1).$$

From the vanishing of the element in (3) in which we set $(\ell, \underline{z}, \underline{v}) = (\ell_1, \underline{z}_1, \underline{v}_1)$ it follows $\wedge(\ell_1 \underline{z}_1, \underline{v}_1) = a^{-1} \cdot \wedge(\ell_1 \underline{z}_0, \underline{v}_1) \cdot \wedge(\ell_0 \underline{z}_1, \underline{v}_0)$. We get three similar relations by setting $(\ell, \underline{z}, \underline{v})$ equal respectively to $(\ell_2, \underline{z}_2, \underline{v}_2)$, $(\ell_2, \underline{z}_1, \underline{v}_2)$ and $(\ell_1, \underline{z}_2, \underline{v}_1)$. Substituting these four relations in (6.2) we get 0. \square

Theorem 6.5. *Let A be a local k -algebra and $\alpha: \text{Spec}(A) \rightarrow \mathbf{E}$. Then*

α factorizes through $\mathbf{E} \cap \Delta$ if and only if, for a general $\ell \in S_1$, the quotient $J_\ell := S_{A,R}/(I_{A,R+1} : \ell)$ is free of rank $p(R)$ and does not depend on ℓ .

Proof. By hypothesis and Theorem 6.2, $S_{A,R+1}/I_{A,R+1}$ is free of rank $p(R+1)$ and J_ℓ is free of rank $p(R)$ for a general $\ell \in S_1$. Therefore, it remains to prove that α factorizes also through Δ if and only if, for a general $\ell \in S_1$, J_ℓ does not depend on ℓ and for this we will exploit Lemma 6.4.

For a general $\ell \in S_1$, we can identify J_ℓ through its Plücker coordinates in $\mathbf{Gr}_{S_R}^{p(R)}$: we choose an isomorphism $f_\ell: \wedge^{p(R)} J_\ell \xrightarrow{\sim} A$ and set, for every tuple \underline{z} of $p(R)$ monomials in S_R , $P_{\underline{z}}(J_\ell) = f_\ell(\wedge \underline{z})$.

Let us choose a tuples $\underline{z}_0, \underline{v}_0$ such that $\wedge(\ell\underline{z}_0, \underline{v}_0)$ is invertible for general $\ell \in S_1$ and let ℓ_0 be one of them (Lemma 6.3). Then, also \underline{z}_0 is a basis for J_ℓ and $P_{\underline{z}_0}(J_\ell)$ is invertible. For every tuple \underline{z} there is a suitable matrix $\mathcal{D}_{\ell, \underline{z}}$ with entries in A such that $\underline{z} = \mathcal{D}_{\ell, \underline{z}} \cdot \underline{z}_0$ in J_ℓ . As a consequence

$$(6.3) \quad \wedge \underline{z} = \det(\mathcal{D}_{\ell, \underline{z}}) \wedge \underline{z}_0 \text{ in } \wedge^{p(R)} J_\ell \quad \text{and} \quad P_{\underline{z}}(J_\ell) = \det(\mathcal{D}_{\ell, \underline{z}}) \cdot P_{\underline{z}_0}(J_\ell) \text{ in } A$$

and, for every tuple \underline{v} of $p(R+1) - p(R)$ monomials in S_{R+1} (especially, for $\underline{v} = \underline{v}_0$).

$$(6.4) \quad \wedge(\ell\underline{z}, \underline{v}) = \det(\mathcal{D}_{\ell, \underline{z}}) \wedge(\ell\underline{z}_0, \underline{v}) \quad \text{in} \quad \wedge^{p(R+1)} S_{A, R+1}/I_{A, R+1} \simeq A.$$

By substitution in (6.4) we obtain

$$(6.5) \quad \wedge(\ell\underline{z}, \underline{v}) = P_{\underline{z}}(J_\ell) \cdot P_{\underline{z}_0}(J_\ell)^{-1} \cdot \wedge(\ell\underline{z}_0, \underline{v}) \quad \forall \underline{z}, \underline{v}, \text{ and } \ell \text{ general.}$$

We use this equality to replace $(\ell\underline{z}, \underline{v})$ and $(\ell_0\underline{z}, \underline{v}_0)$ in $\wedge(\ell\underline{z}, \underline{v}) \cdot \wedge(\ell_0\underline{z}_0, \underline{v}_0) - \wedge(\ell_0\underline{z}, \underline{v}_0) \cdot \wedge(\ell\underline{z}_0, \underline{v})$ and find that the condition (3) of Lemma 6.4 is equivalent to the vanishing of

$$(6.6) \quad (P_{\underline{z}}(J_\ell) \cdot P_{\underline{z}_0}(J_\ell)^{-1} - P_{\underline{z}}(J_{\ell_0}) \cdot P_{\underline{z}_0}(J_{\ell_0})^{-1}) \cdot \wedge(\ell\underline{z}_0, \underline{v}) \cdot \wedge(\ell_0\underline{z}_0, \underline{v}_0).$$

If we assume that J_ℓ does not depend on ℓ for a general ℓ , then in particular $J_\ell = J_{\ell_0}$ (recall that ℓ_0 is general too) and (6.6) vanishes. Hence, α factorizes through Δ since the condition of Lemma 6.4(3) is fulfilled.

On the other hand, if we assume that α factorizes through Δ , then (6.6) vanishes for every tuple \underline{v} and for general $\ell \in S_1$. We consider the special case of (6.6) with $\underline{v} = \underline{v}_0$ and denote by U a non-empty open subset of S_1 such that for all $\ell \in U$, the following two conditions are satisfied:

$$\begin{aligned} (P_{\underline{z}}(J_\ell) \cdot P_{\underline{z}_0}(J_\ell)^{-1} - P_{\underline{z}}(J_{\ell_0}) \cdot P_{\underline{z}_0}(J_{\ell_0})^{-1}) \cdot \wedge(\ell\underline{z}_0, \underline{v}_0) \cdot \wedge(\ell_0\underline{z}_0, \underline{v}_0) &= 0 \\ \wedge(\ell\underline{z}_0, \underline{v}_0) &\text{ is invertible.} \end{aligned}$$

Therefore, $P_{\underline{z}}(J_\ell) \cdot P_{\underline{z}_0}(J_\ell)^{-1} - P_{\underline{z}}(J_{\ell_0}) \cdot P_{\underline{z}_0}(J_{\ell_0})^{-1} = 0$, for every tuple \underline{z} and for every $\ell \in U$. Thus J_ℓ and J_{ℓ_0} coincide since they have the same Plücker coordinates up to an invertible element. We conclude that for every ℓ in U , J_ℓ does not depend on ℓ . \square

Remark 6.6 (Cross product remark). In the previous theorem, we proved that the quadratic equations of Δ characterize that J_l does not depend on l . In this remark, we explain heuristically why it is clear that the independency of J_l with respect to l can be characterized by quadratic equations.

Consider the generic linear form with indeterminate coefficients ie. $L = a_0x_0 + \dots + a_nx_n$. Then the Plucker coordinates of J_L are computed with the indeterminates, ie. they are elements in $k(L) = k(a_0, \dots, a_n)$. If $J_l = J$ is independent of l , then $J_L = J_l = J$ and the coordinates of J_L turn out to be in k rather than in $k(L)$. Now, a $k(L)$ -point of \mathbb{P}^n is a k -point when quadratic cross product equations hold. For instance, the point $P = (3a_1 + 2a_0a_2 : 6a_1 + 4a_0a_2 : 9a_1 + 6a_0a_2) = (1 : 2 : 3) \in \mathbb{P}^2$ is a k -point. The coefficients $(3 : 6 : 9)$ and $(2 : 4 : 6)$ of the graded parts are proportional. This is measured by determinants of order 2. Similarly, the equations of Δ are the equations of degree 2 corresponding to the proportionality of the graded parts of J_L , hence they characterize that J_L has coefficients in k .

