Computing limit linear series with infinitesimal methods

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Abstract:

Alexander and Hirschowitz [1] determined the Hilbert function of a generic union of fat points in a projective space when the number of fat points is much bigger than the greatest multiplicity of the fat points. Their method is based on a lemma which determines the limit of a linear system depending on fat points approaching a divisor.

Other Hilbert functions were computed previously by Nagata [15]. In connection with his counter example to Hilbert's fourteenth problem, Nagata determined the Hilbert function $H(d)$ of the union of k^2 points of the same multiplicity m in the plane up to degree $d = km$.

We introduce a new method to determine limits of linear systems. This generalizes the result by Alexander and Hirschowitz. Our main application of this method is the conclusion of the work initiated by Nagata: we compute $H(d)$ for all d. As a second application, we compute collisions of fat points in the plane.

1 Introduction

Fixing general points p_i in a projective space, what is the dimension d of the space of hypersurfaces of degree δ having multiplicity m_i at p_i for each i ? This simple question is related to numerous other problems: Hilbert's fourteenth problem, Waring's problem, ample bundles on surfaces, symplectic packing ... ([16], [18], [3], [13], [5]). Surprisingly, the question is still open when the projective space has dimension at least two.

This problem is usually attacked using specialisation methods. There is an expected dimension d_e with $d_e \leq d$. The points p_i are moved to a special position. One computes the dimension d' in this special position and checks $d' = d_e$. By semi-continuity, $d_e \leq d \leq d'$ hence $d = d_e$. The difficulty with this approach is to find a good specialisation. Possible methods are the Horace method [12], collisions of fat points [8], or degenerations of the projective space [6].

The drawback of all these methods is that they are hardly usable when there are few points with high mutiplicities because of inevitable numerical difficulties. In this article, we introduce a method of specialisation which tackles the numerical difficulties appearing in these difficult cases. Then we apply the method to complete a result by Nagata and to compute collisions of fat points.

We put the problem in a general context. Let X be a (quasi-)projective scheme, $\mathcal{L}$ a linear system on X and $Z \subset X$ a generic 0-dimensional subscheme of degree d, ie. a subscheme parametrised by a non closed point $p_Z \in Hilb^d(X)$. We address the problem of determining the dimension $\dim \mathcal{L}(-Z)$.

In this introduction, we suppose for simplicity that $Z = Z_t$ is the generic fiber of a subscheme $F \subset X \times \mathbb{A}^1$ flat and finite over $\mathbb{A}^1 = Spec\ k[t]$ and such that the support of the fiber $F(t)$ approaches a divisor D when $t \rightarrow 0$. Our strategy is the following. We specialize Z_t to the subscheme $Z_0 = F(0) = \lim_{t\rightarrow 0} Z_t$. Accordingly, the linear system $\mathcal{L}(-Z_t)$ specialises to a system $\lim_{t\rightarrow 0} \mathcal{L}(-Z_t)$. The subspace $\mathcal{L}(-Z_t) \subset \mathcal{L}$ is associated to a (non closed) point p_t in a Grassmannian $\mathbb{G}$ of subspaces of $\mathcal{L}$. The limit $\lim_{t\to 0}\mathcal{L}(-Z_t)$ is by definition parametrised by the point $p_0 = \lim_{t\to 0} p_t \in \mathbb{G}$. By construction, the dimension of the linear system is preserved under specialisation: $\dim \lim_{t\to 0}\mathcal{L}(-Z_t) = \dim \mathcal{L}(-Z_t)$. In particular, if we can compute the dimension of the limit, we obtain the dimension of $\mathcal{L}(-Z_t)$.

To illustrate this idea with an example, let $\mathcal{L}$ be the set of homogeneous polynomials $P \in k[x, y, z]$ of degree 15, $D \subset \mathbb{P}^2$ the line with equation $y = 0$, p_1, p_2, p_3 three points of D, $p_4(t) = [x_4 : t : 1] \in \mathbb{P}^2$ and $p_5(t) = [x_5 : t : 1] \in \mathbb{P}^2$ two points which move to the line D when t tends to 0. Let $\mathcal{L}(-Z_t)$ be the set of homogeneous P of degree 15 which vanish at each of the points $p_i, i \leq 5$, with multiplicity four. The limit linear system $\lim_{t\to 0}\mathcal{L}(-Z_t)$ parametrises the set of reducible curves $C = 3D + C'$, with C' a curve of degree 12 passing through the points $p_1, p_2, p_3, p_4(0), p_5(0)$. The dimension of the limit is 86, hence $\dim \mathcal{L}(-Z_t) = 86$ too.

One could naïvely hope that $lim_{t\to 0}\mathcal{L}(-Z_t) = \mathcal{L}(-Z_0)$. This is not correct. There is a trivial inclusion

$$\lim_{t\to 0}\mathcal{L}(-Z_t) \subset \mathcal{L}(-Z_0)$$

but in general, this is a strict inclusion. For instance, in the above example, $\dim \mathcal{L}(-Z_0) = \dim \mathcal{L}(-Z_t) + 3$. In other words, the dimension of the linear system jumps when Z_t moves to Z_0. Our point is precisely to determine what could be the limit when the dimension jumps and the displayed inclusion is not an equality.

Our result gives an estimate of the limit $lim_{t\to 0}\mathcal{L}(-Z_t)$. More precisely, we introduce a combinatorical procedure to construct a system $\mathcal{L}' \subset \mathcal{L}(-Z_0)$ and we show an inclusion

$$\lim_{t\to 0}\mathcal{L}(-Z_t) \subset \mathcal{L}' \quad (*).$$

The system $\mathcal{L}'$ has the following form : we find an integer r and a residual scheme $Z_{res} \subset Z_0$ such that $\mathcal{L}' = \mathcal{L}(-rD - Z_{res})$.

With concrete examples (see the applications below), the inclusion $(*)$ suffices to compute $\dim \mathcal{L}(-Z_t)$ using the same argument as above: There is an expected dimension d_e which verifies $d_e \leq \dim \mathcal{L}(-Z_t) = \dim \lim_{t\to 0}\mathcal{L}(-Z_t) \leq \dim \mathcal{L}' = d_e$, hence our analysis finally computes the limit linear system and the dimension of the initial linear system:

$$\dim \mathcal{L}(-Z_t) = d_e \quad \text{and} \quad \lim_{t\to 0}\mathcal{L}(-Z_t) = \mathcal{L}'.$$

The method to estimate the limit is infinitesimal in nature. It is based on a study of deformations of a space of sections. There is a unique flat family G over $\mathbb{A}^1$ whose fiber over the generic point $t \in \mathbb{A}^1$ is $\mathcal{L}(-Z_t)$. Our theorem is obtained by a careful analysis of the restrictions $G \times_{\mathbb{A}^1} Spec\ k[t]/(t^{n_i}) \subset G$ for well chosen integers $n_1, \ldots, n_r$.

The inclusion $(*)$ generalizes the main lemma of Alexander-Hirschowitz [1]. Their statement corresponds essentially to ours in the special case $r = 1$. However, the proofs are different. When Alexander-Hirschowitz published their theorem, our theorem did already exist in a weaker version where the 0-dimensional subscheme Z moving to the divisor had to be supported by a unique point. The current version is a merge which contains both our earlier version and Alexander-Hirschowitz version.

As an application, we extend results by Nagata relative to the Hilbert functions of fat points in the plane. We recall that a consequence of Alexander-Hirschowitz [1] is that the Hilbert function of a generic union of k fat points in the plane of multiplicity $m_1, \ldots, m_k$ is $H_Z(d) = min(\frac{(d+1)(d+2)}{2}, \sum_{i=1}^{k} \frac{m_i(m_i+1)}{2})$ provided $k >> max(m_i)$. The "opposite" cases, those with a fixed number of points (≥ 10) and big multiplicities, have been considered by Nagata. As explained above, they are known empirically to be difficult cases. In connection with his construction of the counter example to the fourteenth problem of Hilbert, Nagata proved that the Hilbert function of a generic union Z of k^2 fat points of the same multiplicity m in $\mathbb{P}^2$ is $H_Z(d) = \frac{(d+1)(d+2)}{2}$ if the degree is not too big, namely if $d \leq km$. This result is asymptotically optimal in m in the sense that it is sufficient to compute the Hilbert function up to the critical degree $d = km + [\frac{k-3}{2}]$ to determine the whole Hilbert function. Nagata was just missing the last $[\frac{k-3}{2}]$ cases. We compute the Hilbert function for every degree:

Theorem . $H_Z(d) = min(\frac{(d+1)(d+2)}{2}, k^2\frac{m(m+1)}{2})$.

This result was already proved when the number of points is a power of four in [9] by methods relying on the geometry of integrally closed ideals which we could not push much further.

Putting the result in perspective, there is a conjecture by Harbourne-Hirschowitz relative to the Hilbert function of a generic union of fat points. The above theorem is a new evidence for the conjecture as it involves cases with few points and big multiplicities.

As a second application, we propose a method to compute collisions of fat points in the plane. We recall that a collision of punctual subschemes $Z_1, \dots, Z_s \subset \mathbb{A}^2$ is a subscheme obtained as a flat limit when the support of the Z_i's approach the same point.

The collisions of at most three fat points are known [7]. When there are four points or more, the situation is still largely open: some collisions have been computed by Ciliberto and Miranda [4] and in [9], but most of them remain to be described.

To illustrate our method, we compute the collisions of four fat points of the same multiplicity which approach successivly the origin along a smooth curve (theorem 21). Besides this illustration, it is clear from the proofs that it is possible using the same method to compute an infinite number of collisions.

