The Chow ring of punctual Hilbert schemes on toric surfaces

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Abstract:

Let X be a smooth projective toric surface, and $\mathbb{H}^d(X)$ the Hilbert scheme parametrising the length d zero-dimensional subschemes of X. We compute the rational Chow ring $A^*(\mathbb{H}^d(X))_{\mathbb{Q}}$. More precisely, if $T \subset X$ is the two-dimensional torus contained in X, we compute the rational equivariant Chow ring $A^*_T(\mathbb{H}^d(X))_{\mathbb{Q}}$ and the usual Chow ring is an explicit quotient of the equivariant Chow ring. The case of some quasi-projective toric surfaces such as the affine plane are described by our method too.

Introduction

The Hilbert scheme $\mathbb{H}^d(X)$ parametrising the zero-dimensional subschemes of length d of X is defined for any projective scheme X and is irreducible when X is a smooth surface. For this reason, $\mathbb{H}^d(X)$ has retained more attention when X is a surface. In particular, many attempts have been done to compute its cohomology. First, Ellingsrud and Strømme [5] computed the Betti numbers of $\mathbb{H}^d(X)$ for a rational ruled surface X. The Betti numbers $b_i(\mathbb{H}^d(X))$ for a general surface X were computed by Göttsche [10] who realised them as coefficients of an explicit power series in two variables. This nice and surprising organisation of the Betti numbers as coefficients of a power series was explained by Nakajima in terms of a Fock space structure constructed on the cohomology of the Hilbert schemes [18]. Grojnowski announced similar results [11].

As to the ring structure on the cohomology of $\mathbb{H}^d(X)$, the first steps were done again by Ellingsrud and Strømme [6] (see also Fantechi-Göttsche [8]) . They gave an indirect description of the ring structure in the case $X = \mathbb{P}^2$ in terms of the action of the Chern classes of the tautological bundles. Another indirect description has been given by Lehn when $X = \mathbb{A}^2$ [14] via an identification between the cohomology ring of $\mathbb{H}^d(\mathbb{A}^2)$ and an explicit ring of differential operators on a Fock space. Lehn and Sorger gave a more explicit description in [16]. At the same time, Vasserot [19] described the cohomology ring of $\mathbb{H}^d(\mathbb{A}^2)$ by methods relying on equivariant cohomology. Lehn and Sorger [15] extended their results to the case of $K3$ surfaces. Li,Qin,Wang have computed the ring structure for some toric surfaces [17]. However, the case $X = \mathbb{P}^2$ is not included.

The goal of this work is to compute the Chow ring $A^*(\mathbb{H}^d(X))$ when X is a smooth projective toric surface.

For simplicity, we use the notation $\mathbb{H}^d$ instead of $\mathbb{H}^d(X)$. Though we use the formalism of Chow rings and work over any algebraically closed field k, it should be pointed that when $k = \mathbb{C}$, the Chow ring coïncides with usual cohomology since the action of the two-dimensional torus T on X induces an action of T on $\mathbb{H}^d$ with a finite number of fixed points.

Nakajima's construction [18] has been fundamental and many of the above papers ([19], [14], [16], [15], [17]) rely on it. The present work is independent of Nakajima's framework and uses equivariant Chow rings as the main tool.

Equivariant Chow rings. The construction of an equivariant Chow ring associated with an algebraic space endowed with an action of a linear algebraic group has been settled by Edidin and Graham [3] using Totaro's algebraic approximation of the classifying space. Their construction is modeled after the Borel in equivariant cohomology. Brion pushed the theory further in the case the group is a torus T acting on a variety X [1]. He gave a description of Edidin and Graham's equivariant Chow ring by generators and relations. This alternative construction makes it possible to prove that the usual Chow ring is a quotient of the equivariant Chow ring by an explicit ideal. This is the starting point of this work: to realize the usual Chow ring as a quotient of the equivariant Chow ring. Moreover, over the rationals, the restriction to fixed points $A^*_T(X)_\mathbb{Q} \to A^*_T(X^T)_\mathbb{Q}$ is injective and its image is the intersection of the images of the morphisms $A^*_T(X^{T'})_\mathbb{Q} \to A^*_T(X^T)_\mathbb{Q}$ where T' runs through all one codimensional subtori of T.
Thus the natural context is that of rational Chow rings and we lighten the notations: From now on, the symbols $A^*(X), A^*_T(X)$ will implicitly stand for the rational Chow rings $A^*(X)_\mathbb{Q}, A^*_T(X)_\mathbb{Q}$.
We apply Brion's results to $X = \mathbb{H}^d$. The locus $X^T = \mathbb{H}^{d,T}$ is a finite number of points and the ring $A^*_T(\mathbb{H}^{d,T})$ is a product of polynomial rings. In particular, the ring structure of $A^*_T(\mathbb{H}^d) \subset A^*_T(\mathbb{H}^{d,T})$ is completly determined by the set theoretic inclusion. In view of the above description, the problem of computing $A^*_T(\mathbb{H}^d) \subset A^*_T(\mathbb{H}^{d,T})$ reduces to the computation of $A^*_T(\mathbb{H}^{d,T'}) \subset A^*_T(\mathbb{H}^{d,T})$. The steps are as follows.

- First, we study the geometry of the locus $\mathbb{H}^{d,T'} \subset \mathbb{H}^d$. We identify its irreducible components with products $V_1 \times \ldots \times V_r$ where each term V_i in the product is a projective space or a graded Hilbert scheme $\mathbb{H}^{T',H}$, in the sense of Haiman-Sturmfels [12].

- A graded Hilbert scheme $\mathbb{H}^{T',H}$ appearing as a term V_i is embeddable in a product $\mathbb{G}$ of Grassmannians. A sligth modification of an argument by King-Walter shows that the restriction morphism $A^*_T(\mathbb{G}) \to A^*_T(\mathbb{H}^{T',H})$ is surjective. The idea for this step is that the universal family over $\mathbb{H}^{T',H}$ is a family of $k[x, y]$-modules with a nice resolution.

- It then suffices to compute explicit generators of $A^*_T(\mathbb{G})$ and to put the two above steps together to obtain a description of the equivariant Chow ring $A_T^*(\mathbb{H}^d)$ (Theorem 17).

At this point, the description of the equivariant Chow ring is complete, but the formula involves tensor products, direct sums and intersections. The last step consists in an application of a Bott formula (proved by Edidin and Graham in an algebraic context [4]) to get a nicer description. This is done in Theorem 28: If $\hat{T}$ is the character group of T, $S = Sym(\hat{T} \otimes \mathbb{Q}) \simeq \mathbb{Q}[t_1, t_2]$ is the symmetric $\mathbb{Q}$-algebra over $\hat{T}$, $A_T^*(\mathbb{H}^d) \subset S^{\mathbb{H}^{d,T}}$ is realised as a set of tuples of polynomials satisfying explicit congruence relations. In this setting, the usual Chow ring is the quotient of the equivariant Chow ring by the ideal generated by the elements $(f, \dots, f)$, f homogeneous with positive degree.

The description of the Chow ring of theorem 17 is valid with conditions on the surface X weaker than projectivity: X need only to be filtrable [1]. In particular, the description applies for the affine plane.

The key results about equivariant Chow rings used in the text have been extended to an equivariant K-theory setting by Vezzosi and Vistoli[20]. Thus the method developped in the present paper should generalize to equivariant K-theory as well.

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1 The objects involved

The toric variety X

Let T be a 2-dimensional torus with character group $\hat{T}$. Let $N = Hom_{\mathbb{Z}}(\hat{T}, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $\Delta \subset N_{\mathbb{R}}$ be a fan defining a complete smooth toric variety X. Denote the maximal cones of Δ by $\sigma_1, \dots, \sigma_r$ with the convention $\sigma_{r+1} = \sigma_1$, and by $p_1, \dots, p_r$ the corresponding closed points of X. Assume that the cones are ordered such that $\sigma_i \cap \sigma_{i+1} = \sigma_{i,i+1}$ is a one dimensional cone. Denote respectivly by $U_{i,i+1}, O_{i,i+1}, V_{i,i+1} = \overline{O}_{i,i+1}$ the open subvariety, the orbit and the closed subvariety of X defined by the cone $\sigma_{i,i+1}$. Define similarly $U_i \subset X$ the open subscheme associated with σ_i: $U_i = Spec\ R_i$, with $R_i = k[\sigma_i^\nu \cap \hat{T}] \subset k[\hat{T}]$. There is an isomorphism $U_i \simeq Spec\ k[x, y]$. Moreover, we require $xy = 0$ to be the equation of $V_{i-1,i} \cup V_{i,i+1}$ around p_i. The isomorphism $U_i \simeq Spec\ k[x, y]$ is then defined up to the automorphism of $k[x, y]$ that exchanges the two coordinates.

Subtori and their fixed locus

Let $T' \hookrightarrow T$ be a one dimensional subtorus of T, $\hat{T}'$ be its character group. The action of T' on U_i induces a decomposition $R_i = \sum_{\chi \in \hat{T}'} R_{T',i,\chi}$, where $R_{T',i,\chi} \subset R_i$ is the subvector space on which T' acts through χ.
The torus T' acts on X. One shows easily that the fixed locus $X^{T'}$ admits two

types of connected components. Some components are isolated fixed points . We let

$$PFix(T') = \{p \in X^{T'},\ p\ isolated\}.$$

The other components are projective lines $V_{i,i+1} \simeq \mathbb{P}^1$ joining two points p_i, p_{i+1} of X^T. We let

$$LFix(T') = \{\{p_i, p_{i+1}\},\ p_i, p_{i+1} \text{lie in an invariant } \mathbb{P}^1\}.$$

