Incidence relations among the Schubert cells of equivariant punctual Hilbert schemes

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Abstract: Let $\mathbb{H}_{ab}(H)$ be the equivariant Hilbert scheme parametrizing the zero dimensional subschemes of the affine plane k^2, fixed under the one dimensional torus $T_{ab} = \{(t^{-b}, t^a),\ t \in k^*\}$ and whose Hilbert function is H. This Hilbert scheme admits a natural stratification in Schubert cells which extends the notion of Schubert cells on Grassmannians. However, the incidence relations between the cells become more complicated than in the case of Grassmannians. In this paper, we give a necessary condition for the closure of a cell to meet another cell. In the particular case of Grassmannians, it coincides with the well known necessary and sufficient incidence condition. There is no known example showing that the condition wouldn't be sufficient.

1 Introduction

The problem (general wording).
Fixing an algebraically closed field k, the Hilbert scheme $\mathbb{H}$ parametrizing the zero dimensional subschemes of the affine plane *Spec* $k[x,y]$ admits a natural action of the two dimensional torus $k^* \times k^*$, induced by the linear action $(t_1, t_2).x^\alpha y^\beta = (t_1.x)^\alpha (t_2.y)^\beta$ of $k^* \times k^*$ on $k[x,y]$. If $T_{ab} = \{(t^{-b}, t^a),\ t \in k^*\}$ is a one dimensional sub-torus, the closed subscheme $\mathbb{H}_{ab}$ which parametrizes by definition the zero dimensional subschemes invariant under the action of T_{ab} is a disjoint union of irreducible subschemes $\mathbb{H}_{ab}(H)$. The ideal of a subscheme corresponding to a point of $\mathbb{H}_{ab}(H)$ is quasi-homogeneous and is a direct sum of vector spaces of fixed dimension. Accordingly, each $\mathbb{H}_{ab}(H)$ can be embedded in a product of Grassmannians and inherits the stratification whose cells are the inverse images of the products of the Schubert cells. The problem is to understand the geometry of this stratification, and in particular to describe the incidences between the cells.

Motivations
Before explaining the results and the techniques, let's explain the motivations. Roughly speaking, Grassmannians are easier to study than Hilbert schemes because they are stratified by Schubert cells. Those stratifications enable for instance to compute the intersection rings of Grassmannians, the Hilbert function of their natural embeddings in a projective space ([GH],[M]). On the other hand, Hilbert schemes admit a stratification furnished by the theory of Gröbner bases (or standard bases) which is fussy to describe ([Gr]). The equivariant Hilbert scheme $\mathbb{H}_{ab}(H)$ is an object which both looks like a Hilbert scheme and a Grassmannian, a sort of interface between them. It has two stratifications, the

stratification in Schubert cells introduced above and the Gröbner stratification obtained by restricting the stratification of $\mathbb{H}$ to $\mathbb{H}_{ab}(H)$. Moreover these two stratifications coincide. So the philosophy is to try to describe the geometry of $\mathbb{H}_{ab} = \cup\mathbb{H}_{ab}(H)$, and then to lift the information on $\mathbb{H}$. This is our motivation to study $\mathbb{H}_{ab}(H)$. To see some precise examples where the geometry of $\mathbb{H}_{ab}$ can be used to describe the geometry of $\mathbb{H}$, see $[I]$, where structures of bundles are highlighted, [ES] where the Betti numbers of the connected components of $\mathbb{H}$ are computed via the study of the equivariant inclusion $\mathbb{H}_{ab} \subset \mathbb{H}$ for (a, b) general enough, or [B] which describes links between the equivariant cohomology of $\mathbb{H}_{ab}$ and the equivariant cohomology of $\mathbb{H}$. Beyond this motivation, note that $\mathbb{H}_{ab}$ is a classical object of study ([Gö],[I],[Y] for instance).

Known results and precise formulation of the problem.
For some special Hilbert functions H, the equivariant Hilbert scheme $\mathbb{H}_{ab}(H)$ is a Grassmannian and the cells of $\mathbb{H}_{ab}(H)$ are the Schubert cells (for this reason, we will say a "Schubert cell" to designate a cell on $\mathbb{H}_{ab}(H)$). In this simple case, the closure of a cell is a union of cells and there is an explicit numerical condition to check incidence. This is no longer so simple when $\mathbb{H}_{ab}(H)$ is not a Grassmannian: Yameogo has given in [Y] an example where the closure of a cell is not a union of cells. So the problem splits into two different pieces: the "weak incidence" problem which consists in deciding whether the closure of a cell meets another cell, and the "strong incidence" problem which consists in deciding whether the closure of a cell contains another cell. In [Y], Yameogo faced the weak incidence problem and gave a necessary but not sufficient condition to have a relation $\overline{C} \cap C' \neq \emptyset$. In this paper, we address the weak incidence problem too.