Our motivation for determining the collisions is the following. If $Z = Z_1 \cup \dots \cup Z_s$ is a generic union of fat points, the Hilbert function of Z is determined by the collisions of the Z_i's. Indeed, there exist "universal" collisions C on which one can read off the Hilbert function of Z: $\forall d,\ H_Z(d) = H_C(d)$ [8]. Determining all collisions of any number of fat points is far beyond our knowledge. However, by semi-continuity it would suffice to exhibit a collision with the expected Hilbert function to prove the Harbourne-Hirschowitz conjecture for Z, hence the need to understand the collisions. The computations of the present paper are a step in this direction.

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2 Statement of the theorem

An elementary example

As the statement of the theorem is somehow intricate, we start with an elementary example to understand the kind of result we are looking for. Precise and more formal statements will follow in the next sections.

Let $\mathcal{L}$ be the vector space whose elements are homogeneous polynomials $P(x, y, z)$ of degree 8 vanishing with order 2 on 4 points $p_1, \dots, p_4 \in \mathbb{P}^2$, aligned on the line D with equation $x = 0$, and vanishing on a fifth general point p_5 with order 4. The order of contact between D and a curve $C \in \mathbb{P}(\mathcal{L})$ is $8 = 4.2$. By Bezout, if the contact was 9, then D would be a fixed component of $\mathcal{L}$. Though we miss $1 = 9 - 8$ orders of contact to prove it, suppose that D is a fixed component of $\mathcal{L}$. An equation $f \in \mathcal{L}$ then writes down $f = xg$ where g has degree 7 and the curve C_g passes through $p_1, \dots, p_4$. By Bezout again, we miss $4 = 8 - 4$ orders of contact to show that g vanishes on D. Summing up, we missed $5 = 1 + 4$ orders of contact to show that D is a fixed component with multiplicity 2.

Passing through p_5 with multiplicity 4 is equivalent to containing a scheme Z_5 of length 10. We move $Z_5 = Z_5(t)$ towards the line D when the time t tends to 0. Then $\mathcal{L} = \mathcal{L}(t)$ tends to a system $\mathcal{L}(0)$. We want to prove that in the limit process, Z_5 gives the missing orders of contact to $\mathcal{L}$, ie. $\mathcal{L}(0)$ will have D as a fixed component with multiplicity 2. If $C \in \mathbb{P}(\mathcal{L}(0)) - 2D$ is a curve of degree 6 in the moving part of the limit linear system, C does not contain Z_5 any more, but a subscheme $Z_5' \subset Z_5$ of length $5 = 10 - 5$ obtained in some sense by "taking off" the 5 orders of contact from Z_5 which have been given to $\mathcal{L}$. To say it precisely, we want to prove the existence of a subscheme Z'_5 of length 5 such that $\mathcal{L}(0) \subset \mathcal{M}$, where $\mathcal{M}$ is the set of polynomials f which decompose: $f = x^2 g$ with g vanishing on Z'_5.

To construct Z'_5, we represent Z_5 with a combinatorical diagram and taking off order of contacts corresponds to a suppression of slices in the diagram (see the picture page 5). In this example, one can prove that the limit linear system $\mathbb{P}(\mathcal{L}_0)$ contains the curves $C = 2D + E$ where E is a curve of degree 6 with a cusp tangent to the line D.

It is possible to do the same analysis when several points approach a divisor. But then the limit depends on the speed of each moving point. For instance, if two punctual schemes Z_1 and Z_2 approach the same divisor D and 5 orders of contact are needed to make D a fixed component, it is possible to pick up a orders of contact from Z_1 and b from Z_2 with $a+b=5$. Different choices for the numbers a and b are possible depending on the speeds of the points Z_1 and Z_2.

In the following analysis, we use the language of generic points and specialisation (which is more precise and compact) rather than the language of families and limits.

Notations

We fix a generically smooth quasi-projective scheme X of dimension d, a locally free sheaf L of rank one on X and a sub-vector space $\mathcal{L} \subset H^0(X, L)$. Let $Z \subset X_{k(Z)}$ be a 0-dimensional subscheme parametrised by a non closed point of $Hilb(X)$ with residual field $k(Z)$. Let $\mathcal{L}(-Z) \subset \mathcal{L}$ be the sub-vector space of sections which vanish on Z (see the definition below). Our goal is to give an estimate of the dimension $\dim \mathcal{L}(-Z)$ under suitable conditions.

The generic point $X(E)$

A staircase $E \subset \mathbb{N}^d$ is a subset whose complement $C = \mathbb{N}^d \setminus E$ verifies $\mathbb{N}^d + C \subset C$. We denote by I^E the ideal of $k[x_1, \dots, x_d]$ (resp. of $k[[x_1, \dots, x_d]]$, of $k[[x_1, \dots, x_d]][t] \dots$) generated by the monomials $x_1^{e_1} \cdot \ldots \cdot x_d^{e_d} = x^e$ whose exponent $e = (e_1, \dots, e_d)$ is in C. If E is a finite staircase, the subscheme $Z(E)$ defined by I^E is 0-dimensional and its degree is $\#E$. The map $E \mapsto Z(E)$ is a one-to-one correspondence between the finite staircases of $\mathbb{N}^d$ and the monomial punctual subschemes of $Spec\ k[x_1, \dots, x_d]$. If $E = (E_1, \dots, E_s)$ is a set of finite staircases, if X is irreducible and if $Z(E)$ is the (abstract non embedded) disjoint union $Z(E_1) \coprod \cdots \coprod Z(E_s)$, there is an irreducible scheme $P(E)$ which parametrizes the embeddings $Z(E) \to X_s$, where $X_s \subset X$ is the smooth locus ([11] and [12]). Such an embedding $Z(E) \to X_s$ determines a subscheme of X, thus there is a natural morphism $f : P(E) \to Hilb(X)$ to the Hilbert scheme of X. We denote by $X(E)$ the subscheme parametrised by $f(p)$ where p is the generic point of $P(E)$. We call $X(E)$ the generic union of the schemes $Z(E_1), \dots, Z(E_s)$.

The linear system $\mathcal{L}(-X(E))$

If $Z \subset X$ is a subscheme, denote by $\mathcal{L}(-Z) \subset \mathcal{L}$ the subvector space which contains the elements of $\mathcal{L}$ vanishing on Z. If p is a non closed point of $Hilb(X)$ whose residual field is $k(p)$, and if $Z \subset X \times_k Spec\ k(p)$ is the corresponding subscheme, the definition of $\mathcal{L}(-Z)$ is as follows. Since $\mathcal{L} \otimes k(p) \subset H^0(L \otimes Spec\ k(p), X \times Spec\ k(p))$, it makes sense to consider the vector space $V \subset \mathcal{L} \otimes k(p)$ containing the sections which vanish on Z. Denoting by λ the codimension of V, we may associate with V a $k(p)$-point $g \in Grass_{k(p)}(\lambda, \mathcal{L} \otimes k(p)) = Grass_k(\lambda, \mathcal{L}) \times Spec\ k(p)$ ([10], prop.9.7.6). In particular $\mathcal{L}(-Z)$ is well defined as a (non closed) point of $Grass_k(\lambda, \mathcal{L})$. The goal of the theorem is to give an estimate of $\dim \mathcal{L}(-X(E))$.

Combinatorical constructions

To formulate the theorem, we need some combinatorical notations that we introduce now. The τ^{th} slice of a staircase $E \subset \mathbb{N}^d$ is the staircase $T(E,\tau) \subset \mathbb{N}^d$ defined by:

$$T(E,\tau) = \{(0, a_2, \ldots, a_d) \textit{ such that } (\tau, a_2, \ldots, a_d) \in E\}$$

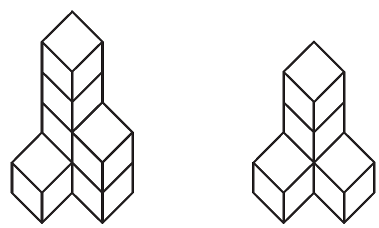
If $E = (E_1, \ldots, E_s)$ is a s-tuple of staircases and $\tau = (\tau_1, \ldots, \tau_s)$, we set

$$T(E,\tau) = (T(E_1,\tau_1), T(E_2,\tau_2), \ldots, T(E_s,\tau_s)).$$

A staircase $E \subset \mathbb{N}^d$ is characterized by a height function $h_E : \mathbb{N}^{d-1} \to \mathbb{N}$ which verifies: $\forall a, b \in \mathbb{N}^{d-1},\ h_E(a+b) \leq h_E(a)$. The staircase E and h_E can be deduced one from the other via the relation: $(a_1, \ldots, a_d) \in E \Leftrightarrow a_1 < h_E(a_2, \ldots, a_n)$. The staircase $S(E,\tau)$ is defined by its height function:

$$\begin{aligned} h_{S(E,\tau)}(a_2, \ldots, a_d) &= h_E(a_2, \ldots, a_d) \text{ if } \tau \geq h_E(a_2, \ldots, a_d) \\ &= h_E(a_2, \ldots, a_d) - 1 \text{ if } \tau < h_E(a_2, \ldots, a_d). \end{aligned}$$

Intuitively, $S(E,\tau)$ is the staircase obtained from E after the suppression of the t^{th} slice, as shown by the following picture.



Staircase Suppression of slice number one

If $E = (E_1, \ldots, E_s)$ is a family of staircases, and $\tau = (\tau_1, \ldots, \tau_s) \in \mathbb{N}^s$, we put:

$$S(E,\tau) = (S(E_1,\tau_1), S(E_2,\tau_2), \ldots, S(E_s,\tau_s)).$$

If $(\tau_1, \ldots, \tau_r) \in (\mathbb{N}^s)^r$, the recursive formula

$$S(E, \tau_1, \ldots, \tau_r) = S(S(E, \tau_1, \ldots, \tau_{r-1}), \tau_r)$$

defines the s-tuple of staircases $S(E, \tau_1, \ldots, \tau_r)$ obtained from the s-tuple $E = (E_1, \ldots, E_s)$ by suppression of r slices in each E_i.