In fact, this is using the ideas of the present remark that we found the equations of Δ , after an explicit computation of J_L in a previous version of this paper (arXiv:1612.03074v3). However, this approach involved many technical details associated with generic points.

In the present version, we have simplified and suppressed generic points. The price paid for this simplification is that we only “check” the equations of Δ by a local computation. It is not clear how one could have guess the equations without the use of generic points.

7. THE PROOF OF THEOREM B

In theorem *B*, the linear equations are the equations of E and the quadratic equations are the equations of Δ . Thus the following claim will conclude the proof of theorem *B*.

Claim The Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^n}^p$ is the subscheme $\mathbf{E} \cap \Delta$ of $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$.

We use the following easy lemma.

Lemma 7.1. *Let P and Q be closed subschemes of a noetherian k -scheme G . Then, P is a subscheme of Q if and only if for every noetherian local k -algebra A every map $\alpha: \text{Spec}(A) \rightarrow G$ that factorizes through Q also factorizes through P .*

Proof. Recall that as proved in Theorem 6.2, a map $\alpha_{R+1}: \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ (A local) factorizes through \mathbf{E} if and only if for a general linear form $\ell \in S_1$ the colon ideal $I_{A,R,\ell} := (I_{A,R+1} : \ell)$ gives a map $\alpha_{R,\ell}: \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_R}^{p(R)}$. Furthermore, α_{R+1} also factorizes through Δ if and only if for ℓ general $I_{A,R,\ell}$ is independent of ℓ .

As the equality of subschemes of a given scheme is a local property (Lemma 7.1), we consider a noetherian local k -algebra A and a map $\alpha: \text{Spec}(A) \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ and prove that α factorizes through $\mathbf{Hilb}_{\mathbb{P}^n}^p$ if and only if α factorizes through \mathbf{E} and $(I_{A,R+1} : \ell) = (I_{A,R+1} : S_1)$ for a general $\ell \in S_1$ (Theorem 6.5).

First, we assume that α factorizes through $\mathbf{Hilb}_{\mathbb{P}^n}^p$.

Following the description of the Hilbert scheme given in Theorem 2.3, to $\alpha: \text{Spec}(A) \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p \rightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ corresponds $\alpha_R: \text{Spec}(A) \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p \rightarrow \mathbf{Gr}_{S_R}^{p(R)}$ given by a submodule $I_{A,R}$ of $S_{A,R}$ such that $S_{A,R}/I_{A,R}$ is (locally) free of rank $p(R)$ and $S_1 I_{A,R} \subset I_{A,R+1}$.

Then, $(I_{A,R+1} : \ell) \supset (I_{A,R+1} : S_1) \supset I_{A,R}$, so that there is a surjective map $S_{A,R}/I_{A,R} \rightarrow S_{A,R}/(I_{A,R+1} : \ell)$. As the computation of exterior powers preserves the surjectivity [11, Proposition A2.2 d], we obtain the exact sequence $0 = \wedge^{p(R)+1} S_{A,R}/I_{A,R} \rightarrow \wedge^{p(R)+1} S_{A,R}/(I_{A,R+1} : \ell) \rightarrow 0$ and then the vanishing of $\wedge^{p(R)+1} S_{A,R}/(I_{A,R+1} : \ell)$. By Definition 5.1 and Notation 6.1 the map α also factorizes through \mathbf{E} .

Now we prove the equalities $(I_{A,R+1} : \ell) = (I_{A,R+1} : S_1) = I_{A,R}$ for every general ℓ in S_1 . If I_A is the saturation of the ideal generated by $I_{A,R+1}$, then $I_{A,R}$ and $I_{A,R+1}$ are its homogeneous components of degree R and $R + 1$ respectively. By the assumption on the noetherianity of A , the ideal I_A has a primary reduced decomposition $I_A = \bigcap \mathfrak{q}_i$. Moreover, no associated prime $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ contains all the variables x_0, \dots, x_n , since I_A is homogeneous and saturated. Hence, the open subset of linear forms $U = \bigcap_i (S_1 \setminus \mathfrak{p}_i)$ is non-empty and for every $\ell \in U$ we have $(I_A : \ell) = (\bigcap \mathfrak{q}_i : \ell) = \bigcap (\mathfrak{q}_i : \ell) = \bigcap \mathfrak{q}_i = I_A$.

Note that α_R is indeed the map $\alpha_{R,\ell}$ for every general $\ell \in S_1$.

Therefore, for a general $\ell \in S_1$, $J_\ell := S_{A,R}/(I_{A,R+1} : \ell) = S_{A,R}/I_{A,R}$ does not depend on ℓ .

To prove the converse we assume that α factorizes through $\mathbf{E} \cap \Delta$ and prove that it also factorizes through $\mathbf{Hilb}_{\mathbb{P}^n}^p$ exploiting the description given in Theorem 2.3.

By Theorems 6.2 and 6.5, for a general $\ell \in S_1$ the module $J_\ell := S_{A,R}/(I_{A,R+1} : \ell)$ is (locally) free of rank $p(R)$ and does not depend on ℓ : let us denote it by J and by $\tilde{I}_{A,R}$ the kernel of the canonical map $S_{A,R} \rightarrow J$. Then, $\ell\tilde{I}_{A,R} \subseteq I_{A,R+1}$ for every ℓ in a suitable open subset U of S_1 . As the ground field k is infinite, U is not contained in a proper k -subvector space of S_1 , hence U contains a basis $\ell_0, \ell_1, \dots, \ell_n$ of S_1 . By construction $\ell_i\tilde{I}_{A,R} \subseteq I_{A,R+1}$, so that $S_1\tilde{I}_{A,R} \subseteq I_{A,R+1}$.

As $S_{A,R}/\tilde{I}_{A,R} = J$ is free of rank $p(R)$ and $S_1\tilde{I}_{A,R} \subseteq I_{A,R+1}$, by Theorem 2.3 the pair $(\tilde{I}_{A,R}, I_{A,R+1})$ corresponds to $\alpha: \text{Spec}(A) \rightarrow \mathbf{Hilb}_{\mathbb{P}^n}^p$.

Finally, we conclude the proof of Theorem B, (proved so far for k algebraically closed), holds for any base field k .

Let K be the algebraic closure of k and consider the inclusion $P_k := H_*^0 \mathcal{O}_{\mathbb{P}_k^{N-1}} \subset P_K := H_*^0 \mathcal{O}_{\mathbb{P}_K^{N-1}}$. According to [21, Proposition 1.3.10], $\mathbf{Hilb}_{\mathbb{P}_K}^p = \mathbf{Hilb}_{\mathbb{P}_k}^p \times_{\text{Spec}(k)} \text{Spec}(K)$, hence any set of equations in P_k which define the Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}_K}^p \subset \mathbb{P}_K^{N-1}$ over K are equations defining the Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}_k}^p \subset \mathbb{P}_k^{N-1}$ over k . Since the equations of Theorem B are defined over \mathbb{Z} or $\mathbb{Z}/n\mathbb{Z}$ (n being the characteristic of k), hence they belong to P_k , and are valid on the algebraic closure K , the equations also define the Hilbert scheme over k . \square

8. EXTENSIONS OF THE THEOREM AND AN EXAMPLE

The goal of this section is to discuss the hypothesis $R \geq r$ that we assume in Theorem B. We prove that it is not possible to use the same set of equations for $R = r - 1$ and we explain this phenomenon in terms of Castelnuovo-Mumford regularity.