By construction,

$$X^T = LFix(T') \cup PFix(T').$$

Staircases and Hilbert functions

A staircase $E \subset \mathbb{N}^2$ is a subset whose complement $C = \mathbb{N}^2 \setminus E$ satisfies $C + \mathbb{N}^2 \subset C$. In our context, the word staircase will stand for finite staircase. By extension, a staircase $E \subset k[x, y]$ is a set of monomials $m_i = x^{a_i} y^{b_i}$ such that the the set of exponents (a_i, b_i) is a staircase of $\mathbb{N}^2$. The automorphism of $k[x, y]$ exchanging x and y preserves the staircases. In particular it makes sense to consider staircases in R_i, though the automorphism $R_i \simeq k[x, y]$ is not canonical.
A staircase $E \subset R_i$ defines a monomial zero-dimensional subcheme $Z(E) \subset U_i$ whose ideal is generated by the monomials $m \in R_i \setminus E$. A multistaircase is a r-tuple $\underline{E} = (E_1, \ldots, E_r)$ of staircases with $E_i \subset R_i$. It defines a subscheme $Z(\underline{E}) = \coprod Z(E_i)$.
In our context, a T'-Hilbert function is a function $H : \hat{T}' \to \mathbb{N}$ such that $\#H = \sum_{\chi \in \hat{T}'} H(\chi)$ is finite. A T'-Hilbert multifunction is a collection of T'-Hilbert functions H_C parametrized by the connected components C of $X^{T'}$. Its cardinal is by definition

$$\#\underline{H} = \sum_C \#H_C.$$

Equivalently, a T'-Hilbert multifunction is a r-tuple $\underline{H} = (H_1, \ldots, H_r)$ of Hilbert functions such that $H_i = H_{i+1}$ if $\{p_i, p_{i+1}\} \in LFix(T')$.
If $Z \subset X$ is a zero-dimensional subscheme fixed under T', then $H^0(Z, \mathcal{O}_Z)$ is a representation of T' which can be decomposed as $\oplus V_\chi$ where $V_\chi \subset H^0(Z, \mathcal{O}_Z)$ is the subspace on which T' acts through χ. The T'-Hilbert function associated with Z is by definition $H_{T',Z}(\chi) = \dim V_\chi$. We also define a Hilbert multifunction $\underline{H}_{T',Z}$ as follows. If $p_i \in PFix(T')$, let $Z_i \subset Z$ the component of Z located on p_i and $H_i = H_{T',Z_i}$. If $\{p_i, p_{i+1}\} \in LFix(T')$, let $Z_i = Z_{i+1} \subset Z$ the component of Z located on $V_{i,i+1}$ and $H_i = H_{i+1} = H_{T',Z_i}$. The Hilbert multifunction associated to Z is

$$\underline{H}_{T',Z} = (H_1, \ldots, H_r).$$

By construction, we have the equality

$$\#\underline{H}_{T',Z} = length(Z).$$

A partition of $n \in \mathbb{N}$ is decreasing sequence $n_1, n_2, \ldots$ of integers ($n_i \geq n_{i+1} \geq 0$) with $n_i = 0$ for $i >> 0$, and $\sum n_i = n$. The number of parts is the number of i such that $n_i \neq 0$.
We will denote by

- $Part(n)$ the set of partitions of n, $Part = \coprod Part(n)$,
- $\mathcal{E}$ the set of staircases of $\mathbb{N}^2$,
- $\mathcal{ME}$ the set of multistaircases,
- $\mathcal{H}(T')$ the set of T'-Hilbert functions,
- $\mathcal{MH}(T')$ the set of T'-Hilbert multifunctions.

Hilbert schemes and Grassmannians

We denote by $\mathbb{H}$ the Hilbert scheme parametrizing the 0-dimensional subschemes of X. It is a disjoint union $\mathbb{H} = \coprod \mathbb{H}^d$, where $\mathbb{H}^d$ parametrizes the subschemes of length d. We denote by $\mathbb{H}_i \subset \mathbb{H}$ the open subscheme parametrizing the subschemes whose support is in U_i, and $\mathbb{H}_{i,i+1} = \mathbb{H}_i \cup \mathbb{H}_{i+1}$.
The action of the torus T on X induces an action of T on $\mathbb{H}$. We denote by $\mathbb{H}^T \subset \mathbb{H}$ the fixed locus under this action. If $T' \subset T$ is a one dimensional subtorus, and $\underline{H}$ is a T'-Hilbert multifunction, $\mathbb{H}^{T',\underline{H}} \subset \mathbb{H}$ parametrizes by definition the subschemes Z, T' fixed, with T'-Hilbert multifunction $\underline{H}_{T',Z} = \underline{H}$. Define similarly $\mathbb{H}^{T',H}$ for a T'-Hilbert function H.
We will freely mix the above notations by intersecting the subschemes when we gather the indexes. For instance, $\mathbb{H}_i^T = \mathbb{H}_i \cap \mathbb{H}^T$, $\mathbb{H}^{T',H,T} = \mathbb{H}^{T',H} \cap \mathbb{H}^T$, $\mathbb{H}_{i,i+1}^{T'} = \mathbb{H}_{i,i+1} \cap \mathbb{H}^{T'}$ etc ... To avoid ambiguity, the formula is

$$\mathbb{H}_s^{s_1,\ldots,s_k} = \mathbb{H}_s \cap \mathbb{H}^{s_1} \cap \cdots \cap \mathbb{H}^{s_k},$$

where

$$s \in \{i, \{i, i+1\}\}, \ s_i \in \{d, T', (T', \underline{H}), (T', H), T\}.$$

If $T' \hookrightarrow T$ is a one dimensional subtorus and if $(i, \chi, h) \in \{1, \ldots, r\} \times \hat{T}' \times \mathbb{N}$, we denote by

$$\mathbb{G}_{T',i,\chi,h}$$

the Grassmannian parametrising the subspaces of $R_{T',i,\chi}$ of codimension h. If H is a T'-Hilbert function, $\mathbb{G}_{T',i,H} = \prod_{\chi \in \hat{T}'} \mathbb{G}_{T',i,\chi,H(\chi)}$. It is a well defined finite product since $G_{T',i,\chi,H(\chi)}$ is a point for all but finite values of χ.

2 Description of the fixed loci

Let $T' \hookrightarrow T$ be a one dimensional subtorus. The goal of this section is to give a description of the irreducible components of $\mathbb{H}^{T'}$.

Theorem 1.

$$\mathbb{H}_i^{T'} = \bigcup_{H \in \mathcal{H}(T')} \mathbb{H}_i^{T',H}$$

is the decomposition of $\mathbb{H}_i^{T'}$ *into smooth disjoint irreductible components.*

Proof. This is proved in [7]. ■

Remark 2. *Depending on* H, $\mathbb{H}_i^{T',H}$ *may be empty so the result is that the irreducible components are in one-to-one correspondance with the set of possible Hilbert functions* H*. Throwing away the empty sets in the above decomposition is possible: There is an algorithmic procedure to detect the emptiness of* $\mathbb{H}_i^{T',H}$ *([7], remark 23).*

Now the goal is to prove that $\mathbb{H}_{i,i+1}^{T',\underline{H}}$ is empty or a product P of projective spaces. An embedding $P \to \mathbb{H}^{T'}$ is constructed in the next proposition. Then it will be shown that a non empty $\mathbb{H}_{i,i+1}^{T',\underline{H}} \subset \mathbb{H}^{T'}$ is the image of such an embedding.

Let $\pi = (\pi_1, \pi_2, \dots) \in \mathcal{P}art(d)$ be a partition. Let $(n_1, \dots, n_s, 0)$ be the finite subsequence with no repetition obtained from π with the removal of duplicates. Denote by d_l the number of indexes j with $\pi_j = n_l$. In other words,

$$\pi = (\underbrace{n_1, \dots, n_1}_{d_1\ times}, \underbrace{n_2, \dots, n_2}_{d_2\ times}, \dots, \underbrace{n_s, \dots, n_s}_{d_s\ times}, 0, \dots).$$

Let $\{p_i, p_{i+1}\} \in LFix(T')$. If $p \in V_{i,i+1} \subset U_i \cup U_{i+1}$, one may suppose by symmetry that $p \in U_i \simeq Spec\ k[x,y]$. Exchanging the roles of x and y, we may suppose that $V_{i,i+1}$ is defined by $y = 0$ in U_i. We denote by $Z_{p,k}$ the subscheme with equation $(x - x(p), y^k)$. Intrinsecally, it is characterized as the only length k curvilinear subscheme $Z \subset X$ supported by p, T'-fixed, such that $Z \cap V_{i,i+1} = p$ as a schematic intersection. The rational function

$$\begin{array}{rcl} \varphi_\pi : Sym^{d_1} V_{i,i+1} \times \dots \times Sym^{d_s} V_{i,i+1} & \to & \mathbb{H}^{d,T'} \\ (p_{11}, \dots, p_{1d_1}), \dots, (p_{s1} \dots p_{sd_s}) & \mapsto & \displaystyle\coprod_{i \le s, j \le d_i} Z_{p_{ij}, n_i} \end{array}$$

is well defined on the locus where all the points $p_{a,b} \in V_{i,i+1}$ are distinct. In fact, it is regular everywhere.

Proposition 3. *The function* φ_π *extends to a regular embedding* $Sym^{d_1} V_{i,i+1} \times \dots \times Sym^{d_s} V_{i,i+1} \to \mathbb{H}^{d,T'}$.

Proof. The extension property is local thus it suffices to check it on an open covering. The covering $V_{i,i+1} = (V_{i,i+1} \cap U_i) \cup (V_{i,i+1} \cap U_{i+1}) = W_i \cup W_{i+1}$ of $V_{i,i+1}$ induces a covering of the symmetric products $Sym^d V_{i,i+1}$. All the open sets in this covering play the same role. Thus by symmetry, it suffices to define an embedding

$$\psi_\pi : Sym^{d_1} W_i \times \dots \times Sym^{d_s} W_i \to \mathbb{H}^{T'}$$

which generically coïncides with φ_π.
Let $Z(p_{11},\dots,p_{sd_s}) \subset U_i$ be the subscheme defined by the ideal

$$I_Z = (y^{n_1}, y^{n_2}\prod_{\beta\le d_1}(x - x(p_{1\beta})),\dots, y^{n_s}\prod_{\alpha<s}^{\beta\le d_\alpha} x - x(p_{\alpha\beta}), \prod_{\alpha\le s}^{\beta\le d_\alpha} x - x(p_{\alpha\beta})\ ).$$

Let

$$\begin{array}{rcl} \psi_\pi : Sym^{d_1}W_i \times \ldots \times Sym^{d_s}W_i & \to & \mathbb{H}^{T'} \\ (p_{11},\dots,p_{1d_1}),\dots,(p_{s1}\dots p_{sd_s}) & \mapsto & Z(p_{11},\dots,p_{sd_s}). \end{array}$$

Clearly, $Z(p_{11},\dots,p_{sd_s})$ is T'-fixed since T' does not act on x. Thus, ψ_π is a well defined morphism which extends φ_π. Now, for $1\le\alpha\le s$, the transporter $(I_Z + y^{n_{\alpha+1}+1} : y^{n_{\alpha+1}})$ defines a subscheme Z_α of $V_{i,i+1}$ of length $d_1+\dots+d_\alpha$. Since $Z_{\alpha-1}\subset Z_\alpha$ the residual scheme $Z'_\alpha = Z_\alpha \setminus Z_{\alpha-1}$ is well defined for $\alpha\ge 2$. Consider the morphism

$$\begin{array}{rcl} \rho : Im(\psi_\pi) & \to & Sym^{d_1}W_i \times \ldots \times Sym^{d_s}W_i \\ Z & \mapsto & (Z_1, Z'_2, Z'_3,\dots,Z'_s). \end{array}$$

The composition $\rho\circ\psi_\pi$ is the identity. Thus, ψ_π is an embedding, as expected.