The generic point $X_\varphi(E,t,v)$

Suppose that E is a staircase. We want to give an upper bound to $\dim \mathcal{L}(-X(E))$. A specialisation $X_\varphi(E,t,v)$ of $X(E)$ is introduced. By semi-continuity, $\dim \mathcal{L}(-X(E)) \leq \dim \mathcal{L}(-X_\varphi(E,t,v))$ thus it will suffice to give an upper bound for $\dim \mathcal{L}(-X_\varphi(E,t,v))$.

Whereas the generic point $X(E)$ corresponds to a monomial subscheme with staircase E which can move generically, $X_\varphi(E,t,v)$ is more special and corresponds to a monomial subscheme which can move only in a prescribed way with respect to a coordinate patch φ. We construct a family whose base is *Spec* $k[[t]]$ by explicit equations in the coordinate patch φ. The equations depend on the time t and v is a control parameter for the velocity of the moving subscheme. The subscheme $X_\varphi(E,t,v)$ is the generic fiber of this explicit family.

To be precise, the subscheme $X_\varphi(E,t,v)$ is defined as follows. If $p \in X$ is a smooth point, a formal neighborhood of p is a morphism $\varphi : \textit{Spec } k[[x_1, \ldots, x_d]] \to X$ which induces an

isomorphism between $Spec\ k[[x_1,\dots,x_d]]$ and the completion $\widehat{\mathcal{O}_p}$ of the local ring of X at p. If $p=(p_1,\dots,p_s)$ is a s-tuple of smooth distinct points, a formal neighborhood of p is a morphism $(\varphi_1,\dots,\varphi_s): U \to X$ from the disjoint union $U = V_1 \coprod \dots \coprod V_s$ of s copies of $Spec\, k[[x_1,\dots,x_d]]$ to X, where $\varphi_i : V_i \to X$ is a formal neighborhood of p_i. If D is a divisor on X, we say that φ and D are compatible if D is defined by the equation $x_1=0$ around each p_i (in particular, p_i is a smooth point of D and X).
Consider the translation morphism:

$$\begin{array}{rcl} Tr_{v_1}: k[[x_1,\dots,x_d]] & \to & k[[x_1,\dots,x_d]]\otimes k[[t]] \\ x_1 & \mapsto & x_1\otimes 1 - 1 \otimes t^{v_1} \\ x_i & \mapsto & x_i \otimes 1 \text{ if } i>1 \end{array}$$

If E_1 is a staircase, the ideal

$$J(E_1,v_1) = Tr_{v_1}(I^{E_1})k[[x_1,\dots,x_d]]\otimes k[[t]] \subset k[[x_1,\dots,x_d]]\otimes k[[t]]$$

defines a flat family F_1 of subschemes of $Spec\ k[[x_1,\dots,x_d]]$ parametrised by $Spec\ k[[t]]$. This corresponds geometrically to the family whose fiber over t is obtained from $Z(E_1)$ by the translation $x_1 \mapsto x_1 - t^{v_1}$. If φ_1 is a formal neighborhood of p_1, F_1 can be seen as a flat family of subschemes of X via φ_1, thus it defines a morphism $Spec\ k[[t]] \to Hilb(X)$. We denote by $X(\varphi_1,E_1,t,v_1)$ the non closed point of $Hilb(X)$ parametrised by the image of the generic point. The first coordinate does not play any specific role. Thus more generally, if $E=(E_1,\dots,E_s)$ is a family of staircases, if $\varphi=(\varphi_1,\dots,\varphi_s)$ is a formal neighborhood of $(p_1,\dots,p_s)$, if $v=(v_1,\dots,v_s)\in\mathbb{N}^s$, one defines similarly families $F_i \subset X\times Spec\ k[[t]]$ flat over $Spec\ k[[t]]$. The disjoint union $F=F_1\cup\dots\cup F_s$ is still flat over $Spec\ k[[t]]$ and corresponds to a morphism $Spec\ k[[t]] \to Hilb(X)$. We denote by $X_\varphi(E,t,v)$ the image of the generic point and by $X_\varphi(E)=X_\varphi(E,0,v)$ the image of the special point (which does not depend on v).

Notation

We denote by $[x]$ the integer part of a real x. If J is an ideal of a ring R, and $s\in R$, we denote $(J:s)=\{r\in R,\ sr\in J\}$.

Statement of the theorem

We are now ready to state the theorem. By the above, $\mathcal{L}(-X_\varphi(E,t,v))$ corresponds to a morphism $Spec\, k((t)) \to \mathbb{G}$ to a Grassmannian $\mathbb{G}$, which extends to a morphism $Spec\, k[[t]] \to \mathbb{G}$ by valuative properness. The theorem gives a control of the limit obtained under suitable conditions.

The formulation is more transparent when there is a unique point moving towards the divisor ($s=1$). The speed v of the point is chosen to be 1.

Theorem 1. *Let D be an effective divisor on a quasi-projective scheme X, $p\in X$, φ a formal neighborhood of p compatible with D, E a staircase with slices $T_0,T_1,\dots$. Let $T_{n_1},\dots,T_{n_r}$ be slices of E with associated subschemes $Z_i=X_\varphi(T_{n_i})$ and $n_1>n_2>\dots>n_r$. Let $F=E\setminus T_{n_1},\dots,T_{n_r}$ be the staircase obtained after suppression of the slices T_{n_i} in E. If*

$$\forall i, 1\leq i\leq r, \mathcal{L}(-(i-1)D - Z_i) = \mathcal{L}(-iD),$$

then

$$\lim_{t\to 0}\mathcal{L}(-X_\varphi(E,t,v=1)) \subset \mathcal{L}(-rD - X_\varphi(F))$$

In the more general version, there are several moving points and one needs to be careful about the speed of each point to describe the limit.

Theorem 2. *Let D be an effective divisor on a quasi-projective scheme X, $p = (p_1, \ldots, p_s)$ be a s-tuple of smooth points of X, φ a formal neighborhood of p compatible with D, $v = (v_1, \ldots, v_s) \in \mathbb{N}^s$ a speed vector, $E = (E_1, \ldots, E_s)$ be staircases and $X_\varphi(E, t, v)$ the generic union of subschemes defined by φ. Suppose that one can find integers $n_1 > \cdots > n_r$ such that:*

- $\forall k$, $n_k - n_{k+1} \geq max(v_i)$,
- $\forall i$, $1 \leq i \leq r$, $\mathcal{L}(-(i-1)D - Z_i) = \mathcal{L}(-iD)$

where $\tau_i = ([\frac{n_i-1}{v_1}], \ldots, [\frac{n_i-1}{v_s}])$, $T_i = T(E, \tau_i)$ and $Z_i = X_\varphi(T_i)$. Then

$$\lim_{t\to 0} \mathcal{L}(-X_\varphi(E, t, v)) \subset \mathcal{L}(-rD - X_\varphi(S(E, \tau_1, \ldots, \tau_r)))$$

Remark 3. *The main lemma 2.3 of [1] corresponds essentially to the above theorem with $r = 1$. Our theorem also generalizes to the vertically graded subschemes considered in [1] instead of monomial subschemes.*

If X is irreducible, $X(E)$ is well defined and it specializes to $X_\varphi(E, t, v)$. Thus we get by semi-continuity the inequality

$$\dim \mathcal{L}(-X(E)) \leq \dim \mathcal{L}(-X_\varphi(E, t, v)) = \dim \lim_{t\to 0} \mathcal{L}(-X_\varphi(E, t, v)).$$

Combining this inequality with the theorem, we obtain the following estimate of $\dim \mathcal{L}(-X(E))$ in terms of a linear system of smaller degree.

Corollary 4. $\dim \mathcal{L}(-X(E)) \leq \dim \mathcal{L}(-rD - X_\varphi(S(E, \tau_1, \ldots, \tau_r)))$

Remark 5. *In case $\mathcal{L}$ is infinite dimensional, the theorem still makes sense since Grassmannians of finite codimensional vector spaces of $\mathcal{L}$ are still well defined and the limit makes sense in such a Grassmannian.*

2.1 Comment on the conditions of the theorem and plan of the proof

In this section, we explain the technical conditions $n_k - n_{k+1} \geq max(v_i)$ and $\tau_i = ([\frac{n_i-1}{v_1}], \ldots, [\frac{n_i-1}{v_s}])$ appearing in the statement of the theorem, and we give a very rough plan of the proof.

Consider the example at the beginning of section 2. There is a subscheme $Z_5(t)$ corresponding to a point of multiplicity 4, which moves towards the line D as t tends to 0. When $t \neq 0$, $Z_5(t) \cap D = \emptyset$. When $t = 0$, $Z_5(0) \cap D$ is a scheme of length 4. To apply Bezout properly, we need to find a t such that $Z_5(t) \cap D$ is a scheme of length 1. Thus neither $t = 0$ nor $t \neq 0$ are suitable. The idea is then to choose $t \neq 0$ but $t^n = 0$, which rigorously corresponds to a restriction over the base *Spec* $k[t]/(t^n)$. The point is to understand how we choose n to get the required intersection. On this example, if $t^4 = 0$, $Z_5(t) \cap D$ "is" a scheme of length 1, as required.

What do we mean when we compute the intersection $Z(t) \cap D$ for $t^n = 0$? Consider a monomial ideal I^E and make the change of variable $x_1 \mapsto x_1 - t^v$ to get the ideal $J(t)$ of $X_\varphi(E, t, v)$. For instance, suppose that the staircase E is defined by $I^E = (x_1^2, x_1x_2^2, x_2^3)$. In other words, E is made from two slices T_0, T_1 corresponding to the subschemes with ideals $I^{T_0} = (x_1, x_2^3)$ and $I^{T_1} = (x_1, x_2^2)$, and $J(t) = (x_1 - 2t^v x_1 + t^{2v}, (x_1 - t^v)x_2^2, x_2^3)$. When we work over *Spec* $k[t]/t^n$, formally, we replace t^i by zero if $i \geq n$. It is easy to see on our example that if $t^v = 0$, $J(t) \subset I^{T_0}$ and if $t^{2v} = 0$, then $J(t) \subset I^{T_1}$. Geometrically, this means that if t is in the infinitesimal neighborhood *Spec* $k[t]/(t^n)$, $n \leq (i+1)v$, then the intersection $X_{\varphi(E,t,v)} \cap D$ contains the subscheme associated with slice number i.