We first show that in general the minimal standard embedding $j_r: \mathbf{Hilb}_{\mathbb{P}_k}^p \hookrightarrow \mathbf{Gr}_{S_r}^{p(r)}$ cannot be defined by quadratic equations. Therefore, a fortiori, the bound we assume on R is sharp.

We present in details an explicit example.

Example 8.1. Each closed point of the Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$ is either the disjoint union of a line and a point like the one given by the ideal (xy, xz) , or a line with an embedded point like the one given by the ideal (x^2, xy) .

Note that for any given line ℓ and point $P \in \ell$, there is only one saturated ideal with Hilbert polynomial $t+2$ whose associated primes are those corresponding to ℓ and P . For instance (x^2, xy) is the only one with associated primes (x) and (x, y) .

Therefore, by easy arguments, we can see that $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$ is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^{2^\vee}$. Hence, it can be embedded in a projective space as a subscheme cut by quadrics, as for instance in \mathbb{P}^8 by the Segre embedding. However, here we are interested in the standard embedding $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2} \hookrightarrow \mathbf{Gr}_{S_2}^4 \hookrightarrow \mathbb{P}^{14}$ of Theorem 2.3.

Using the computational methods developed in [6] and [7] we obtain that $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$ is the subscheme of \mathbb{P}^{14} defined by an ideal \mathcal{I} in $k[\Delta]$ (where Δ denotes the 15 Plücker variables) having as a set of minimal generators 15 Plücker relations, 15 additional quadrics and 28 cubics. In the appendix we list this set of equations (except the Plücker ones).

By standard computational methods, we checked that \mathcal{I} is saturated, hence it contains all the quadrics of \mathbb{P}^{14} that vanish on $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$, and computed the Hilbert polynomial of

$k[\Delta]/\mathcal{I}$ (namely the Hilbert polynomial of $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$ in \mathbb{P}^{14}):

$$P_{\mathcal{I}}(Z) = Z^4 + \frac{9}{2} Z^3 + 7 Z^2 + \frac{9}{2} Z + 1.$$

We also computed the Hilbert polynomial of $k[\Delta]/\mathcal{Q}$, where $\mathcal{Q} := (\mathcal{I}_2)$ is the ideal generated by all the 30 quadrics vanishing on $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$, founding a different polynomial

$$P_{\mathcal{Q}}(Z) = Z^4 + \frac{15}{2} Z^3 + \frac{3}{2} Z^2 + 3 Z + 2.$$

This shows not only that $\mathcal{Q} \neq \mathcal{I}$, but also that $\mathcal{Q}^{sat} \neq \mathcal{I}$, namely that $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$ embedded in \mathbb{P}^{14} is not cut out by quadrics, while by Theorem B this is true if we consider any standard embedding $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2} \hookrightarrow \mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ with $R+1 \geq r+1 = 3$.

We highlight two interesting aspects of this example and the related computations presented in the appendix.

The first one is about the minimal embedding to which the equations obtained in this paper apply, namely $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2} \hookrightarrow \mathbf{Gr}_{S_3}^5 \hookrightarrow \mathbb{P}^{251}$. After Theorem B we know that $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$ is a degenerate subscheme of \mathbb{P}^{251} contained in the hyperplanes defined by the linear forms $E(\underline{x}, \underline{z}, \underline{v})$. A priori there are $126 = 21 \cdot 6$ of them, since we can choose the monomial $\underline{x} \in k[x, y, z]_5$ in 21 ways and the set \underline{z} of 5 distinct monomials in $k[x, y, z]_2$ in 6 ways (while \underline{v} is empty). In the paper we do not prove that the linear forms $E(\underline{x}, \underline{z}, \underline{v})$ are always independent. However, in the present case they are, as we checked by explicit computations.

In the appendix we list a basis for the vector space generated by the 126 linear forms.

The second remark concerns the embedding $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2} \hookrightarrow \mathbf{Gr}_{S_2}^4 \hookrightarrow \mathbb{P}^{14}$. In Example 8.1 we proved that the quadratic equations are not sufficient, but we also show that 15 quadratic equations (independent from the Plücker ones) do exist. Nevertheless, the sets of equations given by Theorem B 2), 3) in this case are empty. Indeed, it is easy to check even from the simplified description given in the introduction, that for every subvector space W in S_2 of codimension $2 + 2 = 4$ a general linear forms ℓ satisfies the equality $(W : \ell) = (W : S_1) = \{0\}$, and the codimension of $(W : \ell)$ in S_1 is 3; hence all the points of the Grassmannian satisfy the conditions described by the equations of Theorem B. Therefore, the equations given by 2), 3) are not sufficient to describe $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2}$ in $\mathbf{Gr}_{S_2}^4$, since they do not exclude for instance the point of the Grassmannian corresponding to $W = (x^2, z^2)_2$, though the Hilbert polynomial of $\text{Proj}(k[x, y, z]/(x^2, z^2))$ is 4 and not $t+2$. The following proposition generalizes this observation.

Proposition 8.2. *Let $p(t)$ be any Hilbert polynomial of subschemes of \mathbb{P}^n , with the sole exclusion of the Hilbert polynomial of $\text{Proj}(k[x_0, \dots, x_n]/(x_0, \dots, x_{s-1}, x_s^r))$, for every s, r .*

Then, the equations of Theorem B in the case $R = r - 1$ define a subscheme Y of $\mathbf{Gr}_{S_r}^{p(r)}$ that properly contains $\mathbf{Hilb}_{\mathbb{P}^n}^p$ as a set.

Proof. First of all we observe that also for $R = r - 1$ the equations are satisfied by all the points of the Hilbert scheme. Indeed, for every saturated ideal \mathfrak{b} , every integer m and general linear form ℓ , then $(\mathfrak{b}_m : S_1) = (\mathfrak{b}_m : \ell)$ as shown in the proof of theorem B.

Moreover, if \mathfrak{b} is a saturated ideal with Hilbert polynomial p , then its regularity is $\leq r$ and its Hilbert function coincides with the Hilbert polynomial also in degree $r - 1$ (see for instance [10, Remark 2.4]). We then prove the statement showing that there is an

ideal I in S whose Hilbert polynomial is different from p , while I_r satisfies the conditions corresponding to the equations of Theorem B in the case $R = r - 1$.

Let \prec be the term order Lex in $k[x_0, \dots, x_n]$ with $x_0 \succ \dots \succ x_n$ and let J be the saturated (non-irrelevant) Lex-segment ideal with Hilbert polynomial p . It is well known that J has regularity exactly r and that no monomial in its minimal monomial basis B is divisible by x_n ; moreover the \prec -minimal monomial x^α in B has degree r and minimal variable x_m larger than x_n (x^α is a minimal generator of a saturated monomial ideal) and strictly lower than the maximal variable x_M of x^α (the cases with $x_m = x_M$ are those excluded in the statement).

Let I' be the ideal generated by $B^* := B \setminus \{x^\alpha\}$ and let I be the ideal generated by $B' := B^* \cup \{x_m^r\}$. By construction I' is a saturated lex-segment ideal and $I, J \supset I'$ with equalities $I_s = J_s = I'_s$ for every $s \leq r - 1$.