■

Remark that the T'-Hilbert function $H_{T',Z}$ is constant when Z moves in a connected component of $\mathbb{H}^{T'}$. In particular, it is constant on $Im(\varphi_\pi)$ and φ_π factorizes:

$$\varphi_\pi : \prod_{\alpha\le s} Sym^{d_\alpha}V_{i,i+1} \to \mathbb{H}^{T',H_\pi}_{i,i+1}$$

for a uniquely defined T'-Hilbert function $H_{\pi,T',i,i+1}$ that we note H_π for simplicity.

Proposition 4. *Let $\{p_i,p_{i+1}\} \in LFix(T')$, H be a T'-Hilbert function. If $H = H_\pi$ for some $\pi\in\mathcal{P}art$, then $\mathbb{H}^{T',H}_{i,i+1}$ is a product of projective spaces, thus irreducible. If $H\ne H_\pi$ then $\mathbb{H}^{T',H}_{i,i+1}=\emptyset$.*

Proof. If $H = H_\pi$ for some $\pi\in\mathcal{P}art$, it suffices to prove that

$$\varphi_\pi : \prod_{\alpha\le s} Sym^{d_\alpha}V_{i,i+1} \to \mathbb{H}^{T',H_\pi}_{i,i+1}$$

is an isomorphism. We already know that φ_π is an embedding thus we need surjectivity. Let $Z\in\mathbb{H}^{T',H_\pi}_{i,i+1}$. We may suppose without loss of generality that $Z\subset U_i = Spec\ k[x_i,y_i]$ and that $V_{i,i+1}$ is defined by $y_i=0$ in U_i. Since the ideal I of Z is T'-invariant, it is generated by elements $y_i^kP(x_i)$, where P is a polynomial. The power k being fixed, the polynomials P such that $y_i^kP(x_i)\in I$

form an ideal in $k[x_i]$ generated by a polynomial P_k. The condition for I to be an ideal implies the divisibility relation $P_k|P_l$ for $l < k$. Since Z is 0-dimensional, $P_i = 1$ for $i >> 0$. Let t be the smallest integer such that $P_t = 1$: $1 = P_t|P_{t-1}|\dots|P_0$. In particular the sequence

$$D = (D_1, D_2, \dots) = (deg(P_0), deg(P_1), \dots)$$

is a partition. Let $D^\nu \in \mathcal{P}art$ be the partition conjugate to D, ie. $D^\nu(k) = \#\{j \ s.t.\ D_j \geq k\}$. By construction, $D^\nu = \pi$. Let $(d_1, d_2, \dots, d_s)$ be the list obtained from the list $(D_t - D_{t+1}, D_{t-1} - D_t, \dots, D_1 - D_2)$ by suppression of the zeros. Then $d_\alpha = deg(P_{j-1}) - deg(P_j)$ for some j and we let $p_{\alpha,1}, \dots, p_{\alpha,d_\alpha}$ be the zeros of the polynomial $\frac{P_{j-1}}{P_j}$. By definition of φ_π, we have the equality $Z = \varphi_\pi(p_{11}, \dots, p_{sd_s})$, which shows the expected surjectivity.
If $\mathbb{H}_{i,i+1}^{T',H}$ is non empty, it contains a subscheme $Z \subset X$ fixed under the action of T. Such a $Z = Z(E_i) \cup Z(E_{i+1})$ is characterized by a pair (E_i, E_{i+1}) of staircases in R_i and R_{i+1}. Suppose as before that $V_{i,i+1}$ is defined by $y_i = 0$ around p_i and by $y_{i+1} = 0$ around p_{i+1}. Using these coordinates, E_i (resp E_{i+1}) is associated with a partition π^i (resp. π^{i+1}) defined by $x_i^a y_i^b \in E_i \Rightarrow b < \pi^i_{a+1}$ (resp. $x_{i+1}^a y_{i+1}^b \in E_{i+1} \Rightarrow b < \pi^{i+1}_{a+1}$). Let $\pi = (\pi^{i\ \nu} + \pi^{i+1\ \nu})^\nu$. Then

$$H = H_{T',Z} = H_{T',Z(E_i)} + H_{T',Z(E_{i+1})} = H_{\pi^i} + H_{\pi^{i+1}} = H_\pi.$$

■

Knowing that $\mathbb{H}_i^{T',H}$ is empty or irreducible (theorem 1) and that $\mathbb{H}_{i,i+1}^{T',H}$ is empty or a product of projective spaces (proposition 4), we obtain easily the irreducible components of $\mathbb{H}^{d,T'}$: According to the last but one item of the next proposition, $\mathbb{H}^{T',\underline{H}}$ is empty or irreducible. Thus, the last item is the decomposition of $\mathbb{H}^{d,T'}$ into irreducible components (in fact into empty or irreducible components and we know which terms in the union are empty).

Proposition 5.

- $\mathbb{H}^T = \coprod_{\underline{E} \in \mathcal{ME}} Z(\underline{E})$.
- $\mathbb{H}^{T'} = \prod_{p_i \in PFix(T')} \mathbb{H}_i^{T'} \times \prod_{\{p_i,p_{i+1}\} \in LFix(T')} \mathbb{H}_{i,i+1}^{T'}$.
- $\mathbb{H}^{T',\underline{H}} = \prod_{p_i \in PFix(T')} \mathbb{H}_i^{T',H_i} \times \prod_{\{p_i,p_{i+1}\} \in LFix(T')} \mathbb{H}_{i,i+1}^{T',H_i}$.
- $\mathbb{H}^{d,T'} = \coprod_{\underline{H} \in \mathcal{MH}(T'), \#\underline{H} = d} \mathbb{H}^{T',\underline{H}}$.

Proof. The first point is well known. As to the second point, since the support of a subscheme $Z \subset X$ parametrised by $p \in \mathbb{H}_i^{T'}$ (resp by $p \in \mathbb{H}_{i,i+1}^{T'}$) is p_i (resp. is on $V_{i,i+1}$) and since the various $p_i, V_{i,i+1}$ do not intersect, the union morphism is a well defined embedding

$$\prod_{i \in PFix(T')} \mathbb{H}_i^{T'} \times \prod_{\{p_i,p_{i+1}\} \in LFix(T')} \mathbb{H}_{i,i+1}^{T'} \to \mathbb{H}^{T'}.$$

Since the support of a subscheme $Z \in \mathbb{H}^{T'}$ is included in $X^{T'}$, surjectivity is obvious. The third point follows from the second. The last point is easy. ■

3 Equivariant Chow rings of products of Grassmannians

Let V be a vector space with base $\mathcal{B} = \{e_0, \dots, e_n\}$. Let $\chi_0, \dots, \chi_n \in \hat{T}$ be characters of T. These characters define an action of T on V by the formula $t.(v_0, \dots, v_n) = (\chi_0(t)v_0, \dots, \chi_n(t)v_n)$. The T-action on V induces a T-action on the Grassmannian $\mathbb{G}(d, V)$ parametrising the d-dimensional quotients of V. In this section, we compute the T-equivariant Chow ring of $\mathbb{G}(d, V)$ and of products of such Grassmannians.

Equivariant Chow ring of $\mathbb{G}(d, V)$

First, we recall the definition of equivariant Chow rings in the special case of a T-action (To keep constant the conventions of the paper, we work with rational Chow groups though it is not necessary in this section).
Let $U = (k^r \setminus 0) \times (k^r \setminus 0)$. The torus $T \simeq k^* \times k^*$ acts on U by the formula $(t_1, t_2)(v, w) = (t_1 v, t_2 w)$. If $\mathcal{X}$ is a T-variety, the quotient $(U \times \mathcal{X})/T = U \times^T \mathcal{X}$ admits a projection to $U/T = \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$. The Chow group $A_{l+2r-2}(U \times^T \mathcal{X})$ does not depend on the choice of r provided that r is big enough (explicitly $r > \dim \mathcal{X} - l$) and this Chow group is by definition the equivariant Chow group $A_l^T(\mathcal{X})$. In case $\mathcal{X}$ is smooth, we let $A_T^l(\mathcal{X}) = A^T_{\dim \mathcal{X} - l}(\mathcal{X})$ and this makes $A_T^*(\mathcal{X}) = \oplus_{l \geq 0} A_T^l(\mathcal{X})$ a ring.
An equivariant vector bundle $F \to \mathcal{X}$ defines equivariant Chern classes: $F \times^T U$ is a vector bundle on $\mathcal{X} \times^T U$ and by definition $c_i^T(F) = c_i(F \times^T U)$.
The equivariant Chow ring $A_T^*(Spec\ k)$ of a point is a polynomial ring by the above description. A more intrinsec description is as follows. A character $\chi \in \hat{T}$ defines canonically an equivariant line bundle V_χ over $Spec\ k$. The map

$$\begin{array}{ccc} \hat{T} & \to & A_T^*(Spec\ k) \\ \chi & \mapsto & c_1^T(V_\chi) \end{array}$$

extends to an isomorphism

$$S = Sym(\hat{T} \otimes \mathbb{Q}) \to A_T^*(Spec\ k)$$

where $Sym(\hat{T} \otimes \mathbb{Q})$ is the symmetric $\mathbb{Q}$-algebra over $\hat{T}$.
The morphism $\mathcal{X} \to Spec\ k$ induces by pullback a S-algebra structure over $A_T^*(\mathcal{X})$.

Let us now turn to the case $\mathcal{X} = \mathbb{G}(d, V)$. We denote by $\mathcal{O}(\chi)$ the line bundle $V_\chi \times^T U \to U/T$. One checks easily that $\mathbb{G}(d, V) \times^T U \to U/T$ is the Grassmann bundle $\mathbb{G}(d, \mathcal{O}(\chi_0) \oplus \dots \oplus \mathcal{O}(\chi_n))$. The universal rank d quotient bundle $Q_T \to \mathbb{G}(d, \mathcal{O}(\chi_0) \oplus \dots \oplus \mathcal{O}(\chi_n))$ over the Grassmann bundle has total space $Q_T = Q \times^T U$ where $Q \to \mathbb{G}(d, V)$ is the universal quotient bundle over the Grasmanniann. In particular $c_i^T(Q) = c_i(Q_T)$.