The general case is similar to this example: for any staircase E, if we restrict to the infinitesimal neighborhood $t^n = 0$, the trace $X_\varphi(E, t, v) \cap D$ contains the subscheme associated with the slice number $[\frac{n-1}{v}]$.

Consider finally the case with several staircases $E_1, \dots, E_s$ and associated subschemes $Z(E_1), \dots, Z(E_s)$ moving with speed $v_1, \dots, v_s$ towards the divisor D, and $X_\varphi(E,t,v) = X_{\varphi_1}(E_1,t,v_1) \coprod \cdots \coprod X_{\varphi_s}(E_s,t,v_s)$. When we make $t^{n_i} = 0$, we see that $X_\varphi(E,t,v) \cap D$ contains a union of subschemes $Z_i = R_1 \coprod \cdots \coprod R_s$ where R_k is defined by the slice number $[\frac{n_i-1}{v_k}]$ of E_k. In other words, the coordinates of $\tau_i = ([\frac{n_i-1}{v_1}], \dots, [\frac{n_i-1}{v_s}])$ are the index of the slices corresponding to the intersection $X_\varphi(E,t,v) \cap D$ when we consider the restriction $t^{n_i} = 0$.

Now the condition $n_1 \geq n_2 \geq n_3 ...$ is clear. It comes from the fact that our analysis uses smaller and smaller restrictions. We restrict over *Spec* $k[t]/(t^{n_1})$ to get the required order of contact $X_\varphi(E,t,v) \cap D$ and we make an analysis of the situation. Then we restrict to a smaller infinitesimal neighborhood *Spec* $k[t]/(t^{n_2})$ and so on.

To explain why the hypothesis required in the theorem is $n_i - n_{i+1} \geq max(v_i)$, which is a bit more than the natural inequality $n_i \geq n_{i+1}$, we look more precisely at the plan of the proof.

Suppose that we have of family of sections $s(t)$ of L vanishing on a moving punctual subscheme $Z(t) = X_\varphi(E,t,v)$ whose support $p(t)$ tends to $p(0)$ as t tends to 0. Using local coordinates around $p(0)$, the sections of L can be considered as functions and the vanishing on $Z(t)$ translates to $s(t) \in J(t)$ where $J(t)$ is the ideal of $Z(t)$. Denote by J_{n_1} the restriction of $J(t)$ to the infinitesimal neighborhood *Spec* $k[t]/t^{n_1}$ of $t = 0$. As explained above, we put $t^{n_1} = 0$ in $J(t)$ and we see that the functions in J_{n_1} vanish on Z_1. In particular, if $t^{n_1} = 0$, $s(t)$ is a family of sections vanishing on Z_1. Then it is a family of sections vanishing on D since by hypothesis a section which vanishes on Z_1 vanishes on D. If D is defined locally by the equation $x_1 = 0$, this means that $s(t) = x_1 s'(t)$ with $s'(t) \in (J_{n_1} : x_1)$. Restrict now to the smaller infinitesimal neighborhood *Spec* $k[t]/t^{n_2}$ and denote by $(J_{n_1} : x_1)_{n_2}$ the restriction of $(J_{n_1} : x_1)$. The restriction of $s'(t)$ to *Spec* $k[t]/t^{n_2}$ is an element of $(J_{n_1} : x_1)_{n_2}$, and a computation shows that it vanishes on Z_2. Then by hypothesis, $s'(t)$ vanishes on D. Using local coordinates, this means that if $t^{n_2} = 0$, $s'(t) = x_1 s''(t)$, with $s''(t) \in ((J_{n_1} : x_1)_{n_2} : x_1)$ and $s(t) = x_1 s'(t) = x_1^2 s''(t)$. Then we put $t = 0$ and we get $s(0) = x_1^2 s''(0)$ where $s''(0) \in ((J_{n_1} : x_1)_{n_2} : x_1)(0)$. A computation shows that $((J_{n_1} : x_1)_{n_2} : x_1)(0) = I^{S(E,\tau_1,\tau_2))}$. The control we get in this way of any element $s(0) \in \lim_{t\to 0} \mathcal{L}(-X_\varphi(E,t,v))$ corresponds to the inclusion

$$\lim_{t \longrightarrow 0} \mathcal{L}(-X_\varphi(E,t,v)) \subset \mathcal{L}(-2D - X_\varphi(S(E,\tau_1,\tau_2)))$$

given by the theorem in the case $r = 2$.

For a general r, the proof follows the same lines. We simply do r restrictions instead of two and we need to control a more complicated ideal $(((J_{n_1} : x_1)_{n_2} : x_1)....)_{n_r} : x_1))$ instead of $((J_{n_1} : x_1)_{n_2} : x_1)$. The computation of the ideal $(((J_{n_1} : x_1)_{n_2} : x_1)....)_{n_r} : x_1))$ is difficult in general, but it simplifies if $n_k - n_{k+1} \geq max(v_i)$. Thus the condition $n_k - n_{k+1} \geq max(v_i)$ is a technical condition to make possible the computations of the ideals involved in the proof.

3 Proof of theorem 2

In the context of the theorem, we are given a set of staircases $E = (E_1, \dots, E_s)$, a vector $v = (v_1, \dots, v_s)$, a divisor D, a formal neighborhood φ of $(p_1, \dots, p_s)$ in which D is given by the equation $x_1 = 0$ around each p_i, and integers $n_1, \dots, n_r$. For $n > 0$, we put $R_n = k[[x_1, \dots, x_d]]^s \otimes k[[t]]/(t^n)$ and $R_\infty = k[[x_1, \dots, x_d]]^s \otimes k[[t]]$. The formal neighborhood $\varphi = (\varphi_1, \dots, \varphi_s)$ is viewed as a map *Spec* $R_1 \to X$. We denote by $\psi_{np} : R_n \to R_p$ the natural projections, which exist for $p \leq n \leq \infty$. If $J \subset R_\infty$ is an ideal, we define recursively the ideals $J_{n_k} \subset R_{n_k}$ and $J_{n_k:} \subset R_{n_k}$ (mind the semicolon in the subscript) by the formulas

- $J_{n_1} = \psi_{\infty n_1}(J)$,
- $J_{n_k:} = (J_{n_k} : x_1)$,
- $J_{n_k} = \psi_{n_{k-1} n_k}(J_{n_{k-1}:})$

As explained before, the vector space $\mathcal{L}(-X(\varphi, E, t, v))$ corresponds to a morphism $Spec\ k((t)) \to \mathbb{G}$ (where $\mathbb{G}$ is a Grassmannian of subvector spaces of $\mathcal{L}$) which extends to a morphism $Spec\ k[[t]] \to \mathbb{G}$. The universal family over the Grassmannian $\mathbb{G}$ pulls back to a family $\tilde{U} \subset Spec\ k[[t]] \times \mathcal{L}$. If $V \subset \mathcal{L}$ is a subvector space, we can define its base locus $B_V \subset X$. In the relative situation, the flat family $\tilde{U}$ of subvector spaces parametrised by $Spec\ k[[t]]$ defines a family of base loci $B_{\tilde{U}}(t) \subset Spec\ k[[t]] \times X$. Since we are interested in the part of the base locus contained in the formal neighborhood $\varphi : Spec\ R_1 \to X$, we consider the intersection $B_{\tilde{U}}(t) \cap (Spec\ k[[t]] \times Spec\ R_1)$ which is defined by an ideal $U \subset R_\infty$. The theorem will be proved if we show that the special fiber $\tilde{U}(0)$ contains only sections vanishing r times on D and if, in local coordinates, $U(0)$ is included in $x_1^r I^{S(E,\tau_1,\ldots,\tau_r)}$.
Let us denote by $\tilde{U}_{n_i} \subset Spec\ k[t]/t^{n_i} \times \mathcal{L}$ and $U_{n_i} \subset R_{n_i}$ the restrictions of $\tilde{U}$ and U over the subscheme $Spec\ k[[t]]/t^{n_i}$. We show by induction that:

$$\forall i \geq 1,\ U_{n_i} \subset x_1^i J_{n_i}:$$

where $J = J(E_1, v_1) \oplus \cdots \oplus J(E_s, v_s) \subset R_\infty$. The proof relies on the following two lemmas whose proof is postponed. These lemmas control the ideal J_{n_k} and the restriction of $J_{n_k}:$ to the special fiber $t = 0$.

Lemma 6. $J_{n_k} \subset I^{T_k}$

Lemma 7. $J_{n_k}:(0) = I^{S(E,\tau_1,\ldots,\tau_k)}$.