Now we prove that I is saturated. Let x^γ be a monomial in the saturation of I . Then, for v sufficiently large, $x^\gamma x_n^v$ is divisible by some monomial in B' . If $x^\gamma x_n^v$ is divisible by a monomial in B^* , then it is an element of the saturated ideal I' , so that $x^\gamma \in I' \subset I$. On the other hand, if $x^\gamma x_n^v$ is divisible by x_m^r , then also x^γ is, so that $x^\gamma \in I$. Therefore I coincides with its saturation.

By construction, I_r is the vector space generated by $\dim(S_r) - p(r)$ monomials of degree r (those of J_r with the only exception that x_m^r replaces x^α), so that its codimension is $p(r)$. Moreover, for a general linear form ℓ since I is saturated, and $(I_r : S_1) = (I'_r : S_1) = (I'_{r-1} : S_1) = (J_{r-1} : S_1)$ since the only monomial in $I_r \setminus I'_r$ is a minimal generator of I , hence it cannot be divisible by a monomial in $I_{r-1} = I'_{r-1}$. Since $(I_r : S_1) \subset (I_r : \ell)$, whereas $\dim(I_r : S_1) \geq (I_r : \ell)$ by Green's theorem and the above computation, we have $(I_r : S_1) = (I_r : \ell)$, ie. I_r admits general linear forms in the sense of Theorem 6.5.

We conclude showing that the scheme X defined by I is not a k -point of $\mathbf{Hilb}_{\mathbb{P}^n}^{p(t)}$. If the Hilbert polynomial of X were $p(t)$, then the dimension of I_{r+1} should be equal to that of J_{r+1} . We now prove that, on the contrary, $\dim(I_{r+1}) > \dim(J_{r+1})$. As I and J are monomial ideals, we compare the two dimensions by comparing the number of monomials in I_{r+1} and J_{r+1} . Recall that the monomial bases of J and I are $B = B^* \cup \{x^\alpha\}$ and $B' = B^* \cup \{x_m^r\}$ and that J is the saturated lexsegment ideal with regularity r , x^α has degree r and it is the \prec -minimal monomial in B , its maximal variable being \prec -larger than the minimal variable x_m .

Both J_{r+1} and I_{r+1} contains the monomials $S_1 B^*$. Then it is sufficient to consider those that are not in this set. As J is a lexsegment ideal, we know that there are exactly $n - m + 1$ monomials in $J_{r+1} \setminus S_1 B^*$ (they are $x_j x^\alpha$ with $j = m, \dots, n$).

We now observe that these monomials are \prec -lower than all those in $S_1 B^*$ and that by construction $x_m x^\alpha \succeq x_M x_m^r$. Therefore, the $n - M + 1$ monomials $x_j x_m^r \in I_{r+1}$ with $j = x_M, \dots, x_n$, do not belong to $S_1 B^*$. Then $\dim(I_{r+1}) - \dim(J_{r+1}) \geq m - M > 0$. \square

In the proof of Proposition 8.2 we present for almost every Hilbert polynomial p an explicit example of an ideal I in S such that I_r is a k -point of $\mathbf{Gr}_{S_r}^{p(r)} \setminus \mathbf{Hilb}_{\mathbb{P}^n}^p$ that satisfies the equations of Theorem B in the case $R = r - 1$. We observe that the regularity of the scheme defined by I is equal to or larger than the Gotzmann number r of p . It is not by chance that this happens; indeed the same happens for every ideal I such that I_r is a k -point of $\mathbf{Gr}_{S_r}^{p(r)} \setminus \mathbf{Hilb}_{\mathbb{P}^n}^p$ that satisfies the equations given by Theorem B for $R = r - 1$. In fact the codimension of I_t in S_t coincides with $p(t)$ for both $t = r - 1$ and $t = r$; if

we also assume that the regularity of I is at most $r - 1$, the above conditions imply that the Hilbert polynomial of $k[x_0, \dots, x_n]/I$ is p . Therefore, in a suitable open subset of the Grassmannian $\mathbf{Gr}_{S_r}^{p(r)}$ the equations given in Theorem B in the case $R = r - 1$ define the open subscheme of Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^n}^p$ where the regularity is upper bounded by $r - 1$.

Following [5], we denote by $\mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']}$ the *Hilbert scheme with regularity upper bounded by r'* , namely the open subscheme of $\mathbf{Hilb}_{\mathbb{P}^n}^p$ that parametrises the flat families with regularity lower than or equal to r' ; for the main features of this scheme we refer to [5] and the references therein.

Theorem 8.3. *Let p be any Hilbert polynomial of subschemes in \mathbb{P}^n and r be its Gotzmann number. Let moreover r' and R be integers such that $r' \leq R \leq r - 1$.*

Then, the Hilbert scheme $\mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']}$ is the locally closed subscheme of $\mathbb{P}^{D(R+1)}$ defined by the equations given in Theorem B and a suitable set of linear inequalities.

Proof. The result is a straightforward consequence of the results of [5] and of §7 about the meaning of the linear and quadratic equations of Theorem B. We outline the proof simply considering k -points, but the arguments can be generalized to families.

By [5, Theorem 1.2 (ii)], for every integer $m \geq r'$, $\mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']}$ can be embedded as a closed subscheme of $\mathbf{Gr}_{S_m}^{p(m)} \setminus L_p^{r',m}$ where $L_p^{r',m}$ is a subscheme of $\mathbf{Gr}_{S_m}^{p(m)}$ cut by a linear space under the Plücker embedding.

Moreover, by [5, Lemma 7.1], the k -points of $\mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']}$ are exactly the k -points V of $\mathbf{Gr}_{S_m}^{p(m)} \setminus L_p^{r',m}$ such that the ideal $I := (V)^{sat} \subset k[x_0, \dots, x_n]$ satisfy the condition $\text{codim}(I_t) = p(t)$ for every integer $t \geq r'$ (so that in particular $I_m = V$); this condition is equivalent to the apparently slighter condition $\text{codim}(I_t) = p(t)$ for two consecutive integers $t = t_0 \geq r'$ and $t = t_0 + 1$. Here we are interested in the case $m = R + 1$ and describe the k -points of $\mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']}$ as k -points V of $\mathbf{Gr}_{S_{R+1}}^{p(R+1)} \setminus L_p^{r',R+1}$ such that $I := (V)^{sat}$ satisfy the above condition for $t = t_0 = R$ and $t = t_0 + 1 = R + 1$.

In §7 we proved that a k -point V of $\mathbf{Gr}_{S_{R+1}}^{p(R+1)}$ satisfies the equations of Theorem B if and only if the ideal $I := (V)^{sat}$ satisfies the conditions $\text{codim}(I_{R+1}) = p(R + 1)$ and $\text{codim}(I_R) = \text{codim}(I_{R+1} : S_1) = \text{codim}((I_{R+1} : \ell)) = p(R)$ for a general $\ell \in S_1$.