If $\lambda = (\lambda_1, \dots, \lambda_{\dim V - d}, 0, \dots) \in \mathcal{P}art$, let us denote by

$$D_\lambda = det(c^T_{\lambda_i + s - i}(Q))_{1 \leq i, s \leq \dim V - d}$$

the associated Schur polynomial in the equivariant Chern classes of Q.

Proposition 6. *The elements D_λ generate the S-module $A^*_T(\mathbb{G}(d,V))$.*

Proof. Let $\delta \in \mathbb{N}$, $A_T^{\leq\delta} \subset A^*_T(\mathbb{G}(d,V))$ be the submodule defined by the elements of degree at most δ. It suffices to prove that every class in $A_T^{\leq\delta}$ is a linear combination of D_λ's with coefficients in S. By definition, $A_T^{\leq\delta} = A^{\leq\delta}(U \times^T \mathbb{G}(d,V))$ with $U = (k^n \setminus \{0\}) \times (k^n \setminus \{0\})$ and $n >> 0$. As explained, the quotient $U \times^T \mathbb{G}(d,V)$ is a Grassmann bundle over U/T and the result follows from [9], Proposition 14.6.5 and Example 14.6.4, which describe the Chow ring of Grassmann bundles. ■

Remark 7. *The number of generators in the last proposition is finite since $D_\lambda = 0$ for $\lambda_1 > d$.*

To realize $A^*_T(\mathbb{G}(d,V))$ as an explicit S-subalgebra of $S^{\mathbb{G}(d,V)^T}$, we recall from [1], the following result:

Proposition 8. *If $\mathcal{X}$ is a projective non singular variety, the inclusion map $i : \mathcal{X}^T \to \mathcal{X}$ induces an injective S-algebra homomorphism $i^*_T : A^*_T(\mathcal{X}) \to A^*_T(\mathcal{X}^T)$.*

In fact, Brion proved the injectivity of i^*_T when $\mathcal{X}$ is a smooth filtrable variety and projective varieties are filtrable.
In the present situation, $\mathcal{X} = \mathbb{G}(d,V)$. A point $p_\Sigma \in \mathbb{G}(d,V)^T$ is characterized by a subset $\Sigma = \{e_{i_1}, \ldots, e_{i_d}\} \subset \mathcal{B}$ of cardinal d: if $W \subset V$ is the vector space generated by $\{e_j, e_j \notin \Sigma\}$, then

$$p_\Sigma = V/W.$$

Let σ_i be the i-th symmetric polynomial in d variables. Let

$$c_{i,\Sigma} = \sigma_i(\chi_{i_1}, \ldots, \chi_{i_d}) \in S$$

and

$$c_i \in S^{\mathbb{G}(d,V)^T} = (c_{i,\Sigma})_{\Sigma \subset B, \#\Sigma = d}.$$

Proposition 9. *Let $i^*_T : A^*_T(\mathbb{G}(d,V)) \to A^*_T(\mathbb{G}(d,V)^T) = S^{\mathbb{G}(d,V)^T}$ be the restriction morphism induced by the inclusion $i : \mathbb{G}(d,V)^T \hookrightarrow \mathbb{G}(d,V)$. Then $i^*_T(c_i^T(Q)) = c_i$.*

Proof. The fiber of the universal quotient bundle $Q \to \mathbb{G}(d,V)$ over p_Σ, $\Sigma = \{e_{i_1}, \ldots, e_{i_d}\}$, is a direct sum of one dimensional representations with characters $\chi_{i_1}, \ldots, \chi_{i_d}$ thus its equivariant total Chern class is $c^T(Q) = \prod_{j \leq d}(1 + \chi_{i_j})$. ■

Notation 10. *If* $\lambda = (\lambda_1, \dots, \lambda_{\dim V - d}, 0, \dots) \in \mathcal{P}art$*, let*

$$\Delta_\lambda = det(c_{\lambda_i + s - i})_{1 \le i, s \le \dim V - d} \in S^{\mathbb{G}(d,V)^T}$$

As in remark 7, only a finite number of Δ_λ *are non zero.*

Corollary 11. *The* S*-algebra* $A^*_T(\mathbb{G}(d,V)) \subset S^{\mathbb{G}(d,V)^T}$ *is generated as an* S*-module by the elements* Δ_λ*.*

Proof. The restriction morphism $A^*_T(\mathbb{G}(d,V)) \to A^*_T(\mathbb{G}(d,V)^T) = S^{\mathbb{G}(d,V)^T}$ is injective and gives the inclusion. Since $c_i^T(Q)$ restricts to c_i, the generators D_λ of the S-module $A^*_T(\mathbb{G}(d,V))$ restrict to Δ_λ. The result follows. ∎

Products of Grassmannians

In the sequel, we will need to compute equivariant Chow ring of products of Grassmannians, and of products in general. The following result explains how to deal with these products.

Notation 12. *If* P *and* Q *are two finite sets,* $M \subset S^P$ *and* $N \subset S^Q$ *are two* S*-modules, we denote by* $M \otimes N$ *the* S*-submodule of* $S^{P \times Q}$ *image of* $M \otimes N$ *under the natural isomorphism* $S^{P \times Q} \simeq S^P \otimes S^Q$*.*

Proposition 13. *Let* X *and* Y *be smooth projective* T*-varieties with a finite number of fixed points. Let* $A^*_T(X) \subset S^{X^T}$ *and* $A^*_T(Y) \subset S^{Y^T}$ *be their equivariant Chow rings. Then* $A^*_T(X \times Y) \subset S^{(X \times Y)^T}$ *identifies to* $A^*_T(X) \otimes A^*_T(Y)$*.*

Lemma 14. *If* X *and* Y *are two smooth varieties with cellular decompositions, then* $A^*_T(X \times Y) = A^*_T(X) \otimes A^*_T(Y)$*.*

Proof. Let F be a smooth variety with a cellular decomposition and B be a smooth variety. According to [2], prop. 2, if $\mathcal{F} \xrightarrow{\pi} B$ is a locally trivial fibration with fiber F, there is a non canonical isomorphism of $A^*(B)$-modules

$$\varphi : A^*(\mathcal{F}) \to A^*(B) \otimes A^*(F).$$

Explicitly, let us denote by $f_i \in A^*(F)$ the classes of the closures of the cells of F. They form a base of $A^*(F)$. If $F_i \in A^*(\mathcal{F})$ is such that $F_i \cdot F = f_i$ then

$$\varphi^{-1}(b \otimes f_i) = \pi^* b \cdot F_i.$$

In our case, X and Y admit cellular decompositions whose cells are the Bialynicki-Birula strata associated with the action of a general one parameter subgroup $T' \hookrightarrow T$. Let us denote by $V_i \subset X$ and $W_i \subset Y$ be the closures of these cells. Let

$$X_i = V_i \times^T U \subset X \times^T U$$

and

$$Y_i = W_i \times^T U \subset Y \times^T U.$$

By the above result about fibrations, we have:

$$A_T^*(X) \simeq A^*(X) \otimes S, \quad A_T^*(Y) \simeq A^*(Y) \otimes S,$$

and the isomorphisms identify $[X_i]$ with $[V_i] \otimes 1$, and $[Y_i]$ with $[W_i] \otimes 1$. The left arrow of the diagram

$$\begin{array}{ccc} (X \times Y \times U)/T & \to & X \times^T U \\ \downarrow & & \downarrow \\ Y \times^T U & \to & U/T \end{array}$$

yields an identification

$$\psi : A_T^*(X \times Y) \to A_T^*(Y) \otimes A^*(X) \to A^*(Y) \otimes S \otimes A^*(X).$$

Consider the natural S-module morphism (see [3])

$$K : A_T^*(X) \otimes_S A_T^*(Y) \to A_T^*(X \times Y).$$

The composition

$$\psi \circ K : A_T^*(X) \otimes_S A_T^*(Y) \to A^*(X) \otimes S \otimes A^*(Y)$$

sends the base $[X_i] \otimes [Y_j]$ to the base $[V_i] \otimes 1 \otimes [W_j]$, thus K is an isomorphism.

■

Now the proposition follows from the lemma and the commutativity of the following diagram.

$$\begin{array}{ccc} A_T^*(X) \otimes A_T^*(Y) & \to & A_T^*(X \times Y) \\ \downarrow i_X^* \otimes i_Y^* & & \downarrow i_{X \times Y}^* \\ A_T^*(X^T) \otimes A_T^*(Y^T) & \to & A_T^*(X^T \times Y^T) \end{array}$$

4 Chow rings of graded Hilbert schemes

Let $R = k[x, y]$. In this section, the toric variety X is not projective since we consider the case $X = Spec\ R$. The torus $T \simeq k^* \times k^*$ acts on X by

$$(t_1, t_2).(x^\alpha y^\beta) = (t_1 x)^\alpha (t_2 y)^\beta.$$

Let $T' \hookrightarrow T$ be a one dimensional subtorus such that $X^{T'} = (0, 0)$. Let $H \in \mathcal{H}(T')$ be a Hilbert function. The aim of this section is the computation of the image of the restriction morphism $A_T^*(\mathbb{H}^{T',H}) \to A_T^*(\mathbb{H}^{T',H,T})$ (Corollary 16).

A point $p \in \mathbb{H}^{T',H}$ parametrizes a T'-stable ideal

$$I = \bigoplus_{\chi \in \hat{T}'} I_\chi \subset R,$$

where T' acts with character χ on I_χ. There is a T-equivariant embedding

$$\begin{array}{rcl} \mathbb{H}^{T',H} & \overset{l}{\hookrightarrow} & \mathbb{G}_{T',H} = \prod_{\chi\in\hat{T}'} \mathbb{G}(H(\chi), R_\chi) \\ I & \mapsto & (I_\chi). \end{array}$$

Proposition 15. *1. $l^* : A^*(\mathbb{G}_{T',H}) \to A^*(\mathbb{H}^{T',H})$ is surjective.*

2. $l_T^* : A_T^*(\mathbb{G}_{T',H}) \to A_T^*(\mathbb{H}^{T',H})$ is surjective.