The fibers of $\tilde{U}$ contain sections of $\mathcal{L}$ which vanish on $X_\varphi(E, t, v)$. Since J is the ideal of $X_\varphi(E, t, v)$, this implies the inclusion $U \subset J$, hence $U_{n_1} \subset J_{n_1}$. By lemma 6, this inclusion implies that the fibers of $\tilde{U}_{n_1}$ are elements of $\mathcal{L}$ which vanish on Z_1, hence they vanish on D by hypothesis. It follows that elements of U_{n_1} are divisible by x_1 and we can then write: $U_{n_1} \subset x_1 J_{n_1}:$. Suppose now that $U_{n_i} \subset x_1^i J_{n_i}:$. Then $U_{n_{i+1}} \subset x_1^i J_{n_{i+1}}$. By lemma 6, this inclusion implies that the fibers of $\tilde{U}_{n_{i+1}}$ are elements of $\mathcal{L}(-iD)$ which vanish on Z_{i+1}, hence they vanish on D by hypothesis. It follows that elements of $U_{n_{i+1}}$ are divisible by x_1^{i+1} and we can write $U_{n_{i+1}} \subset x_1^{i+1} J_{n_{i+1}}:$. This ends the induction on i. In particular, for $i = r$, using lemma 7 for the last equality, we have the required inclusion:

$$U(0) = U_{n_r}(0) \subset x_1^r J_{n_r}:(0) = x_1^r I^{S(E,\tau_1,\ldots,\tau_r)}.\blacksquare$$

We now turn to the proof of the lemmas 6 and 7 on which the above proof relies. Note that $J = (J^1, \ldots, J^s)$ and $I^{T_k} = ((I^{T_k})^1, \ldots, (I^{T_k})^s)$ are defined componentwise, the component number i corresponding to the study around the point p_i. Thus lemmas 6 and 7 below can be proved for each component and one may suppose $s = 1$ to prove them. We thus suppose for the rest of this section that $s = 1$, that $E = (E_1, \ldots, E_s)$ is a staircase given by a height function h, and that $v = (v_1, \ldots, v_s) \in \mathbb{N}$.
Let $\mathcal{B}$ (resp. $\mathcal{C}$) be the set of elements $m = (m_2, \ldots, m_d) \in \mathbb{N}^{d-1}$ such that $h(m) \neq 0$ (resp. $h(m) = 0$). Remark that $\mathcal{B}$ is finite due to the finitness of E. We denote by

- $C(t) \subset R_n$ the $k[[x_1]] \otimes k[[t]]$ sub-module containing the elements $\sum a_{m_1 m_2 \ldots m_d} x_1^{m_1} x_2^{m_2} \ldots x_d^{m_d} \otimes f(t)$, where $f(t) \in k[[t]]/t^n$ and $(m_2, \ldots, m_d) \in \mathcal{C}$
- $C(0) \subset R_1 = k[[x_1, \ldots, x_d]]$ the $k[[x_1]]$ sub-module containing the series $\sum a_{m_1 m_2 \ldots m_d} x_1^{m_1} x_2^{m_2} \ldots x_d^{m_d}$ where $(m_2, \ldots, m_d) \in \mathcal{C}$
- $B(m) \subset R_n$ the $k[[x_1]] \otimes k[[t]]$ sub-module generated by $f_m = (x_1 - t^v)^{h(m)} x_2^{m_2} \ldots, x_d^{m_d}$,
- $B(m, 0) \subset R_1 = k[[x_1, \ldots, x_d]]$ the $k[[x_1]]$ sub-module generated by $f_m(0) = (x_1)^{h(m)} x_2^{m_2} \ldots, x_d^{m_d}$,
- $B_{n_k}(m) \subset R_n$ the $k[[x_1]] \otimes k[[t]]$ sub-module generated by the elements f_m, $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i}$, $1 \leq i \leq k$, where $\alpha_i = max(0, n_i - vh(m))$ for $i > 0$. In particular, for $k = 0$, $B_{n_k}(m) = B(m)$.

To simplify the notations, we have adopted above the same notation for distinct submodules (living in distinct ambiant modules). The following lemma says that the module $B_{n_k}(m)$ is well defined as a sub-module of R_j for $j \leq n_k$.

Lemma 8. *Let $j \leq n_k$. If $i \leq k$, the element $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i} \in R_j$. In particular $B_{n_k}(m) \subset R_j$ is well defined for $j \leq n_k$. If in addition, $j \leq n_{k+1}$, then $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i}$ is a multiple of x_1.*

Proof. First, if $l < i$, the coefficient of x_1^l in $t^{\alpha_{k-i+1}} f_m$ is a multiple of $t^{\alpha_{k-i+1}} t^{v(h(m)-l)}$. This term is zero in R_j since the exponent of t is at least $n_{k-i+1} - vl \geq n_k + (i-1)v - vl \geq n_k \geq j$. It follows that $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i} \in R_j$ is well defined. A similar estimate shows that for $l \leq i$, the coefficient of x_1^l in $t^{\alpha_{k-i+1}} f_m$ is zero in R_j for $j \leq n_{k+1}$. Thus $\frac{t^{\alpha_{k-i+1}} f_m}{x_1^i}$ is a multiple of x_1. ■

Lemma 9.
- *As $k[[x_1]]$-modules, $I^E = \bigoplus_{m \in \mathcal{B}} B(m,0) \oplus C(0) \subset k[[x_1, \dots, x_d]]$*
- *As $k[[x_1]] \otimes k[[t]]$-modules, $J = \bigoplus_{m \in \mathcal{B}} B(m) \oplus C(t) \subset R_\infty$*

Proof: This is a straightforward verification left to the reader. ■

Lemma 10. *We have the equality of $k[[x_1]] \otimes k[[t]]$-modules:*
- $J_{n_k} = \bigoplus_{m \in \mathcal{B}} B_{n_{k-1}}(m) \oplus C(t) \subset R_{n_k}$
- $J_{n_k:} = \bigoplus_{m \in \mathcal{B}} B_{n_k}(m) \oplus C(t) \subset R_{n_k}$

Proof. Let us say that the index i of J_{n_k} and $J_{n_k:}$ is respectivly $2k-1$ and $2k$. We prove the lemma by induction on the index i. If $i = 1$, we get from the preceding lemma the equality

$$\begin{aligned} J_{n_1} = \psi_{\infty n_1}(J) &= \sum_{m \in \mathcal{B}} \psi_{\infty n_1}(B(m)) + \psi_{\infty n_1}(C(t)) \\ &= \sum_{m \in \mathcal{B}} B(m) + C(t) \text{ in } R_{n_1}. \end{aligned}$$

The last sum is obviously direct, thus it is the required equality.
Suppose now that we want to prove the lemma for $i = 2k-1$. This is exactly the same reasoning as in the case $i = 1$, substituting J_{n_k}, $J_{n_{k-1}:}$ and $\psi_{n_{k-1} n_k}$ for J_{n_1}, J, and ψ_{∞, n_1}.
Consider now the case $i = 2k$. Taking the conductor from the expression of J_{n_k} coming from induction hypothesis, we get:

$$J_{n_k:} = \bigoplus_{m \in \mathcal{B}} (B_{n_{k-1}}(m) : x_1) \oplus (C(t) : x_1)$$

The equality $(C(t) : x_1) = C(t)$ is obvious, so we are done if we prove the equality $(B_{n_{k-1}}(m) : x_1) = B_{n_k}(m)$ in the ambiant module R_{n_k}. The inclusion $\supset$ is clear since for every generator g of $B_{n_k}(m)$, $x_1 g$ is a multiple of one of the generators of $B_{n_{k-1}}(m)$. As for the reverse inclusion, if $z \in (B_{n_{k-1}}(m) : x_1)$, one can write down

$$x_1 z = \sum_{1 \leq i \leq k-1} P_i \frac{t^{\alpha_{k-i}} f_m}{x_1^i} + x_1 P_0 f_m + Q_0 f_m \quad (*)$$

where $P_i \in k[[x_1]] \otimes k[[t]]$ and $Q_0 \in k[[t]]$. By lemma 8, the terms $\frac{t^{\alpha_{k-i}} f_m}{x_1^i} \in R_{n_k}$ are divisible by x_1, thus x_1 divides $Q_0 f_m$. It follows that the coefficient $Q_0 t^{vh(m)} x_2^{m_2} \dots x_d^{m_d}$ of x_1^0 in $Q_0 f_m$ is zero, which happens only if Q_0 is a multiple of $t^{max(0, n_k - vh(m))} = t^{\alpha_k}$. Writing down $Q_0 = \lambda t^{\alpha_{k-1+1}}$ and dividing the displayed equality $(*)$ by x_1 shows that

$z \in B_{n_k}$, as expected. ■

Lemma 6. $J_{n_k} \subset I^{T_k}$

Proof. In view of the previous lemma, and since the inclusion $C(t) \subset I^{T_k}$ is obvious, one simply has to check that the generators of $B_{n_{k-1}}(m)$ verify the inclusion. If $h(m) \leq [\frac{n_k-1}{v}]$, $x_2^{m_2} \dots x_d^{m_d} \in I^{T_k}$. Since every generator of $B_{n_{k-1}}(m)$ is a multiple of $x_2^{m_2} \dots x_d^{m_d}$, it is in I^{T_k}. If $h(m) > [\frac{n_k-1}{v}]$, then $x_1 x_2^{m_2} \dots x_d^{m_d} \in I^{T_k}$. According to lemma 8, every generator of $B_{n_{k-1}}(m)$ is a multiple of x_1, hence is in I^{T_k} as a multiple of $x_1 x_2^{m_2} \dots x_d^{m_d}$. ■

Lemma 7. $J_{n_k}:(0) = I^{S(E,\tau_1,\dots,\tau_k)}$.

Proof. According to lemmas 10 and 9, it suffices to show that $B_{n_k}(m,0) \subset k[[x_1]]$ is the submodule generated by $x_1^{h(m)-p(m)} x_2^{m_2} \dots x_d^{m_d}$ where $p(m)$ is the number of τ_i's verifying $\tau_i < h(m)$, $1 \leq i \leq k$. Since the generators of $B_{n_k}(m)$ are explicitely given, the lemma just comes from the evaluation of these generators at $t = 0$. We have $f_m(0) = x_1^{h(m)} x_2^{m_2} \dots x_d^{m_d}$. By definition of $p(m)$, for $1 \leq i \leq k$, $\tau_i \geq h(m)$ if and only if $i \leq k - p(m)$. In particular $\alpha_i = 0$ if and if $i > k - p(m)$. We now evaluate the generators of $B_{n_k}(m)$ using this information on α_i. If $i \leq p(m)$, $\frac{t^{\alpha_{k-i+1}} f_m(0)}{x_1^i} = \frac{t^0 f_m}{x_1^i}(0) = x_1^{h(m)-i} x_2^{m_2} \dots x_d^{m_d}$. If $i > p(m)$, $\frac{t^{\alpha_{k-i+1}} f_m(0)}{x_1^i} = 0$. Thus $B_{n_k}(m,0)$ is generated by $x_1^{h(m)-p(m)} x_2^{m_2} \dots x_d^{m_d}$ as expected. ■

4 The Hilbert function of k^2 fat points in $\mathbb{P}^2$

In this section, we compute the Hilbert function of the generic union of k^2 fat points in $\mathbb{P}^2$ of the same multiplicity m.