Therefore, for every k -point V of $\mathbf{Gr}_{S_{R+1}}^{p(R+1)} \setminus L_p^{r',R+1}$, we see that V is a point of $\mathbf{Hilb}_{\mathbb{P}^n}^{p,[r']}$ if and only if $I := (V)^{sat}$ satisfies the equations of Theorem B. \square

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9. APPENDIX

The following are 15 quadrics F_i and 28 cubics G_j that, together with the Plücker relations, generate the saturated ideal \mathcal{I} of $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2} \hookrightarrow \mathbb{P}^{14}$. Each Plücker variable corresponds to the choice of 4 monomials in $k[x, y, z]_2$: we denote it by the position of the 2 missing monomials in the list ordered in decreasing degrevlex order $x^2, xy, y^2, xz, yz, z^2$.

$$\begin{aligned}
 F_1 &= -P_{3,5}P_{5,6} + P_{3,6}^2 \\
 F_2 &= -P_{2,5}P_{5,6} + 2P_{2,6}P_{3,6} - P_{3,4}P_{5,6} - P_{3,5}P_{4,6} \\
 F_3 &= -2P_{2,3}P_{5,6} + P_{2,5}P_{3,6} - 2P_{3,4}P_{3,6} - P_{3,5}P_{4,5}
 \end{aligned}$$

$$\begin{aligned}
 F_4 &= -P_{1,5}P_{5,6} + 2P_{1,6}P_{3,6} - P_{2,4}P_{5,6} - P_{2,5}P_{4,6} + P_{2,6}^2 - P_{3,4}P_{4,6} \\
 F_5 &= -4P_{1,3}P_{5,6} + 2P_{1,5}P_{3,6} - 3P_{2,4}P_{3,6} + P_{2,5}P_{2,6} - P_{2,5}P_{4,5} - P_{3,4}P_{4,5} \\
 F_6 &= -2P_{1,3}P_{3,6} + P_{1,5}P_{3,5} + P_{2,3}P_{2,6} - P_{2,3}P_{4,5} - P_{2,4}P_{3,5} + P_{3,4}^2 \\
 F_7 &= -P_{1,4}P_{5,6} - P_{1,5}P_{4,6} + 2P_{1,6}P_{2,6} - P_{2,4}P_{4,6} \\
 F_8 &= -3P_{1,2}P_{5,6} - 2P_{1,3}P_{4,6} - 2P_{1,4}P_{3,6} + 3P_{1,5}P_{2,6} - P_{1,5}P_{4,5} - P_{2,4}P_{2,6} - P_{2,4}P_{4,5} \\
 F_9 &= -4P_{1,2}P_{3,6} + 2P_{1,3}P_{2,6} - 3P_{1,3}P_{4,5} + P_{1,5}P_{2,5} - P_{2,4}P_{2,5} + P_{2,4}P_{3,4} \\
 F_{10} &= 2P_{1,2}P_{3,5} - P_{1,3}P_{2,5} - 2P_{1,3}P_{3,4} + P_{2,3}P_{2,4} \\
 F_{11} &= -P_{1,4}P_{4,6} + P_{1,6}^2 \\
 F_{12} &= -P_{1,2}P_{4,6} - P_{1,4}P_{2,6} - P_{1,4}P_{4,5} + 2P_{1,5}P_{1,6} \\
 F_{13} &= -P_{1,2}P_{2,6} - P_{1,2}P_{4,5} + 2P_{1,3}P_{1,6} - P_{1,4}P_{2,5} + P_{1,4}P_{3,4} + P_{1,5}^2 \\
 F_{14} &= -P_{1,2}P_{2,5} + 2P_{1,2}P_{3,4} + 2P_{1,3}P_{1,5} - P_{1,3}P_{2,4} \\
 F_{15} &= -P_{1,2}P_{2,3} + P_{1,3}^2 \\
 G_1 &= P_{1,6}P_{5,6}^2 - P_{2,6}P_{4,6}P_{5,6} + P_{3,6}P_{4,6}^2 \\
 G_2 &= 2P_{1,6}P_{3,6}P_{5,6} - P_{5,6}P_{2,5}P_{4,6} - P_{3,4}P_{4,6}P_{5,6} + P_{3,5}P_{4,6}^2 \\
 G_3 &= -4P_{1,3}P_{5,6}^2 + 4P_{1,5}P_{3,6}P_{5,6} - 2P_{5,6}P_{2,4}P_{3,6} - P_{2,5}P_{4,5}P_{5,6} - P_{3,4}P_{4,5}P_{5,6} + P_{3,5}P_{4,5}P_{4,6} \\
 G_4 &= -P_{1,3}P_{3,6}P_{5,6} + P_{1,5}P_{3,5}P_{5,6} + P_{2,3}P_{3,6}P_{4,6} - P_{2,3}P_{4,5}P_{5,6} - P_{5,6}P_{2,4}P_{3,5} + P_{3,4}^2P_{5,6} \\
 G_5 &= P_{1,4}P_{5,6}^2 - P_{1,5}P_{4,6}P_{5,6} + 2P_{1,6}P_{3,6}P_{4,6} - P_{2,4}P_{4,6}P_{5,6} \\
 G_6 &= -2P_{1,3}P_{4,6}P_{5,6} + P_{1,4}P_{3,6}P_{5,6} + 3P_{1,5}P_{3,6}P_{4,6} - P_{1,5}P_{4,5}P_{5,6} - P_{2,4}P_{2,6}P_{5,6} - P_{5,6}P_{2,4}P_{4,5} \\
 G_7 &= P_{5,6}P_{1,2}P_{4,6} - P_{1,4}P_{2,6}P_{5,6} + 2P_{1,4}P_{3,6}P_{4,6} - P_{1,4}P_{4,5}P_{5,6} \\
 G_8 &= P_{1,2}P_{3,6}P_{5,6} - P_{1,3}P_{2,6}P_{5,6} + P_{1,3}P_{3,6}P_{4,6} \\
 G_9 &= P_{1,2}P_{3,6}P_{4,6} - P_{1,3}P_{1,6}P_{5,6} \\
 G_{10} &= 2P_{1,3}P_{3,5}P_{5,6} - P_{2,3}P_{2,5}P_{5,6} - P_{2,3}P_{3,4}P_{5,6} + P_{2,3}P_{3,5}P_{4,6} \\
 G_{11} &= 2P_{3,5}P_{1,2}P_{5,6} - P_{1,3}P_{2,5}P_{5,6} - P_{1,3}P_{3,4}P_{5,6} + P_{1,3}P_{3,5}P_{4,6} \\
 G_{12} &= P_{1,2}P_{3,4}P_{5,6} + P_{1,2}P_{3,5}P_{4,6} - P_{1,3}P_{2,4}P_{5,6} \\
 G_{13} &= P_{1,2}P_{2,6}P_{4,6} - P_{1,4}P_{1,5}P_{5,6} + P_{1,4}P_{3,4}P_{4,6} \\
 G_{14} &= -P_{2,5}P_{1,2}P_{4,6} - 2P_{1,2}P_{3,4}P_{4,6} + 2P_{1,3}P_{1,4}P_{5,6} + P_{1,3}P_{2,4}P_{4,6} \\
 G_{15} &= P_{1,4}P_{1,2}P_{5,6} - P_{1,2}P_{1,5}P_{4,6} - P_{2,4}P_{1,2}P_{4,6} + 2P_{1,3}P_{1,4}P_{4,6} \\
 G_{16} &= -P_{1,3}P_{3,5}P_{3,6} + P_{2,3}^2P_{5,6} + P_{2,3}P_{3,4}P_{3,6} \\
 G_{17} &= -P_{1,2}P_{3,5}P_{3,6} + P_{1,3}P_{2,3}P_{5,6} + P_{1,3}P_{3,4}P_{3,6} \\
 G_{18} &= -2P_{1,2}P_{3,4}P_{3,6} - P_{3,5}P_{1,2}P_{4,5} + P_{1,3}P_{2,4}P_{3,6} \\
 G_{19} &= -2P_{1,2}P_{2,4}P_{3,6} - P_{2,5}P_{1,2}P_{4,5} - 2P_{1,2}P_{3,4}P_{4,5} + 4P_{1,3}P_{1,4}P_{3,6} + P_{1,3}P_{2,4}P_{4,5} \\
 G_{20} &= -P_{1,3}P_{3,5}^2 + P_{2,3}^2P_{3,6} + P_{2,3}P_{3,4}P_{3,5} \\
 G_{21} &= -P_{1,2}P_{3,5}^2 + P_{1,3}P_{2,3}P_{3,6} + P_{1,3}P_{3,4}P_{3,5} \\
 G_{22} &= 2P_{1,2}P_{2,3}P_{3,6} - P_{3,5}P_{1,2}P_{2,5} + P_{1,3}P_{2,4}P_{3,5} \\
 G_{23} &= P_{1,2}P_{1,3}P_{3,6} - P_{1,2}P_{1,5}P_{3,5} + P_{1,3}P_{1,4}P_{3,5} \\
 G_{24} &= P_{1,2}P_{1,3}P_{4,6} - 2P_{1,2}P_{1,4}P_{3,6} + P_{1,3}P_{1,4}P_{2,6} - P_{1,3}P_{1,4}P_{4,5} \\
 G_{25} &= -P_{1,2}^2P_{3,6} + P_{1,2}P_{1,3}P_{2,6} - P_{1,4}P_{1,2}P_{3,5} + P_{1,3}P_{1,4}P_{3,4} \\
 G_{26} &= P_{1,2}^2P_{4,6} - P_{1,2}P_{1,4}P_{2,6} - P_{1,2}P_{1,4}P_{4,5} + 2P_{1,3}P_{1,4}P_{1,6} \\
 G_{27} &= 2P_{1,2}P_{1,3}P_{1,6} - P_{1,4}P_{1,2}P_{2,5} + P_{1,3}P_{1,4}P_{2,4} \\
 G_{28} &= P_{1,2}^2P_{1,6} - P_{1,2}P_{1,4}P_{1,5} + P_{1,3}P_{1,4}^2.
 \end{aligned}$$