Proof. The surjectivity of l^* has been shown by King and Walter [13] when $T' = \{(t,t)\}$. Their argument is valid for any T' with minor modifications. We recall briefly their method which uses ideas from [6]. Let $\mathcal{S}$ be an associative k-algebra and M be a fine moduli space whose closed points parametrize a class $\mathcal{C}$ of $\mathcal{S}$-modules. Denote by $\mathcal{A}$ the universal $\mathcal{S}\otimes\mathcal{O}_M$-module associated with the moduli space. King and Walter exhibit generators of $A^*(M)$ when $\mathcal{A}$ admits a nice resolution and some cohomological conditions are satisfied.
In the case $\mathcal{S} = R$,$T' = \{(t,t)\}$, $M = \mathbb{H}^{T',H}$, $\mathcal{I}$ the universal ideal over M, $\mathcal{A} = (R\otimes\mathcal{O}_{\mathbb{H}^{T',H}})/\mathcal{I} = \oplus\mathcal{A}_n$, the resolution is

$$0 \to \bigoplus_n R(-n-2)\otimes_k\mathcal{A}_n \to \bigoplus_n R(-n-1)^2\otimes_k\mathcal{A}_n \to \bigoplus_n R(-n)\otimes_k\mathcal{A}_n \to \mathcal{A} \to 0.$$

Consider now a general T'. For $\chi \in \hat{T}'$, we denote by $R_\chi \subset R$ the subvector space on which T' acts through χ and by $R(\chi)$ the $\hat{T}'$-graded R-module defined by $R(\chi)_{\chi'} = R_{\chi+\chi'}$. Let as above $\mathcal{A} = (R\otimes\mathcal{O}_{\mathbb{H}^{T',H}})/\mathcal{I} = \oplus_{\chi\in\hat{T}'}\mathcal{A}_\chi$. The torus T' acts on x and y with characters χ_x, χ_y. Multiplications by x and y define morphisms $\xi : \mathcal{A}_\chi \to \mathcal{A}_{\chi+\chi_x}$ and $\eta : \mathcal{A}_\chi \to \mathcal{A}_{\chi+\chi_y}$. The resolution of $\mathcal{A}$ is:

$$0 \to \bigoplus_{\chi\in\hat{T}'} R(-\chi-\chi_x-\chi_y)\otimes_k\mathcal{A}_\chi \xrightarrow{\alpha}$$
$$\bigoplus_{\chi\in\hat{T}'} (R(-\chi-\chi_x)\oplus R(-\chi-\chi_y))\otimes_k\mathcal{A}_\chi \xrightarrow{\beta} \bigoplus_{\chi\in\hat{T}'} R(-\chi)\otimes_k\mathcal{A}_\chi \to \mathcal{A} \to 0.$$

where the morphisms are

$$\alpha = \begin{pmatrix} -y\otimes 1 + 1\otimes\eta \\ x\otimes 1 - 1\otimes\xi \end{pmatrix}, \beta = (x\otimes 1 - 1\otimes\xi \;\; y\otimes 1 - 1\otimes\eta).$$

With this resolution in hand, we can follow the rest of the argument of [13] to conclude that $A^*(\mathbb{H}^{T',H})$ is generated by the Chern classes $c_i(\mathcal{A}_\chi)$, hence l^* is surjective.

As to the second point, remark that the morphism l^* is obtained from l_T^* with the application of the functor $.\otimes S/S^+$, where $S^+ \subset S$ denotes the set of elements of positive degree. Since l^* is surjective, it follows from the graded Nakayama's lemma that l_T^* is surjective. ∎

The commutative diagram

$$\begin{array}{ccc} \mathbb{H}^{T',H} & \stackrel{l}{\hookrightarrow} & \mathbb{G}_{T',H} \\ j\uparrow & & \uparrow m \\ \mathbb{H}^{T',H,T} & \stackrel{n}{\hookrightarrow} & \mathbb{G}^{T}_{T',H} \end{array}$$

induces a map on the level of equivariant Chow rings. Using the surjectivity of l_T^*, we get

Corollary 16. *$Im\ j_T^* = Im\ n_T^* m_T^*$.*

5 The equivariant Chow ring of $\mathbb{H}^d$

Let $T' \subset T$ be a one-dimensional subtorus. In this section, we define finite S-modules

$$M_{T',i,H_i|} \subset S^{\mathbb{H}_i^{T',H_i,T}}$$

and

$$M_{T',i,i+1,H_i} \subset S^{\mathbb{H}_{i,i+1}^{T',H_i,T}}$$

with explicit generators and we prove the formula:

Theorem 17.

$$A_T^*(\mathbb{H}^d) = \bigcap_{T'\subset T} \bigoplus_{\#\underline{H}=d}^{\underline{H}\in\mathcal{MH}(T')} \Big(\bigotimes_{p_i\in PFix(T')} M_{T',i,H_i|} \bigotimes_{\{p_i,p_{i+1}\}\in LFix(T')} M_{T',i,i+1,H_i} \Big)$$

Proof. A large part of the proof consists in collecting the results from the preceding sections using the appropriate notations.
Let T' be a one dimensional subtorus of T. Let $p_i \in PFix(T')$. Denote by $\mathcal{P}_d(R_{T',i,\chi})$, $\chi \in \hat{T}'$ the set of subsets of monomials of $R_{T',i,\chi}$ of cardinal d. A set of monomials $Z \in \mathcal{P}_d(R_{T',i,\chi})$ defines a point $p_Z \in \mathbb{G}^T_{T',i,\chi,d}$ as explained in the preceding sections: the subspace $V_Z \subset R_{T',i,\chi}$ associated to p_Z is generated by the monomials $m \in R_{T',i,\chi} \setminus Z$. If $m \in Z$, it is an eigenvector for the action of T and we denote by χ_m the associated character. Denote by

$$c_{T',i,\chi,d,j,Z} = \sigma_j(\chi_m, m \in Z) \in S$$

the j-th symmetric polynomial in d variables evaluated on the χ_m and by

$$c_{T',i,\chi,d,j} = (c_{T',i,\chi,d,j,Z})_{Z\in\mathcal{P}_d(R_{T',i,\chi})} \in S^{\mathcal{P}_d(R_{T',i,\chi})} = S^{\mathbb{G}^T_{T',i,\chi,d}}.$$

For $\lambda = (\lambda_1, \ldots, \lambda_{\dim R_{T',i,\chi}-d}) \in \mathcal{P}art$, $\lambda_1 \le d$, let

$$\Delta_{T',i,\chi,d,\lambda} = det(c_{T',i,\chi,d,\lambda_r+s-r})_{1\le s,r\le \dim R_{T',i,\chi}-d} \in S^{\mathbb{G}^T_{T',i,\chi,d}}$$

be the associated Schur polynomials. These Schur polynomials generate a S-module

$$M_{T',i,\chi,d} \subset S^{\mathbb{G}^T_{T',i,\chi,d}}$$

By corollary 11, we have

Proposition 18. $A_T^*(\mathbb{G}_{T',i,\chi,d}) \simeq M_{T',i,\chi,d}$.

If H is a T'-Hilbert function, denote

$$M_{T',i,H} = \bigotimes_{H(\chi)\neq 0} M_{T',i,\chi,H(\chi)} \subset S^{\mathbb{G}^T_{T',i,H}}$$

According to the description of equivariant Chow rings of products (proposition 13) and since $\mathbb{G}_{T',i,H} = \prod_{\chi\in\hat{T}'} \mathbb{G}_{T',i,\chi,H(\chi)}$, we have:

Proposition 19. $A_T^*(\mathbb{G}_{T',i,H}) \simeq M_{T',i,H}$.

The equivariant embedding

$$\mathbb{H}_i^{T',H} \hookrightarrow \mathbb{G}_{T',i,H}$$

yields by restriction a morphism

$$S^{\mathbb{G}^T_{T',i,H}} \to S^{\mathbb{H}_i^{T',H,T}}.$$

If $M \subset S^{\mathbb{G}^T_{T',i,H}}$, we denote by $M_|$ the image of M by this restriction.
The section on the Chow ring of graded Hilbert schemes (corollary 16) can be reformulated in this context as:

Proposition 20. *$A_T^*(\mathbb{H}_i^{T',H}) \simeq M_{T',i,H|} \subset S^{\mathbb{H}_i^{T',H,T}}$. In particular, if $\chi_1,\dots,\chi_s \in \hat{T}'$ are the characters such that $H(\chi_i)\neq 0$, the generators of $A_T^*(\mathbb{H}_i^{T',H})$ are the elements*

$$g_{T',i,H,\lambda_1,\dots,\lambda_s} = (\bigotimes_{\chi_j} \Delta_{T',i,\chi_j,H(\chi_j),\lambda_j})_|.$$

Now we come to the description of $A_T^*(\mathbb{H}_{i,i+1}^{T',H})$ when $\{p_i,p_{i+1}\} \in LFix(T')$. Remember that we have associated a T'-Hilbert function H_π to a partition π such that $\mathbb{H}_{i,i+1}^{T',H} \neq \emptyset$ iff $H = H_\pi$ for some π. Thus we are interested in the case $H = H_\pi$ and we start with the case $\pi = \pi(d,k) = (k,k,\dots,k,0,\dots)$ where k appears d times. In this case, a point $p \in \mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}$ parametrizes a subscheme $Z = Z_i \cup Z_{i+1}$ where $Z_i \in \mathbb{H}_i^T$ and $Z_{i+1} \in \mathbb{H}_{i+1}^T$ are characterized by the integers $l_{i,Z} = length(Z_i \cap V_{i,i+1})$ and $l_{i+1,Z} = length(Z_{i+1}\cap V_{i,i+1}) = d - l_{i,Z}$ (in local coordinates around p_i (resp. around p_{i+1}) $I_{Z_i} = (y^k, x^{l_{i,Z}})$ (resp. $I_{Z_{i+1}} = (y^k, x^{l_{i+1,Z}})$)).
There is an action of T on $V_{i,i+1}$ and we let χ_i (resp. $\chi_{i+1} = -\chi_i$) the character of T which acts on the tangent space of $p_i \in V_{i,i+1}$ (resp. of p_{i+1}).
For $Z \in \mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}$, we define

$$c_{i,i+1,\pi(d,k),Z} = \frac{l_{i,Z}(l_{i,Z}+1)}{2}\chi_i + \frac{l_{i+1,Z}(l_{i+1,Z}+1)}{2}\chi_{i+1} \in S.$$

Then we put

$$c_{i,i+1,\pi(d,k)} = (c_{i,i+1,\pi(d,k),Z}) \in S^{\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}}$$

and we define

$$M_{T',i,i+1,H_{\pi(d,k)}} \subset S^{\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}}$$

to be the S-module generated by the powers $c^j_{i,i+1,\pi(d,k)}$, $0 \leq j \leq d$.