We work over a field of characteristic 0.

Definition 11. *If $Z \subset \mathbb{P}^2$ is a zero-dimensional subscheme of degree $deg(Z)$, we denote by $H_v(Z) : \mathbb{N} \to \mathbb{N}$ the virtual Hilbert function of Z defined by the formula $H_v(Z,d) = min(\frac{(d+1)(d+2)}{2}, deg(Z))$. The critical degree for Z, denoted by $d_c(Z)$ is the smallest integer d such that $H_v(Z,d) > deg(Z)$. We denote by $H(Z)$ the Hilbert function of Z.*

Theorem 12. *Let Z be the generic union of k^2 fat points of multiplicity m in $\mathbb{P}^2$. Then $H(Z) = H_v(Z)$.*

Let us recall the following well known lemma:

Lemma 13. *If $H(Z,d) \geq H_v(Z,d)$ for $d = d_c(Z)$ and $d = d_c(Z) - 1$, then $H(Z) = H_v(Z)$.*

Definition 14. *The regular staircase $R_m \subset \mathbb{N}^2$ is the set defined by the relation $(x,y) \in R_m \Leftrightarrow x + y < m$. A quasi-regular staircase E is a staircase such that $R_m \subset E \subset R_{m+1}$ for some m. A right specialized staircase is a staircase such that $((x,y) \in E$ and $y > 0) \Rightarrow (x+1, y-1) \in E$. A monomial subscheme of $\mathbb{P}^2$ with staircase E is a punctual subscheme supported by a point p which is defined by the ideal I^E in some formal neighborhood of p.*

Example 15. *A fat point of multiplicity m is a monomial subscheme with staircase R_m.*

Our first intermediate goal is lemma 17 which says that under suitable conditions, if $Z = L \cup R \subset \mathbb{P}^2$ is a subscheme with L included in a line, the Hilbert function of Z is determined by that of R.

We recall that a collision of punctual subschemes $Z_1, \dots, Z_s \subset \mathbb{A}^2$ is a subscheme obtained as a flat limit when the support of the Z_i's approach the same point (see [8]).

Proposition 16. *Let Z be a generic union of fat points. The following conditions are equivalent.*

- $H(Z) = H_v(Z)$
- *there exists a quasi-regular right-specialized staircase E and a collision C of the fat points which is monomial with staircase E.*
- *there exists a quasi-regular staircase E and a collision C of the fat points which is monomial with staircase E.*

Proof. 1 $\Rightarrow$ 2. Let ρ_t be the automorphism of $\mathbb{P}^2 = Proj(k[X,Y,H])$ defined for $t \neq 0$ by $f_t : X \mapsto \frac{X}{t}, Y \mapsto \frac{Y}{t}, H \mapsto H$. Consider the collision $C = \lim_{t\to 0} f_t(Z)$. It is a subscheme of the affine plane $Spec\ k[x = \frac{X}{H}, y = \frac{Y}{H}]$ supported by the origin $(0,0)$. It is shown in [8] that if $H(Z) = H_v(Z)$, then there is an integer m such that the ideal of C verifies $I^{R_{m+1}} \subset I(C) \subset I^{R_m}$. Thus $I(C) = V \oplus k[x,y]_{\geq m+1}$ where $k[x,y]_{\geq m+1}$ stands for the vector space generated by the monomials of degree at least $m+1$, and $V \subset k[x,y]_m$. Let now $g_t : x \mapsto x - ty, y \mapsto y$. Then the ideal of $D = \lim_{t\to\infty} g_t(C)$ is $I(D) = W \oplus k[x,y]_{\geq m+1}$ where $W = \lim_{t\to\infty} g_t(V)$ is a vector space which admits a base of the form $y^m, xy^{m-1}, \ldots, x^k y^{m-k}$. Thus $I(D) = I^E$ for some quasi-regular right-specialized staircase E. And D is a collision of the fat points since it is a specialisation of the collision C and since being a collision is a closed condition.
2 $\Rightarrow$ 3 is obvious.
3 $\Rightarrow$ 1. If there exists a collision C associated with a quasi-regular staircase E, then by semi-continuity $H(Z,d) \geq H(C,d) = min(\frac{(d+1)(d+2)}{2}, \#E) = min(\frac{(d+1)(d+2)}{2}, deg(C)) = min(\frac{(d+1)(d+2)}{2}, deg(Z)) = H_v(Z,d)$. Since the well known reverse inequality $H_v(Z,d) \geq H(Z,d)$ is always true, we have the required equality $H_v(Z,d) = H(Z,d)$. ■

Lemma 17. *Let $R \subset \mathbb{P}^2$ be a generic union of fat points, $D \subset \mathbb{P}^2$ be a generic line, $L \subset D$ be a subscheme whose support is generic in D. Let $Z = R \cup L$ and suppose that the degree of L satisfies $deg(L) \leq d_c(R)$. Then $H(R) = H_v(R)$ implies $H(Z) = H_v(Z)$.*

Proof. By the above lemma and its proof, there exists a quasi-regular right specialized staircase E and a collision C of the fat points supported by the origin of $\mathbb{A}^2 = Spec\ k[x,y]$ such that the ideal of $C \subset \mathbb{A}^2$ is $I(C) = I^E$. By the genericity hypothesis, L can be specialized to the subscheme $L(t)$ with equation $(y - t, x^{deg(L)})$. Obviously $L(t)$ is monomial with staircase $F = \{(0,0), (1,0), \ldots, (deg(L)-1, 0)\}$. Let $D = \lim_{t\to 0} C \cup L(t)$. By [12], $I(D) = I^G$ for some monomial staircase G. Moreover, the explicit description of G given in [12] (G is the "vertical collision" of E and F) and the inequality $deg(L) \leq d_c(R)$ shows that G is quasi-regular. Since $Z = R \cup L$ can be specialized to a scheme D defined by a quasi regular staircase, $H(Z) = H_v(Z)$. ■

Lemma 18. *Let $Z \subset \mathbb{P}^2$ be a union of k^2 fat points of multiplicity m with $k \geq 4$. The critical degree $d_c(Z)$ verifies $km + 1 < d_c(Z) \leq km + k - 2$.*

Proof: Direct calculation. ■

Proof of theorem 12.
We show by induction on k that the Hilbert function of the generic union Z of k^2 fat points of multiplicity m is the virtual Hilbert function $H_v(Z)$. If $k \leq 3$, this is known by [14]. So we may suppose $k \geq 4$. According to lemma 13, we only need to check that $H(Z,d) \geq H_v(Z,d)$ for $d = d_c(Z)$ or $d = d_c(Z) - 1$, and, by lemma 18, such a d verifies $d = km + s$ for some s satisfying $0 \leq s \leq k-2$. By semi-continuity, it suffices to specialize Z to a scheme Z' with $H(Z',d) \geq H_v(Z,d)$. First, we choose a generic line D and generic points $p_1, \ldots, p_{2k-1}$ on D. We divide the k^2 fat points into three subsets E_1, E_2, E_3 of respective cardinal

$k, k-1, (k-1)^2$. We specialize the k fat points of E_1 on the points $p_k, \ldots, p_{2k-1}$. We leave the generic $(k-1)^2+(k-1)$ points of $E_3 \cup E_2$ in their generic position. We denote by $\mathcal{L}$ the set of sections of $\mathcal{O}(d)$ vanishing on the fat points of $E_1 \cup E_3$. Since the points of E_1 have been specialised, we have by semi-continuity the inequality:

$$(*)\ H(Z,d) \geq \frac{(d+1)(d+2)}{2} - \dim \mathcal{L}(-X(E))$$

where

$$E = (\underbrace{R_m, \ldots, R_m}_{(k-1)\ copies}).$$

We now make a further specialisation, moving the $k-1$ fat points of E_2 on the points $p_1, \ldots, p_{k-1}$ using theorem 2. To this end, we fix the notations. We choose a formal neighborhood φ of $p = (p_1, \ldots, p_{k-1})$, a number $N >> 0$ and we take the speed vector

$$v = (\underbrace{N, \ldots, N}_{k-s-2\ times}, \underbrace{N+1, \ldots, N+1}_{s+1\ times}).$$

Finally, we let

$$n_i = (N+1)(m-i+1), 1 \leq i \leq m.$$

Let us check that the conditions of theorem 2 apply. The condition $n_k - n_{k+1} \geq max(v_i)$ is obviously satisfied. As for the remaining condition, remark that $\mathcal{L}(-(i-1)D)$ is a set of sections of $\mathcal{O}(d-i+1)$ vanishing on $p_k^{m-i+1}, \ldots, p_{2k-1}^{m-i+1}$. In particular, if Z_i is a punctual subscheme of D of degree $d-i+2-k(m-i+1) = s+1+(i-1)(k-1)$ whose support does not meet the union $p_k \cup \cdots \cup p_{2k-1}$, then $\mathcal{L}(-iD - Z_i) = \mathcal{L}(-(i+1)D)$. In our case, Z_i is a union of one-dimensional fat points of the line D. Let us compute its degree. The subscheme Z_i is supported by $p_1 \cup \cdots \cup p_{k-1}$ and we denote by d_j the degree of the part $(Z_i)_{p_j}$ supported by p_j. By definition of Z_i, d_j is the cardinal $m - [\frac{n_i-1}{v_j}]$ of the slice $T(R_m, [\frac{n_i-1}{v_j}])$. Since $N >> 0$, $d_j = i-1$ if $j \leq k-s-2$ and $d_j = i$ if $k-s-1 \leq j \leq k-1$. Thus $deg(Z_i) = \sum d_j = s+1+(i-1)(k-1)$. We can then apply theorem 2 and its corollary. We conclude that:

$$(**)\ \dim \mathcal{L}(-X(E)) \leq \dim \mathcal{L}(-mD - X_\varphi(S(E, \tau_1, \ldots, \tau_m)).$$