The following are 126 independent linear forms L_i that belong to the saturated ideal of $\mathbf{Hilb}_{\mathbb{P}^2}^{t+2} \hookrightarrow \mathbb{P}^{251}$. Each Plücker variable corresponds to the choice of 5 monomials in $k[x, y, z]_3$: we denote it by the position of these 5 monomials in the list ordered in decreasing degrevlex order.

$$L_1 = P_{1,2,3,5,6}, \quad L_2 = P_{1,2,3,5,8}, \quad L_3 = P_{1,2,3,6,8}, \quad L_4 = P_{1,2,5,6,8}, \quad L_5 = P_{1,3,5,6,8},$$

$$\begin{aligned}
L_6 &= P_{2,3,4,6,7}, \quad L_7 = P_{2,3,4,6,9}, \quad L_8 = P_{2,3,4,7,9}, \quad L_9 = P_{2,3,5,6,8}, \quad L_{10} = P_{2,3,6,7,9}, \\
L_{11} &= P_{2,4,6,7,9}, \quad L_{12} = P_{3,4,6,7,9}, \quad L_{13} = P_{5,6,7,8,9}, \quad L_{14} = P_{5,6,7,8,10}, \quad L_{15} = P_{5,6,7,9,10}, \\
L_{16} &= P_{5,6,8,9,10}, \quad L_{17} = P_{5,7,8,9,10}, \quad L_{18} = P_{6,7,8,9,10}, \quad L_{19} = 3P_{1,2,4,5,6} + 2P_{1,2,3,5,7}, \\
L_{20} &= -P_{1,2,5,6,7} + 6P_{1,2,4,5,8}, \quad L_{21} = P_{1,2,5,6,7} + 2P_{1,2,3,5,9}, \quad L_{22} = -P_{1,2,5,7,8} + 3P_{1,2,3,5,10}, \\
L_{23} &= 2P_{1,2,5,7,8} + P_{1,2,5,6,9}, \quad L_{24} = -2P_{1,2,5,8,9} + 3P_{1,2,5,6,10}, \quad L_{25} = 2P_{2,3,4,5,7} + 3P_{1,3,4,6,7}, \\
L_{26} &= -2P_{2,5,8,9,10} + 3P_{1,6,8,9,10}, \quad L_{27} = -P_{3,4,5,6,7} + 2P_{2,3,4,7,8}, \quad L_{28} = P_{3,4,5,6,7} + 6P_{1,3,4,7,9}, \\
L_{29} &= -P_{3,4,5,7,8} + 3P_{1,3,4,7,10}, \quad L_{30} = -P_{3,4,6,7,8} + 6P_{2,3,4,7,10}, \quad L_{31} = P_{3,4,6,7,8} + 2P_{3,4,5,7,9}, \\
L_{32} &= -2P_{3,4,7,8,9} + 3P_{3,4,6,7,10}, \quad L_{33} = -2P_{3,5,8,9,10} + P_{2,6,8,9,10}, \\
L_{34} &= -P_{3,5,8,9,10} + 3P_{1,7,8,9,10}, \quad L_{35} = -6P_{4,5,8,9,10} + P_{3,6,8,9,10}, \\
L_{36} &= -3P_{4,5,8,9,10} + P_{2,7,8,9,10}, \quad L_{37} = -3P_{4,6,8,9,10} + 2P_{3,7,8,9,10}, \\
L_{38} &= 3P_{1,3,4,5,6} + 6P_{1,2,4,5,7} + P_{1,2,3,6,7}, \quad L_{39} = -3P_{1,4,5,6,8} + P_{1,2,6,7,8} + 2P_{1,2,5,7,9}, \\
L_{40} &= 3P_{1,4,5,6,8} - P_{1,2,6,7,8} + 6P_{1,2,4,5,10}, \quad L_{41} = 3P_{1,4,5,6,8} + 2P_{1,3,5,7,8} + P_{1,3,5,6,9}, \\
L_{42} &= -P_{1,5,6,7,8} - P_{1,2,6,8,9} + 3P_{1,2,5,7,10}, \quad L_{43} = P_{1,5,6,7,8} - 2P_{1,3,5,8,9} + 3P_{1,3,5,6,10}, \\
L_{44} &= P_{1,5,6,8,9} - 3P_{1,2,6,8,10} + 6P_{1,2,5,9,10}, \quad L_{45} = P_{2,3,4,5,6} + 6P_{1,3,4,5,7} + 3P_{1,2,4,6,7}, \\
L_{46} &= P_{2,3,4,6,8} + 2P_{2,3,4,5,9} + 3P_{1,3,4,6,9}, \quad L_{47} = -P_{2,4,5,6,7} + P_{2,3,4,6,8} + 3P_{1,3,4,7,8}, \\
L_{48} &= 3P_{2,4,5,6,8} + 2P_{2,3,5,7,8} + P_{2,3,5,6,9}, \quad L_{49} = P_{2,5,6,7,8} - 2P_{2,3,5,8,9} + 3P_{2,3,5,6,10}, \\
L_{50} &= P_{2,5,6,8,10} - 3P_{1,5,6,9,10} + 6P_{1,2,8,9,10}, \quad L_{51} = -2P_{3,4,5,7,8} - P_{3,4,5,6,9} + 3P_{1,4,6,7,9}, \\