Proposition 21. $A^*_T(\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)}}) \simeq M_{T',i,i+1,H_{\pi(d,k)}} \subset S^{\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}}$.

Proof. We know by proposition 4 that $\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)}} \simeq Sym^d V_{i,i+1}$. Denote by V the vector space with $\mathbb{P}(V) = V_{i,i+1}$ and by P_i, P_{i+1} a base of V with $k.P_i = p_i$, $k.P_{i+1} = p_{i+1}$. The action of T on $V_{i,i+1}$ lifts to an action of T on V with characters 0 on P_i and χ_i on P_{i+1}. The action on $\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)}} \simeq Sym^d V_{i,i+1} = \mathbb{P}(Sym^d(V))$ is induced by the characters $0, \chi_i, \ldots, d\chi_i$ on $P_i^d, P_i^{d-1}P_{i+1}, \ldots, P_{i+1}^d$. Under the above identifications, a point $Z \in \mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}$ corresponds to the line $kP_i^{l_{i,Z}} P_{i+1}^{d-l_{i,Z}} \subset Sym^d(V)$. In particular, the universal quotient bundle $Q = V/\mathcal{O}(-1)$ restricts on Z with equivariant Chern class

$$c_1^T Q_Z = \sum_{0\leq j \leq d, j\neq d-l_{i,Z}} j.\chi_i = (\frac{d(d+1)}{2} - (d-l_{i,Z}))\chi_i.$$

If we call c_1 the tuple $(c_1^T Q_Z)_{Z\in\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}}$, there is a constant $a = d+1$ such that all the coordinates of

$$ac_1 - c_{i,i+1,\pi(d,k)} \in S^{\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}}$$

are equal to a constant $b \in \mathbb{Z}\chi_i$, independent of Z. The equivariant Chow ring $A^*_T(\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)}}) \subset S^{\mathbb{H}_{i,i+1}^{T',H_{\pi(d,k)},T}}$ is the S-module generated by the powers c_1^j, $0 \leq j \leq d$, which is also the module generated by the powers $c^j_{i,i+1,\pi(d,k)}$. ■

Let now π be any partition and call d_j then number of parts of π whose value is j. Then $H_\pi = \sum_{j>0} H_{\pi(d_j,j)}$. Consider the decomposition (proposition 4)

$$\mathbb{H}_{i,i+1}^{T',H_\pi} \simeq \prod_{j>0}^{d_j>0} \mathbb{H}_{i,i+1}^{T',H_{\pi(d_j,j)}}.$$

If we adopt the convention that

$$M_{T',i,i+1,H_\pi} = \bigotimes_{j>0}^{d_j>0} M_{T',i,i+1,H_{\pi(d_j,j)}} \subset S^{\mathbb{H}_{i,i+1}^{T',H_\pi,T}}.$$

and if we denote by $n_1 > n_2 > \cdots > n_s$ the integers such that $d_{n_i} > 0$, the formula for the equivariant Chow ring of a product yields:

Proposition 22. *If* $\{p_i, p_{i+1}\} \in LFix(T')$, $A_T^*(\mathbb{H}_{i,i+1}^{T',H_\pi}) \simeq M_{T',i,i+1,H_\pi} \subset S^{\mathbb{H}_{i,i+1}^{T',H_\pi,T}}$ *where* $M_{T',i,i+1,H_\pi}$ *is the submodule generated by the elements*

$$g_{T',i,i+1,l_1,\ldots,l_s} = \bigotimes_{j=1}^{j=s} c_{i,i+1,\pi(d_{n_j},n_j)}^{l_j}, \quad 0 \le l_j \le d_{n_j}.$$

Let $\underline{H} = (H_1, \ldots, H_r)$ be a T'-Hilbert multifunction. The decomposition

$$\mathbb{H}^{T',\underline{H}} \simeq \prod_{p_i \in PFix(T')} \mathbb{H}_i^{T',H_i} \prod_{\{p_i,p_{i+1}\} \in LFix(T')} \mathbb{H}_{i,i+1}^{T',H_i}$$

yields the following formula for $A_T^*(\mathbb{H}^{T',H})$.

Proposition 23. $A_T^*(\mathbb{H}^{T',\underline{H}}) \subset S^{(\mathbb{H}^{T',\underline{H}})^T}$ *identifies to the* S*-module*

$$\bigotimes_{p_i \in PFix(T')} M_{T',i,H_i|} \bigotimes_{\{p_i,p_{i+1}\} \in LFix(T')} M_{T',i,i+1,H_i}.$$

Since

$$\mathbb{H}^{d,T'} = \coprod_{\underline{H} \in \mathcal{MH}(T'), \#\underline{H}=d} \mathbb{H}^{T',\underline{H}},$$

we obtain:

Proposition 24. $A_T^*(\mathbb{H}^{d,T'}) \subset S^{\mathbb{H}^{d,T}}$ *is isomorphic to*

$$\bigoplus_{\underline{H} \in \mathcal{MH}(T'), \#\underline{H}=d} \Big(\bigotimes_{p_i \in PFix(T')} M_{T',i,H_i|} \bigotimes_{\{p_i,p_{i+1}\} \in LFix(T')} M_{T',i,i+1,H_i} \Big)$$

Now by [1], theorem 3.3,

$$A_T^*(\mathbb{H}^d) \simeq \bigcap_{T' \subset T} A_T^*(\mathbb{H}^{d,T'}),$$

which proves the theorem. ∎

6 Description by congruences

In the last section, $A_T^*(\mathbb{H}^d)$ has been described by a formula involving tensor products and intersections. The goal of this section is to give a simpler presentation. Explicitly, we will describe $A_T^*(\mathbb{H}^d) \subset S^{\mathbb{H}^{d,T}}$ as a set of tuples of elements of S satisfying congruence relations. In this section, projectivity is needed, contrary to the preceding sections where only filtrability was required.

The possibility to reformulate the description of the last section with congruence relations was suggested to me by Michel Brion.

Let $\pi : \mathcal{X} \to Spec\ k$ be a smooth projective T-variety with a finite number of fixpoints, and

$$\begin{array}{rcccc} f : A_T^*(\mathcal{X}) & \otimes_S & A_T^*(\mathcal{X}) & \to & S = A_T^*(Spec\ k) \\ x & \otimes & y & \mapsto & \pi_*(x \cdot y). \end{array}$$

Let $Q = Frac(S)$. According to the localisation theorem ([1], cor. 3.2.1) the morphism $i_T^* : A_T^*(\mathcal{X}) \hookrightarrow S^{\mathcal{X}^T}$ becomes an isomorphism

$$i_{T,Q}^* : A_T^*(\mathcal{X})_Q = A_T^*(\mathcal{X}) \otimes_S Q \to Q^{\mathcal{X}^T}$$

after tensorisation with Q.

Proposition 25. *Let $\beta_i = i_{T,Q}^*(\overline{\beta}_i)$ be a set of generators of the S-module $i_T^* A_T^*(\mathcal{X}) \subset S^{\mathcal{X}^T}$ and $\alpha = i_{T,Q}^*(\overline{\alpha}) \in S^{\mathcal{X}^T}$. Then $\alpha \in i_T^* A_T^*(\mathcal{X}) \Leftrightarrow \forall i, f_Q(\overline{\alpha} \otimes \overline{\beta}_i) \in S$.*

Proof. According to lemma 26 applied with $M = i_T^* A_T^*(\mathcal{X})$, it suffices to find bases $B_i, C_j \in A_T^*(\mathcal{X})$ with $f(B_i \otimes C_j) = \delta_{ij}$. Let $\lambda : T' = k^* \subset T$ be a one parameter subgroup with $\mathcal{X}^{T'} = \mathcal{X}^T$. The Bialynicki-Birula cell associated to a point $p_i \in \mathcal{X}^{T'}$ is the set $\{x \in \mathcal{X},\ \lim_{t\to\infty} \lambda(t).x = p_i\}$. We denote by b_i^+ its closure and we let $B_i^+ = b_i^+ \times^T U \subset \mathcal{X} \times^T U$. It follows from the proof of lemma 14 that the elements $[B_i^+] \in A_T^*(\mathcal{X})$ form a S-base. Consider similarly the cells B_i^- defined by the one parameter subgroup $\lambda \circ i$, where $i : k^* \to k^*,\ x \mapsto x^{-1}$. It is a property of the Bialynicki-Birula cells that one can order the points p_i such that:

$$\begin{array}{rcl} b_i^+ \cap b_j^- & \neq & \emptyset \Rightarrow p_j \leq p_i, \\ b_i^+ \cap b_i^- & = & p_i \quad \text{(transversal intersection)}. \end{array}$$

It follows that

$$\begin{array}{rcl} f([B_i^+] \otimes [B_j^-]) & \neq & 0 \Rightarrow p_j \leq p_i, \\ f([B_i^+] \otimes [B_i^-] & = & 1. \end{array}$$

Up to relabelling, one may suppose $p_1 < p_2 < \cdots < p_n$. The matrix $m_{ij} = f([B_i^+] \otimes [B_j^-])$ is a lower triangular unipotent matrix. In particular, there exists a triangular matrix λ_{ij} such that $[C_j] = \sum \lambda_{ij} [B_i^-]$ verifies $f([B_i^+] \otimes [C_j]) = \delta_{ij}$. ∎

Lemma 26. *Let $Q = Frac(S)$, $M \subset S^n$ be a free S-module, $M_Q = M \otimes_S Q$, $f : M \otimes_S M \to S$ be S-linear, $f_Q : M_Q \otimes M_Q \to Q$ be the Q-linear map extending f, and $\beta_1, \ldots, \beta_p$ be generators of M. Suppose that*

- *$M \subset S^n$ yields an isomorphism $M_Q \simeq Q^n$ after tensorisation by Q,*

- *there exist basis $(B_1,\dots,B_n)$, $(C_1,\dots,C_n)$ of M such that $f(B_i\otimes C_j)=\delta_{ij}$.*

Let $\alpha\in S^n$. Then $\alpha\in M \Leftrightarrow \forall i, f_Q(\alpha\otimes\beta_i)\in S$. ■

As a corollary, we get a description of $i_T^*A_T^*(\mathcal{X})\subset S^{\mathcal{X}^T}$ in terms of congruences involving generators and equivariant Chern classes of the restrictions $T_{\mathcal{X},p}$ of the tangent bundle $T_{\mathcal{X}}$ to fixed points.