The linear system $\mathcal{L}(-mD)$ is the set of sections of $\mathcal{O}(d-m)$ which vanish on the union Z' of the fat points of E_3. Moreover, $X_\varphi(S(E, \tau_1, \ldots, \tau_m))$ is the union L of the one-dimensional fat points $p_1^m \cap D, \ldots, p_{k-s-2}^m \cap D$. It follows that

$$(***)\ \dim \mathcal{L}(-mD - X_\varphi(S(E, \tau_1, \ldots, \tau_m)) = \frac{(d-m+2)(d-m+1)}{2} - H(Z' \cup L, d-m).$$

By lemma 17 and the induction, we have

$$(****)\ H(Z' \cup L, d-m) = H_v(Z' \cup L, d-m)$$

Now, by construction (or by an easy direct calculation),

$$(*****)\ H_v(Z' \cup L, d-m) - \frac{(d-m+2)(d-m+1)}{2} = H_v(Z,d) - \frac{(d+2)(d+1)}{2}$$

Putting together the displayed equalities and inequalities (*)...(*****) yields the required inequality $H(Z,d) \geq H_v(Z,d)$. ■

5 Prospects and limits

In this section, we discuss the difficulties for the application of theorem 2, in particular for the application of the method to more general numbers of points.

First, one has to find a divisor D and a good specialisation. For instance, consider the system $\mathcal{L} = H^0\mathcal{O}_{\mathbb{P}^2}(13H - p_1^4 \cdots - p_{10}^4)$ containing the equations of plane curves of degree 13 vanishing on ten generic fat points of multiplicity 4. If the point p_{10} moves to the cubic C containing the points $p_1, \dots, p_9$, it is not possible to compute the limit of the linear system using theorem 2. To apply the theorem in a sensible way, one would need to take subschemes Z_1 and Z_2 of length 3, or equivalently two slices of cardinal 3 in the staircase of p_{10}^4. However, there is only one slice of cardinal 3 in the staircase of the fat point p_{10}^4. We could of course apply the theorem with Z_1 of length 3 and Z_2 of length 4 and conclude that the limit system is included in $\mathcal{G} = f^2H^0\mathcal{O}_{\mathbb{P}^2}(7H - p_1^2 \cdots - p_{10}^2)$, where $p_1, \dots, p_{10}$ are located on the cubic C and f is the equation of C. But the inclusion $\lim \mathcal{L}_t \subset \mathcal{G}$ is strict ($\dim \mathcal{G} = \dim \lim \mathcal{L}_t + 1$).

Even if it is possible to find in the moving points the slices of the required cardinal, it still happens that the inclusion $\lim \mathcal{L}_t \subset \mathcal{G}$ of the theorem is strict because $\mathcal{G}$ is special. For instance, consider the system $\mathcal{L} = H^0\mathcal{O}_{\mathbb{P}^2}(19H - p_1^6 - \dots p_{10}^6)$. One can show that $\mathbb{P}(\mathcal{L})$ is empty. We put the three points p_1, p_2, p_3 on a line D with equation $f = 0$. Then, we apply the theorem with the points p_4 and p_5 moving to D. The theorem asserts that the limit system is included in $\mathcal{G} = f^6H^0\mathcal{O}_{\mathbb{P}^2}(13H - p_6^6 \cdots - p_{10}^6)$. There are too many conditions on the conic through $p_6, \dots, p_{10}$ and $\dim \mathcal{G} = \dim \lim \mathcal{L}_t + 6$.

However, the above problems are not real obstacles in the application of the method. When the points are in general position, it is *always* possible to find a suitable D and a well chosen number of points moving to D such that the theorem gives a sensible candidate for the limit.

For instance, in the above case, one can move only p_4 to the line D instead of p_4 and p_5. The theorem then says that the limit system is included in $\mathcal{G} = f^3H^0\mathcal{O}_{\mathbb{P}^2}(16H - p_1^3 - p_2^3 - p_3^3 - Z_4 - p_5^6 \cdots - p_{10}^6)$, where $p_1, p_2, p_3 \in D$ and Z_4 is a subscheme of length 9 (Z_4 can be obtained as a collision of two fat points of multiplicity 3 and 2 moving along D). Then, one can do a further specialisation from $\mathcal{G}$ and move some other points to a curve. However, in this new specialisation, p_1, p_2, p_3, Z_4 must move along D, not freely.

At each step of the procedure, there are several possibilities (application of the theorem, collision of points, specialisation of curves, Cremona transformations...). Thanks to this flexibility, on concrete examples, it always seems to be possible to add a new step and to progress. However, if the example is significant, the number of steps to reach a situation where one can conclude is far too big in general. The author gave up some interesting examples in view of the amount of calculation required. It is not possible to progress step by step. One has to imagine a systematic procedure.

Thus, the difficulty is that we start with a general position and we end up with a special position. It makes it hard to perform the computations in a systematic way. This is the main reason why we dealt with k^2 fat points : under this condition, we could fine a systematic procedure.

Obviously, other results can be proved with theorem 2 and ad hoc inductions. For instance, an exploration of the method will at least give a bound for the smallest degree d of a curve passing through k general points in the plane with multiplicity m. The limits of the method are not clear. Is it possible to find the exact value of d along these lines ? We don't know whether the difficulty is to find an induction with the tools developped so far, or if some new tools will be necessary. Computing an example is tedious by hand, and examples are missing to have a guess on this question.

6 Collisions of fat points

This section gives an other application of theorem 2: the computation of collisions of fat points in the plane.

The collisions of at most three fat points are known [7]. When the number of fat points is four or five, the general collisions where all points approach the origin with the same order of speed have been computed by Ciliberto and Miranda [4]. We want to discuss an other type of collision of fat points, namely the collisions where the points approach the origin successivly. It is in some sense the opposite cases compared to those studied by Ciliberto and Miranda: for any pair of points, one of the two points approach the origin infinitly faster than the other one.

We start with a definition of a generic successive collision of fat points in $\mathbb{A}^2$. We proceed by induction. A generic successive collision of one fat point p^m is the fat point itself. Suppose defined the generic successive collision $Z_{m_1\dots m_{k-1}}$ of $p_1^{m_1},\dots,p_{k-1}^{m_{k-1}}$. Let $C(d)$ be the generic curve of degree d containing the support O of $Z_{m_1\dots m_{k-1}}$. Let

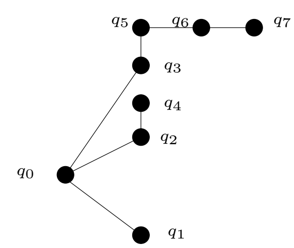
$$Z_{m_1\dots m_k}(d) = \lim_{p\in C(d),\ p\to O} Z_{m_1\dots m_{k-1}} \cup p^{m_k}.$$

Proposition 19. *There exists an integer d_0 such that $\forall d \geq d_0$, $Z_{m_1\dots m_k}(d) = Z_{m_1\dots m_k}(d_0)$. We denote this subscheme by $Z_{m_1\dots m_k}$ and this is by definition the generic successive collision of $p_1^{m_1},\dots,p_k^{m_k}$.*

We omit the proof of the proposition as it will be clear in our context: the integers d_0 which appear in the definition of Z_{mmmm} will always be equal to 1. In other words, it will be clear from the calculations that the generic collision of four fat points will be shown to depend only on the tangent directions of the approaching fat points.

Our goal is to compute the generic collision Z_{mmmm} of 4 fat points of multiplicity m 20 and more generally the successive collsions of four fat points moving along smooth curves 21.

We will describe Z_{mmmm} as a pushforward via a blowup $\pi : \tilde{S} \to \mathbb{A}^2$, where π is the blowup defined by the following Enriques diagram .