L_{52} &= 2P_{3,4,5,7,8} - P_{2,4,6,7,8} + 3P_{2,3,4,6,10}, \quad L_{53} = P_{3,5,6,8,9} - 2P_{2,5,7,8,9} + 3P_{1,6,7,8,9}, \\
L_{54} &= P_{3,5,6,8,10} - P_{2,5,6,9,10} + 3P_{1,3,8,9,10}, \quad L_{55} = P_{3,5,6,8,10} - 2P_{2,5,7,8,10} + 3P_{1,6,7,8,10}, \\
L_{56} &= -P_{4,5,6,7,9} - 2P_{2,4,7,8,9} + 3P_{2,4,6,7,10}, \quad L_{57} = P_{4,5,6,7,9} - P_{3,4,6,8,9} + 3P_{3,4,5,7,10}, \\
L_{58} &= 3P_{4,5,6,8,9} - 2P_{3,5,7,8,9} + P_{2,6,7,8,9}, \quad L_{59} = 3P_{4,5,6,8,10} - P_{3,5,6,9,10} + 6P_{1,4,8,9,10}, \\
L_{60} &= 3P_{4,5,6,8,10} - P_{3,5,6,9,10} + 2P_{2,3,8,9,10}, \quad L_{61} = 3P_{4,5,6,8,10} - 2P_{3,5,7,8,10} + P_{2,6,7,8,10}, \\
L_{62} &= P_{4,6,7,8,9} - 6P_{3,4,7,8,10} + 3P_{3,4,6,9,10}, \quad L_{63} = 3P_{4,6,7,8,10} - P_{3,6,7,9,10} + 6P_{3,4,8,9,10}, \\
L_{64} &= -P_{1,3,5,6,7} + 3P_{1,3,4,5,8} + 3P_{1,2,4,6,8} + P_{1,2,3,7,8}, \\
L_{65} &= -P_{1,3,5,6,7} + 6P_{1,3,4,5,8} + 3P_{1,2,4,6,8} + 3P_{1,2,4,5,9}, \\
L_{66} &= 2P_{1,3,5,6,7} - 6P_{1,3,4,5,8} - 3P_{1,2,4,6,8} + P_{1,2,3,6,9}, \\
L_{67} &= -6P_{1,4,5,6,8} - 2P_{1,3,5,7,8} + P_{1,2,6,7,8} + 3P_{1,2,3,6,10}, \\
L_{68} &= -3P_{1,4,5,6,8} - P_{1,3,5,7,8} + P_{1,2,6,7,8} + P_{1,2,3,8,9}, \\
L_{69} &= 6P_{1,4,5,7,8} + 3P_{1,4,5,6,9} + P_{1,3,6,7,8} + 2P_{1,3,5,7,9}, \\
L_{70} &= -P_{1,5,6,7,8} + P_{1,3,5,8,9} - P_{1,2,6,8,9} + 3P_{1,2,3,8,10}, \\
L_{71} &= 2P_{1,5,7,8,9} + 3P_{1,5,6,7,10} - 3P_{1,3,6,8,10} + 6P_{1,3,5,9,10}, \\
L_{72} &= 2P_{2,3,4,5,8} + 3P_{1,3,4,6,8} + 6P_{1,3,4,5,9} + 3P_{1,2,4,6,9}, \\
L_{73} &= 2P_{2,3,5,8,10} + 3P_{1,5,6,7,10} - 3P_{1,3,6,8,10} + 6P_{1,2,7,8,10}, \\
L_{74} &= P_{2,4,5,6,7} - P_{2,3,4,6,8} - P_{2,3,4,5,9} + 3P_{1,2,4,7,9}, \\
L_{75} &= -P_{2,4,5,6,8} - 6P_{1,4,5,7,8} - 3P_{1,4,5,6,9} + P_{1,2,6,7,9}, \\
L_{76} &= P_{2,4,5,6,8} - P_{1,3,6,7,8} + 6P_{1,3,4,5,10} + 3P_{1,2,4,6,10}, \\
L_{77} &= 2P_{2,4,5,7,8} + P_{2,4,5,6,9} + 3P_{1,4,6,7,8} + 6P_{1,4,5,7,9}, \\
L_{78} &= -P_{3,4,5,6,8} - 2P_{2,4,5,7,8} - P_{2,4,5,6,9} + P_{1,3,6,7,9}, \\
L_{79} &= -2P_{3,4,5,7,8} + P_{3,4,5,6,9} + P_{2,4,6,7,8} + 2P_{2,3,4,8,9}, \\
L_{80} &= 2P_{3,4,5,7,8} + P_{3,4,5,6,9} + P_{2,4,6,7,8} + 2P_{2,4,5,7,9}, \\
L_{81} &= 2P_{3,4,5,8,9} + 3P_{3,4,5,6,10} - P_{2,4,6,8,9} + 6P_{2,3,4,8,10}, \\
L_{82} &= P_{3,5,6,7,10} + 6P_{3,4,5,8,10} - 3P_{2,4,6,8,10} + 2P_{2,3,7,8,10}, \\
L_{83} &= P_{3,5,6,8,10} - P_{2,5,7,8,10} - P_{2,5,6,9,10} + 3P_{1,5,7,9,10}, \\
L_{84} &= -P_{4,5,6,7,8} + 2P_{3,4,5,8,9} - P_{2,4,6,8,9} + P_{2,3,6,7,10}, \\
L_{85} &= -P_{4,5,6,7,9} + P_{3,4,6,8,9} - P_{2,4,7,8,9} + 3P_{2,3,4,9,10}, \\
L_{86} &= P_{4,5,6,8,9} + 6P_{3,4,5,8,10} - 3P_{2,4,6,8,10} + P_{2,3,6,9,10}, \\
L_{87} &= P_{4,5,6,8,9} - P_{3,5,6,7,10} - 6P_{1,4,7,8,10} + 3P_{1,4,6,9,10},
\end{aligned}$$