Corollary 27. *Let $\beta_i=(\beta_{ip})_{p\in\mathcal{X}^T}$ be a set of generators of the S-module $i_T^*A_T^*(\mathcal{X})\subset S^{\mathcal{X}^T}$ and $\alpha=(\alpha_p)\in S^{\mathcal{X}^T}$. Then the following conditions are equivalent.*

- $\alpha\in i_T^*A_T^*(\mathcal{X})$
- $\forall i,\ \sum_{p\in\mathcal{X}^T}(\alpha_p\beta_{ip}\prod_{q\neq p}c^T_{\dim\mathcal{X}}(T_{\mathcal{X},q}))\equiv 0\ (\prod_{p\in\mathcal{X}^T}c^T_{\dim\mathcal{X}}(T_{\mathcal{X},p}))$

Proof. Let us write $\beta_i=i^*_{T,Q}(\overline{\beta}_i)$, $\alpha=i^*_{T,Q}(\overline{\alpha})$. By the integration formula of Edidin and Graham [4], $f_Q(\overline{\alpha}\otimes\overline{\beta}_i)=\sum_{p\in\mathcal{X}^T}\frac{\alpha_p\beta_{ip}}{c^T_{\dim\mathcal{X}}(T_{\mathcal{X},p})}$. Thus, the corollary is nothing but the criteria of the last proposition. ■

We can collect in a set $G(T',\underline{H})\subset S^{\mathbb{H}^{T',\underline{H},T}}$ the generators of $i_T^*A_T^*(\mathbb{H}^{T',\underline{H}})\subset S^{\mathbb{H}^{T',\underline{H},T}}$ constructed in section 5. Explicitly, $G(T',\underline{H})$ contains the elements

$$g_{T',\underline{H},\lambda_{ij},l_{ij}}=\bigotimes_{p_i\in PFix(T')}g_{T',i,H_i,\lambda_{i1},\dots,\lambda_{i,s_i}}\bigotimes_{\{p_i,p_{i+1}\}\in LFix(T')}g_{T',i,i+1,H_i,l_{i1},\dots,l_{i,t_i}}.$$

These generators and the last corollary make it possible to obtain a description of $i_T^*A_T^*(\mathbb{H}^{T',\underline{H}})$ via congruences. To get a description of

$$A_T^*(\mathbb{H}^d)\simeq\bigcap_{T'\subset T}\ \bigoplus_{\mathbb{H}^{T',\underline{H}}\neq\emptyset}i_T^*A_T^*(\mathbb{H}^{T',\underline{H}})$$

we merely have to gather the congruence relations constructed for the various $\mathbb{H}^{T',\underline{H}}$. We finally obtain:

Theorem 28. *The ring $A_T^*(\mathbb{H}^d)\subset S^{\mathbb{H}^{d,T}}$ is the set of tuples $\alpha=(\alpha_p)$ such that, $\forall T'\subset T$ one dimensional subtorus , $\forall\underline{H}\in\mathcal{MH}(T')$ with $\mathbb{H}^{T',\underline{H}}\neq\emptyset$, $\forall g=(g_p)\in G(T',\underline{H})$, the congruence relation*

$$\sum_{p\in\mathbb{H}^{T',\underline{H},T}}(\alpha_p g_p\prod_{q\neq p}c^T_{\dim\mathbb{H}^{T',\underline{H}}}(T_{\mathbb{H}^{T',\underline{H}},q}))\equiv 0\ (\prod_{p\in\mathbb{H}^{T',\underline{H},T}}c^T_{\dim\mathbb{H}^{T',\underline{H}}}(T_{\mathbb{H}^{T',\underline{H}},p}))$$

holds.

Remark 29. *The tangent space at a T-fixed point of $\mathbb{H}_i^{T',H}$ or $\mathbb{H}_{i,i+1}^{T',H}$ is known [7]. In particular, since $\mathbb{H}^{T',\underline{H}}$ is a product of terms isomorphic to $\mathbb{H}_i^{T',H}$ or $\mathbb{H}_{i,i+1}^{T',H}$, the equivariant Chern classes appearing in the theorem are explicitly computable (See the example in the next section).*

The description of the usual Chow ring now follows from [1],cor.2.3.1.

Theorem 30. *The ring $A^*(\mathbb{H}^d)$ is the quotient of $A_T^*(\mathbb{H}^d) \subset S^{\mathbb{H}^{d,T}}$ by the ideal $S^+A_T^*(\mathbb{H}^d)$ generated by the elements $(f,\dots,f)$, $f \in S^+$.*

7 An example

In this section, we compute the Chow ring of the Hilbert scheme $\mathbb{H}^3 = \mathbb{H}^3\mathbb{P}^2$.

First, we fix the notations: $T = k^* \times k^* = Spec\ k[t_1^{\pm 1}, t_2^{\pm 1}]$ and $\mathbb{P}^2 = Proj\ k[x_1, x_2, x_3]$. The torus T acts on $\mathbb{P}^2$ and on itself. The symmetric group S_3 acts on $\mathbb{P}^2$. The action of an element $(a,b) \in T$, $\sigma \in S_3$ is as follows.

$$(a,b).x_1^\alpha x_2^\beta x_3^\gamma = x_1^\alpha (ax_2)^\beta (bx_3)^\gamma$$
$$(a,b).t_1^\alpha t_2^\beta = (at_1)^\alpha (bt_2)^\beta$$
$$\sigma.x_1^\alpha x_2^\beta x_3^\gamma = x_{\sigma(1)}^\alpha x_{\sigma(2)}^\beta x_{\sigma(3)}^\gamma.$$

The equivariant map $T \to \mathbb{P}^2$, $(a,b) \to (1,a,b)$ identifies t_1 with $\frac{x_2}{x_1}$, and t_2 with $\frac{x_3}{x_1}$. We denote by $p_1 = (1:0:0), p_2 = (0:1:0), p_3 = (0:0:1)$ the three toric points of $\mathbb{P}^2$. The plane $\mathbb{P}^2$ is covered by the three affine planes $U_1 = Spec\ k[t_1, t_2] = Spec\ R_1$, $U_2 = Spec\ k[t_1^{-1}, t_1^{-1}t_2] = Spec\ R_2$, $U_3 = Spec\ k[t_2^{-1}, t_1 t_2^{-1}] = Spec\ R_3$. Since S_3 acts on $T = \{x_1x_2x_3 \neq 0\}$, it acts on $\hat{T}$ by $\sigma.\chi(t) = \chi(\sigma^{-1}t)$, and on $S = Sym(\hat{T} \otimes \mathbb{Q})$. If $T' \subset T$ and $H \in \mathcal{H}(T')$ is a T'-Hilbert function, let $\sigma.H \in \mathcal{H}(\sigma.T')$ be the Hilbert function defined by $(\sigma.H)(\chi) = H(\sigma^{-1}.\chi)$. If $\underline{H} = (H_1, H_2, H_3) \in \mathcal{MH}(T')$ is a Hilbert multifonction, let $\sigma.\underline{H} \in \mathcal{MH}(\sigma.T')$ be the Hilbert multifunction with $(\sigma.\underline{H})_i = \sigma.H_j$ where j is such that $\sigma.p_j = p_i$. To each subvariety $\mathbb{H}^{T',\underline{H}} \subset \mathbb{H}$, we have associated a congruence relation R. Explicitly, constants $t \in S$ and $d(q) \in S$ for $q \in \mathbb{H}^{T',\underline{H},T}$ have been defined such that $s \in S^{\mathbb{H}^{T',\underline{H},T}}$ satisfies R if

$$\sum_{q \in \mathbb{H}^{T',\underline{H},T}} d(q)s(q) \equiv 0(t).$$

The subvariety $\sigma.\mathbb{H}^{T',\underline{H}} = \mathbb{H}^{\sigma.T',\sigma.\underline{H}}$ is associated with the congruence relation $\sigma.R$:

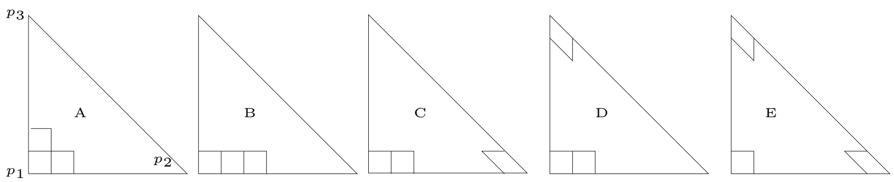
$$\sum_{q \in \mathbb{H}^{T',\underline{H},T}} d(\sigma.q)s(\sigma.q) \equiv 0(t).$$

Summing up, there is an action of S_3 on the set of congruence relations. We will produce the set of relations up to this action.

We list the possible $p \in \mathbb{H}^T$. Let $E_1 = \{1, t_1, t_2\} \subset R_1$, $E_2 = \{1, t_1, t_1^2\} \subset R_1$, $E_3 = \{1, t_1,\} \subset R_1$, $E_4 = \{1\} \subset R_1$, $E_5 = \{1\} \subset R_2$, $E_6 = \{1\} \subset R_3$. The multistaircases

$$
\begin{aligned}
\underline{E}_A &= (E_1, \emptyset, \emptyset)\\
\underline{E}_B &= (E_2, \emptyset, \emptyset)\\
\underline{E}_C &= (E_3, E_5, \emptyset)\\
\underline{E}_D &= (E_3, \emptyset, E_6)\\
\underline{E}_E &= (E_4, E_5, E_6)
\end{aligned}
$$

are associated with points $A, B, C, D, E \in \mathbb{H}^T$. Up to the action of S^3, these are the only points of $\mathbb{H}^{3,T}$.