The meaning of the Enriques diagram is explained in [9], but we recall for convenience what this means on this particular example. Let $q_0 \in \mathbb{A}^2$, q_1, q_2, q_3 be three distinct tangent directions at q_0. Let

$$\eta : S_1 \to S_0 = \mathbb{A}^2$$

be the blowup of q_0, and $Q_0 \subset S_1$ the exceptionnal divisor. Let

$$S_2 \to S_1$$

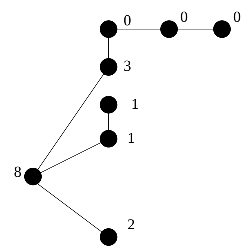
be the blowup of $(q_1 \cup q_2 \cup q_3) \subset Q_0$, and $Q_1, Q_2, Q_3 \subset S_2$ the respective exceptional divisors. If $Q_i \subset S_{n_i}$ is an exceptional divisor, and if $S_j \to S_{n_i}$ is a sequence of blowups, we still denote by $Q_i \subset S_j$ (resp. we denote by $E_i \subset S_j$) the strict transform (resp. the total transform) of Q_i in S_j. With this convention, let $q_4 = Q_0 \cap Q_2 \in S_2$, $q_5 = Q_0 \cap Q_3 \in S_2$. Let

$$S_3 \to S_2$$

be the blowup of $q_4 \cup q_5$, Q_4, Q_5 the corresponding exceptionnal divisors. Let $q_6 = Q_3 \cap Q_5 \in S_3$, $S_4 \to S_3$ the blowup of q_6, Q_6 its exceptional divisor. Let $q_7 = Q_6 \cap Q_3 \in S_4$ and $\tilde{S} = S_5 \to S_4$ the blowup of q_7. We denote by

$$\rho : \tilde{S} \to S_1 \quad and \quad \pi : \tilde{S} \to \mathbb{A}^2$$

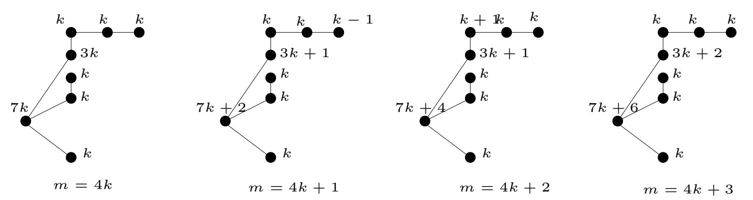
the compositions of the blowups introduced above. As explained, each point q_i defines a divisor $E_i \subset \tilde{S}$. If $(m_0, \ldots, m_7) \in \mathbb{N}^8$, the ideal $\pi_*(\mathcal{O}_{\tilde{S}}(-\sum m_i E_i))$ defines a punctual subscheme supported by q_0 which we will represent graphically with a label m_i at the point of the Enriques diagram corresponding to q_i. For instance, the subscheme $\pi_*(\mathcal{O}_{\tilde{S}}(-8E_0 - 2E_1 - E_2 - E_4 - 3E_3))$ is associated with the following diagram.



Theorem 20. *Let $q_0 \in \mathbb{A}^2$, q_1, q_2, q_3 three distinct tangent directions at q_0 and C_1, C_2, C_3 be three smooth curves passing through p_0 with tangent direction q_1, q_2, q_3. Let Z_{mmmm} be the collision of the fat points $p_0^m, p_1^m, p_2^m, p_3^m$ where:*

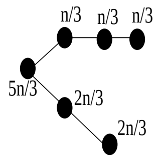
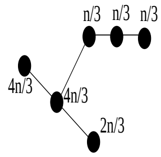
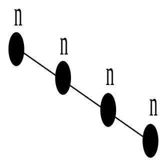
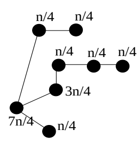
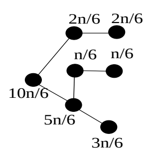
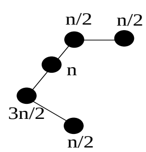
- *p_0 is located at q_0,*
- *p_1 moves on the curve C_1 (resp. p_2 on C_2, p_3 on C_3).*

Then Z_{mmmm} is defined by the following Enriques diagram, which depends on m modulo 4.



Besides theorem 20, many collisions are computable using the same method (in fact, an infinite number). For instance, consider the successive collisions of four fat points $p_1^m, .., p_4^m$ in the plane, ie. the collisions obtained in four steps by moving successivly each of the fat points p_i^m to the origin of $\mathbb{A}^2$ along a curve C_i. It is possible to compute all the successive collisions when the curves C_i are smooth at the origin. The following array sums up the results.

Theorem 21. *The successive collisions of four fat points $p_1^n, \ldots, p_4^n$ moving along smooth curves $C_1, \ldots, C_4$ are defined by integrally closed ideals and the corresponding Enriques diagrams are:*

Enriques diagram	*Equations of curves which realise the collision*
	$C_1, C_2, C_3: \ y = 0$ $C_4: \ x = 0$
	$C_1, C_2, C_3: \ y = 0$ $C_4: \ y = x^2$
	$C_1, C_2, C_3, C_4: \ y = 0$
	$C_1, C_2: \ y = 0$ $C_3: \ x = 0$ $C_4: \ y = x$
	$C_1, C_2, C_4: \ y = 0$ $C_3: \ x = 0$
	$C_1, C_2: \ y = 0$ $C_3, C_4: \ x = 0$

Remark 22. *To avoid too many cases, we supposed that the numbers* $\frac{n}{2}, \frac{n}{3}, \frac{n}{4}, \frac{n}{6}$ *appearing above are integers. Of course, it is possible to write down slightly different formulas when these numbers are not integers as in theorem 20.*

Since the proof of theorem 21 use the same arguments as theorem 20, we just prove theorem 20 for brevity.

Proof: All cases are similar and we only consider the case $m = 4k$. We choose a formal neighborhood ξ of $p = (q_1, q_2, q_3) \in (S_1)^3$ such that $Q_0 \subset S_1$ is defined by the equation $x_1 = 0$ around each q_i and such that C_3 is defined by $x_2 = 0$ around q_3 (this is possible since C_3 is smooth). Let $n = (m-1, m-5, \dots, 3)$. Let F_m be the staircase defined by the height function $h_{F_m}(d) = h_{R_m}([\frac{d}{2}])$, and let $G_m = S(R_m, n)$ be the staircase obtained from R_m by suppression of the slices indexed by n. Let $X_\xi(R_k, F_k, G_m) \subset S_1$ be the subscheme defined by the formal neighborhood ξ and the staircases R_k, F_k, G_m. According to the correspondance between complete ideals and monomial subschemes formulated in [9], if the m_i's are the integers defined in the Enriques diagram,

$$\rho_* \mathcal{O}_{\tilde{S}}(-\sum m_i E_i)) = \mathcal{O}_{S_1}(-m_0 Q_0 - X_\xi(R_k, F_k, G_m))) \quad (*)$$

Let $J(p_3)$ denote the ideal of $Z_{mmm} \cup p_3^m$. I claim that we are done if we prove the inclusion

$$\lim_{p_3 \to p_0} \eta^* J(p_3) \subset H^0(\mathcal{O}_{S_1}(-m_0 Q_0 - X_\xi(R_k, F_k, G_m)) \quad (**).$$

Indeed, we would then have the inclusions

$$\begin{aligned} I_{Z_{mmmm}} &\subset \eta_* \eta^* I_{Z_{mmmm}} = \eta_* \eta^* \lim_{p_3 \to p_0} J(p_3) \\ &\subset \eta_* \lim_{p_3 \to p_0} \eta^* J(p_3) \\ &\subset \eta_* H^0(\mathcal{O}_{S_1}(-7kQ_0 - X_\varphi(R_k, F_k, G_m)) \quad by\ (**) \\ &\subset H^0(\eta_*(\mathcal{O}_{S_1}(-7kQ_0 - X_\varphi(R_k, F_k, G_m)) \\ &\subset H^0(\eta_* \rho_* \mathcal{O}_{\tilde{S}}(-\sum m_i E_i)) \quad by\ (*) \\ &\subset I_Z \text{ where } I_Z = \pi_* \mathcal{O}_{\tilde{S}}(-\sum m_i E_i). \end{aligned}$$

According to [2], since the Enriques diagram defining Z is unloaded, $deg(Z) = \sum \frac{m_i(m_i+1)}{2}$ which is immediately checked to be $4\frac{4k(4k+1)}{2} = deg(Z_{mmmm})$. Summing up, Z and Z_{mmmm} are two punctual subschemes of the same degree with $I_{Z_{mmmm}} \subset I_Z$, thus they are equal. It remains to prove the displayed inclusion $(**)$ using our theorem. By [7] or [17],

$$\eta^* I_{Z_{mmm}} = H^0 \mathcal{O}_{S_1}(-6kQ_0 - X_\psi(R_{2k}, F_{2k}))$$

where ψ is the formal neighborhood of (q_1, q_2) induced by the formal neighborhood ξ of (q_1, q_2, q_3). Thus

$$\lim_{p_3 \to p_0} \eta^* J(p_3) = \lim_{t \to 0} \mathcal{L}(-X_\varphi(R_m, t, v = 1))$$

where φ is the formal neighborhood of q_3 induced by the formal neighborhood ξ of (q_1, q_2, q_3) and $\mathcal{L} = H^0(\mathcal{O}_{S_1}(-6kQ_0 - X_\psi(R_{2k}, F_{2k})))$. To apply theorem 2 with $X = S_1$, $s = 1$,$D = Q_0$, and $n = (m, m-4, \dots, 4)$, the verification $\mathcal{L}((-i+1)D - Z_i) = \mathcal{L}(-iD)$ is needed. Elements of $\mathcal{L}((-i+1)D - Z_i)$ are sections of $\mathcal{O}_{S_1}((-6k - i + 1)Q_0)$ vanishing on

$$X_\psi(R_{2k-i+1}, F_{2k-i+1}) \cup Z_i = X_\xi(R_{2k-i+1}, F_{2k-i+1}, T(R_m, m - 1 - 4(i-1))).$$

Since the intersection

$$Q_0 \cap X_\xi(R_{2k-i+1}, F_{2k-i+1}, T(R_m, m - 1 - 4(i-1)))$$

has degree $3(2k - i + 1) + (4i - 3)$ greater than the degre $6k + i - 1$ of the restriction $\mathcal{O}_S((-6k - i + 1)Q_0)_{|Q_0}$, it follows that any section of $\mathcal{L}((-i+1)D - Z_i)$ vanishes on D. Thus we can apply the theorem and we get:

$$\begin{aligned} \lim_{t \to 0} \mathcal{L}(-X_\varphi(R_m, t, 1)) \quad \subset \quad & \mathcal{L}(-kQ_0 - X_\varphi(S(R_m, n))) \\ & = \\ & H^0(\mathcal{O}_{S_1}(-7kQ_0 - X_\psi(R_k, F_k) - X_\varphi(S(R_m, n)))) \\ & = \\ & H^0(\mathcal{O}_{S_1}(-m_0 Q_0 - X_\xi(R_k, F_k, S(R_m, n))), \end{aligned}$$

which concludes the proof. ■

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