$$\begin{aligned}
 L_{88} &= 3P_{4,5,6,8,10} - P_{3,5,7,8,10} - P_{3,5,6,9,10} + P_{2,5,7,9,10}, \\
 L_{89} &= 6P_{4,5,6,8,10} - 2P_{3,5,7,8,10} - P_{3,5,6,9,10} + 3P_{1,6,7,9,10}, \\
 L_{90} &= 2P_{4,5,7,8,9} - 3P_{4,5,6,7,10} - 6P_{2,4,7,8,10} + 3P_{2,4,6,9,10}, \\
 L_{91} &= 3P_{4,5,7,8,10} - 3P_{4,5,6,9,10} - P_{3,6,7,8,10} + P_{3,5,7,9,10}, \\
 L_{92} &= 6P_{4,5,7,8,10} - 3P_{4,5,6,9,10} - 2P_{3,6,7,8,10} + P_{2,6,7,9,10}, \\
 L_{93} &= 6P_{4,5,7,8,10} - 3P_{4,5,6,9,10} - P_{3,6,7,8,10} + 3P_{2,4,8,9,10}, \\
 L_{94} &= P_{1,5,6,7,9} - 6P_{1,4,5,8,9} + 9P_{1,4,5,6,10} - 2P_{1,3,6,8,9} + 6P_{1,3,5,7,10}, \\
 L_{95} &= -P_{2,3,5,6,7} + 2P_{2,3,4,5,8} - 3P_{1,4,5,6,7} + 6P_{1,3,4,6,8} + 6P_{1,2,4,7,8}, \\
 L_{96} &= -4P_{2,4,5,6,8} - 2P_{2,3,5,7,8} + 3P_{1,4,5,6,9} + 3P_{1,3,6,7,8} + 6P_{1,2,4,8,9}, \\
 L_{97} &= 2P_{2,4,5,8,9} + 6P_{2,4,5,6,10} - P_{2,3,6,8,9} - 3P_{1,4,6,8,9} + 18P_{1,3,4,8,10}, \\
 L_{98} &= -P_{2,5,6,7,8} + 2P_{2,3,5,8,9} + 9P_{1,4,5,6,10} - 3P_{1,3,6,8,9} + 18P_{1,2,4,8,10}, \\
 L_{99} &= -P_{2,5,6,7,8} + 2P_{2,3,5,8,9} + 3P_{1,5,6,7,9} - 3P_{1,3,6,8,9} + 6P_{1,2,7,8,9}, \\
 L_{100} &= P_{2,5,6,7,9} - 2P_{2,4,5,8,9} + 3P_{2,4,5,6,10} - 6P_{1,4,6,8,9} + 18P_{1,4,5,7,10}, \\
 L_{101} &= 2P_{2,5,6,7,10} + 6P_{2,4,5,8,10} - P_{2,3,6,8,10} - 9P_{1,4,6,8,10} + 6P_{1,3,7,8,10}, \\
 L_{102} &= 3P_{3,4,5,6,8} + 6P_{2,4,5,7,8} + 3P_{2,4,5,6,9} + P_{2,3,6,7,8} + 2P_{2,3,5,7,9}, \\
 L_{103} &= -2P_{3,5,6,7,8} + 5P_{2,4,5,8,9} - 3P_{2,4,5,6,10} - 3P_{1,4,6,8,9} + 9P_{1,2,4,9,10}, \\
 L_{104} &= -P_{3,5,6,8,9} + 2P_{2,5,7,8,9} + 3P_{2,5,6,7,10} - 9P_{1,4,6,8,10} + 18P_{1,4,5,9,10}, \\
 L_{105} &= -P_{3,5,6,8,9} + 2P_{2,5,7,8,9} + 3P_{2,5,6,7,10} - 3P_{2,3,6,8,10} + 6P_{2,3,5,9,10}, \\
 L_{106} &= -3P_{4,5,6,7,8} + P_{3,5,6,7,9} + 6P_{3,4,5,8,9} - 3P_{2,4,6,8,9} + 2P_{2,3,7,8,9}, \\
 L_{107} &= -4P_{4,5,6,8,9} + 2P_{3,5,7,8,9} + 3P_{3,5,6,7,10} - 3P_{2,4,6,8,10} + 6P_{2,4,5,9,10}, \\
 L_{108} &= -2P_{4,5,7,8,9} + 6P_{4,5,6,7,10} + P_{3,6,7,8,9} - 3P_{3,4,6,8,10} + 6P_{3,4,5,9,10}, \\
 L_{109} &= P_{2,3,5,6,7} - 4P_{2,3,4,5,8} + 3P_{1,4,5,6,7} - 6P_{1,3,4,6,8} - 6P_{1,3,4,5,9} + 2P_{1,2,3,7,9}, \\
 L_{110} &= -4P_{2,4,5,6,8} - P_{2,3,5,7,8} - 3P_{1,4,5,7,8} + 2P_{1,3,6,7,8} - 9P_{1,3,4,5,10} + 3P_{1,2,3,7,10}, \\
 L_{111} &= -5P_{2,5,6,7,8} + 4P_{2,3,5,8,9} + 18P_{1,4,5,8,9} - 27P_{1,4,5,6,10} - 3P_{1,3,6,8,9} + 18P_{1,2,3,9,10}, \\
 L_{112} &= P_{2,5,6,8,9} + 2P_{2,3,5,8,10} - 2P_{1,5,7,8,9} - 3P_{1,5,6,7,10} - 3P_{1,3,6,8,10} + 3P_{1,2,6,9,10}, \\
 L_{113} &= -5P_{3,4,5,6,8} - 8P_{2,4,5,7,8} + P_{2,3,6,7,8} - 6P_{2,3,4,5,10} + 3P_{1,4,6,7,8} + 18P_{1,2,4,7,10}, \\
 L_{114} &= -P_{3,4,5,6,8} - 2P_{2,4,5,7,8} + 2P_{2,4,5,6,9} + P_{2,3,6,7,8} + 3P_{1,4,6,7,8} + 6P_{1,3,4,8,9}, \\
 L_{115} &= 2P_{3,4,5,6,8} + 2P_{2,4,5,7,8} - P_{2,3,6,7,8} + 6P_{2,3,4,5,10} - 3P_{1,4,6,7,8} + 9P_{1,3,4,6,10}, \\
 L_{116} &= -2P_{3,5,6,7,8} + 2P_{2,5,6,7,9} + 6P_{2,4,5,8,9} - P_{2,3,6,8,9} - 9P_{1,4,6,8,9} + 6P_{1,3,7,8,9}, \\
 L_{117} &= 2P_{3,5,6,7,8} + P_{2,5,6,7,9} - 6P_{2,4,5,8,9} + 9P_{2,4,5,6,10} - 2P_{2,3,6,8,9} + 6P_{2,3,5,7,10}, \\
 L_{118} &= 5P_{3,5,6,8,9} - 4P_{2,5,7,8,9} - 3P_{2,5,6,7,10} + 18P_{2,4,5,8,10} - 27P_{1,4,6,8,10} + 18P_{1,2,7,9,10}, \\
 L_{119} &= -5P_{4,5,6,7,8} - P_{3,5,6,7,9} + 8P_{3,4,5,8,9} - 3P_{3,4,5,6,10} - 6P_{1,4,7,8,9} + 18P_{1,3,4,9,10}, \\
 L_{120} &= -2P_{4,5,6,7,8} - P_{3,5,6,7,9} + 2P_{3,4,5,8,9} - 3P_{3,4,5,6,10} - 6P_{1,4,7,8,9} + 9P_{1,4,6,7,10}, \\
 L_{121} &= P_{4,5,6,7,8} + P_{3,5,6,7,9} - 2P_{3,4,5,8,9} + 3P_{3,4,5,6,10} - 2P_{2,4,6,8,9} + 6P_{2,4,5,7,10}, \\
 L_{122} &= 4P_{4,5,6,8,9} - P_{3,5,7,8,9} - 2P_{3,5,6,7,10} + 3P_{3,4,5,8,10} - 9P_{1,4,7,8,10} + 3P_{1,3,7,9,10}, \\
 L_{123} &= 4P_{4,5,7,8,9} - 6P_{4,5,6,7,10} - P_{3,6,7,8,9} + 3P_{3,4,6,8,10} - 6P_{2,4,7,8,10} + 2P_{2,3,7,9,10}, \\
 L_{124} &= -4P_{2,5,6,7,8} + 2P_{2,3,5,8,9} - 3P_{1,5,6,7,9} + 18P_{1,4,5,8,9} - 27P_{1,4,5,6,10} - 3P_{1,3,6,8,9} + 9P_{1,2,6,7,10}, \\
 L_{125} &= -5P_{3,5,6,7,8} - P_{2,5,6,7,9} + 12P_{2,4,5,8,9} - 9P_{2,4,5,6,10} - P_{2,3,6,8,9} - 9P_{1,4,6,8,9} + 9P_{1,3,6,7,10}, \\
 L_{126} &= 4P_{3,5,6,8,9} - 2P_{2,5,7,8,9} - 3P_{2,5,6,7,10} + 18P_{2,4,5,8,10} - 3P_{2,3,6,8,10} - 27P_{1,4,6,8,10} + 9P_{1,3,6,9,10}.
 \end{aligned}$$