We recall the description of the tangent space at $p \in \mathbb{H}^T$ where p is described by a multistaircase (F_1, F_2, F_3) ([7]). The staircase F_i is a set of monomials in $R_i = k[x, y]$ where x, y are the toric coordinates around p_i. A cleft for F_i is a monomial $m = x^a y^b \notin F_i$ with ($a = 0$ or $x^{a-1}y^b \in F_i$) and ($b = 0$ or $x^a y^{b-1} \in F_i$). We order the clefts of F_i according to their x-coordinates: $c_1 = y^{b_1}, c_2 = x^{a_2}y^{b_2}, \dots, c_p = x^{a_p}$ with $a_1 = 0 < a_2 < \dots < a_p$. An x-cleft couple for F_i is a couple $C = (c_k, m)$, where c_k is a cleft ($k \neq p$), $m \in F_i$, and $mx^{a_{k+1}-a_k} \notin F_i$. The torus T acts on the monomials c_k and m with characters χ_k and χ_m. We let $\chi_C = \chi_m - \chi_k$. By symmetry, there is a notion of y-cleft couple for F_i. The set of cleft couples for p is by definition the union of the (x or y)-cleft couples for F_1, F_2, F_3. The vector space $T_p\mathbb{H}$ is in bijection with the formal sums $\sum \lambda_i C_i$, where C_i is a cleft couple for p. Moreover, under this correspondance, the cleft couple C is an eigenvector for the action of T with respect to the character χ_C.

If $p \in \mathbb{H}^T$, and if $\underline{H}$ is the T'-Hilbert multifunction of the subscheme associated with p, we let $\mathbb{H}^{T',p} = \mathbb{H}^{T',\underline{H}}$. The subvariety $\mathbb{H}^{T',p} \subset \mathbb{H}$ gives a non trivial congruence only if $\mathbb{H}^{T',p}$ is not a point, ie. if $T_p\mathbb{H}^{T',p} \neq 0$. Using the above description of the tangent space, we find for each point p a finite number of possible T'. The results are collected in the following array. Under each point p are listed the couples (a, b) such that $T_{ab} = \{t^a, t^b\} \subset T$ verifies $\mathbb{H}^{T_{ab},p} \neq \{p\}$. For each such (a, b), the corresponding dimension $\dim \mathbb{H}^{T_{ab},p}$ is given.

A		B		C		D		E	
a, b	dim	a, b	dim	a, b	dim	a, b	dim	a, b	dim
1, 0	2	1, 0	1	1, 0	1	1, 0	2	1, 0	2
0, 1	2	0, 1	3	0, 1	3	0, 1	2	0, 1	2
2, 1	1	1, 1	1	1, 1	2	1, 1	2	1, 1	2
1, 2	1	1, 2	1						

For some a, b, p, a', b', p', we have an identification $\mathbb{H}^{T_{ab},p} = \sigma.\mathbb{H}^{T_{a'b'},p'}$ ($\sigma \in S^3$). Explicitly, up to action, we have $H^{T_{10},A} = H^{T_{10},D} = H^{T_{01},A}$, $H^{T_{01},B} = H^{T_{01},C}$, $H^{T_{12},A} = H^{T_{12},B} = H^{T_{21},A}$, $H^{T_{11},C} = H^{T_{11},D}$, $H^{T_{01},D} = H^{T_{01},E} = H^{T_{10},E} = H^{T_{11},E}$. Thus, by symmetry, we only consider (a, b, p) within the following list:

$$\{(0,1,A),(1,2,A),(1,0,B),(0,1,B),(1,1,B),(1,1,C),(1,0,C),(0,1,D)\}.$$

For each of the above values of (a, b, p), we construct the congruence relation associated with the variety $\mathbb{H}^{T_{ab},p}$. The results are summed up in the following array.

a, b, p	$\mathbb{H}^{T_{ab},p}$	$(H^{T_{ab},p})^T$	generators	$c^T_{top}(T_p)$	*relations*
$0, 1, A$	$\mathbb{P}^1 \times \mathbb{P}^1$	A, A_{13}, D, D_{13}	$1 \quad 1 \quad 1 \quad 1$ $t_2 \quad -t_2 \quad -t_2 \quad t_2$ $t_2 \quad -t_2 \quad t_2 \quad -t_2$ $t_2^2 \quad t_2^2 \quad -t_2^2 \quad -t_2^2$	$t_2^2, t_2^2, -t_2^2, -t_2^2$	$a + a_{13} - d - d_{13} \equiv 0(t_2^2)$ $d - d_{13} \equiv 0(t_2)$ $a - a_{13} \equiv 0(t_2)$
$1, 2, A$	$\mathbb{P}^1$	A, B	$1 \quad 1$ $t_2 \quad 2t_1$	$-2t_1 + t_2, 2t_1 - t_2$	$a - b \equiv 0(2t_1 - t_2)$
$1, 0, B$	$\mathbb{P}^1$	B, B_{13}	$1 \quad 1$ $-t_2 \quad t_2$	$-t_2, t_2$	$b - b_{13} \equiv 0(t_2)$
$0, 1, B$	$\mathbb{P}^3$	B, C, C_{12}, B_{12}	$1 \quad 1 \quad 1 \quad 1$ $6t_1 \quad 2t_1 \quad -2t_1 \quad -6t_1$ $36t_1^2 \quad 4t_1^2 \quad 4t_1^2 \quad 36t_1^2$ $216t_1^3 \quad 8t_1^3 \quad -8t_1^3 \quad -216t_1^3$	$-6t_1^3, 2t_1^3, -2t_1^3, 6t_1^3$	$-b + 3c - 3c_{12} + b_{12} \equiv 0(t_1^3)$ $-b + c + c_{12} - b_{12} \equiv 0(t_1^2)$ $3b - c + c_{12} + -3b_{12} \equiv 0(t_1)$
$1, 1, B$	$\mathbb{P}^1$	B, B_{23}	$1 \quad 1$ $t_2 \quad t_1$ $t_2^2 \quad t_1^2$ $t_2 - t_1 \quad t_1 - t_2$ $t_2^2 - t_2 t_1 \quad t_1^2 - t_1 t_2$ $t_2^3 - t_2^2 t_1 \quad t_1^3 - t_1^2 t_2$	$t_1 - t_2, -t_1 + t_2$	$b - b_{23} \equiv 0(t_2 - t_1)$
$1, 1, C$	$\mathbb{P}^1 \times \mathbb{P}^1$	C, D, C_{23}, D_{23}	$1 \quad 1 \quad 1 \quad 1$ $t_2 - t_1 \quad t_1 - t_2 \quad t_1 - t_2 \quad t_2 - t_1$ $t_1 \quad t_1 \quad t_2 \quad t_2$ $t_1(t_2 - t_1) \quad t_1(t_1 - t_2) \quad t_2(t_1 - t_2) \quad t_2(t_2 - t_1)$	$(t_1 - t_2)^2, -(t_1 - t_2)^2, (t_1 - t_2)^2, -(t_1 - t_2)^2$	$c - d + c_{23} - d_{23} \equiv 0((t_1 - t_2)^2)$ $c + d - c_{23} - d_{23} \equiv 0(t_1 - t_2)$ $c_{23} - d_{23} \equiv 0(t_1 - t_2)$
$1, 0, C$	$\mathbb{P}^1$	C, C_{13}	$1 \quad 1$ $-t_2 \quad +t_2$	$t_2, -t_2$	$c - c_{13} \equiv 0(t_2)$
$0, 1, D$	$\mathbb{P}^2$	D, E, D_{12}	$1 \quad 1 \quad 1$ $-3t_1 \quad 0 \quad 3t_1$ $9t_1^2 \quad 0 \quad 9t_1^2$	$2t_1^2, -t_1^2, 2t_1^2$	$d - 2e + d_{12} \equiv 0(t_1^2)$ $d - d_{12} \equiv 0(t_1)$

If $\sigma = (n_1, n_2) \in S_3$ is a permutation and $p \in \mathbb{H}^T$, we have denoted by $p_{n_1 n_2}$ the element $\sigma.p$. We explain how to read the array, taking the second line as an example. The first three column means that $\mathbb{H}^{T_{01},A}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and contains the points A, A_{13}, D, D_{13}. Four generators of $A_T^*(\mathbb{H}^{T_{01},A}) \subset S^{\{A,A_{13},D,D_{13}\}}$ have been constructed in section 5, namely $A+A_{13}+D+D_{13}, \ldots, t_2^2A+t_2^2A_{13}-t_2^2D-t_2^2D_{13}$. The coefficients of these expressions are written down in the fourth column. The top equivariant Chern classes $c_{top}^T(T_A\mathbb{H}^{T_{01},A}), \ldots, c_{top}^T(T_{D_{13}}\mathbb{H}^{T_{01},A})$ are respectivly $t_2^2, \ldots, -t_2^2$. We can construct congruence relations with these data following the procedure of section 6: $A_T^*(\mathbb{H}^{T_{01},A}) \subset S^{\{A,A_{13},D,D_{13}\}}$ is the set of elements $aA + a_{13}A_{13} + dD + d_{13}D_{13}$ whose coefficients $a, \ldots, d_{13}$ verify $a + a_{13} - d - d_{13} \equiv 0(t_2^2)$, $d - d_{13} \equiv 0(t_2)$ and $a - a_{13} \equiv 0(t_2)$. This is the meaning of the last column. We gather the congruence relations constructed in the array, and we obtain:

Theorem 31. *The equivariant Chow ring $A_T^*(\mathbb{H}^3\mathbb{P}^2) \subset \mathbb{Q}[t_1, t_2]^{\{A,A_{12},\ldots,E\}}$ is the set of linear combinations $aA + a_{12}A_{12} + \cdots + eE$ satisfying the relations*

- $a + a_{13} - d - d_{13} \equiv 0(t_2^2)$
- $d - d_{13} \equiv 0(t_2)$
- $a - a_{13} \equiv 0(t_2)$
- $a - b \equiv 0(2t_1 - t_2)$
- $b - b_{13} \equiv 0(t_2)$
- $-b + 3c - 3c_{12} + b_{12} \equiv 0(t_1^3)$
- $-b + c + c_{12} - b_{12} \equiv 0(t_1^2)$
- $3b - c + c_{12} + -3b_{12} \equiv 0(t_1)$
- $b - b_{23} \equiv 0(t_2 - t_1)$
- $c - d + c_{23} - d_{23} \equiv 0((t_1 - t_2)^2)$
- $c + d - c_{23} - d_{23} \equiv 0(t_1 - t_2)$
- $c_{23} - d_{23} \equiv 0(t_1 - t_2)$
- $c - c_{13} \equiv 0(t_2)$
- $d - 2e + d_{12} \equiv 0(t_1^2)$
- $d - d_{12} \equiv 0(t_1)$
- *all relations deduced from the above by the action of the symmetric group S_3.*

The Chow ring $A^*(\mathbb{H}^3\mathbb{P}^2)$ is the quotient of $A_T^*(\mathbb{H}^3\mathbb{P}^2)$ by the ideal generated by the elements $fA + \cdots + fE$, $f \in \mathbb{Q}[t_1, t_2]^+$.

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