The results.
Our main answer will be a necessary condition (theorem 8) for weak incidence. No counterexample to sufficiency is known to the author. The other propositions try to analyse the pertinence of the condition, by comparing it to the previous results (prop. 21, 22) and by testing it on small Hilbert functions (section 5).
We now formulate our results with more details. We associate with any Schubert cell C a combinatorial data and we express the relation $\overline{C} \cap C' \neq \emptyset$ by a combinatorial property linking C and C', as follows. Each Schubert cell is determined by a staircase E, i.e.

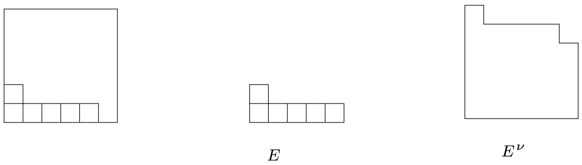
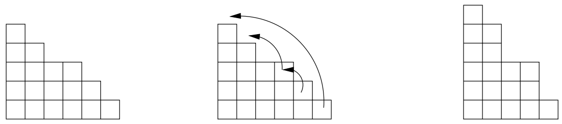


Figure 1

a subset of $\mathbb{N}^2$ whose complementary is stable under the addition of $\mathbb{N}^2$: the ideal I^E generated by the monomials whose exponent is not in E is the unique monomial ideal of the cell. If n is the cardinal of E, E is included in a box of size $n \times n$ and we denote by E^ν the dual staircase defined by the complementary of E in the box (figure 1). A system of arrows S on E is the data for each element p of E of an arrow (p, f_p) with origin p

and end f_p, the set of arrows being compelled to compatibility conditions. Replacing



E — The system S on E. (Arrows of the form (p,p) . have not been drawn) — $S(E)$

Figure 2

each element p of E by f_p gives a new subset of $\mathbb{N}^2$ which turns out to be a staircase thanks to the compatibility conditions. Thinking of S as an operator on E, we denote by $S(E)$ the staircase obtained from E and S by the above procedure. The theorem says that two cells $C(E)$ and $C(F)$ corresponding respectively to staircases E and F verify $\overline{C(E)} \cap C(F) \neq \emptyset$ only if

1. there exists a system of arrows S on E such that $S(E) = F$
2. there exists a system of arrows S^ν on E^ν such that $S^\nu(E^\nu) = F^\nu$

We then compare this condition to the known conditions and we test it on an example. In the particular case of Grassmannians, 1) and 2) are equivalent and reduce to the well known necessary and sufficient incidence condition (prop. 21). In general, they are not equivalent (ex. 10) but any of them imply the condition of [Y] (proposition 22). Finally, we study an example (section 5) which illustrates that when the Hilbert function H is small, we can solve the weak incidence problem because our condition can be shown to be necessary and sufficient.

The methods.
The first step consists in putting the problem in the context of Gröbner bases theory and to describe a cell $C(E)$ as the locus formed by the ideals I having I^E as initial ideal. Then we interpret the incidence relation $\overline{C(E)} \cap C(F) \neq \emptyset$ as the existence of a rational morphism from $\mathbb{A}^1 = Spec\ k[t]$ to $C(E)$ which can be extended in ∞ by putting $f(\infty) = X_F \in C(F)$. The universal ideal on $\mathbb{H}_{ab}(H)$ restricted on the affine curve defines an ideal $I(t) \subset k[x,y,t]$. The key consists in using ideas coming from Gröbner bases theory to exhibit within $I(t)$ a set of elements $P_1, P_2, \ldots$ from which we read the system of arrows on E^ν. Then, using an argument of duality corresponding to the geometric notion of linkage of two zero dimensional schemes in a complete intersection, we get the system of arrows on E.

I thank Michel Brion who told me about the problems that motivated this work and who answered several questions.

2 The stratification on $\mathbb{H}_{ab}(H)$.

2.1 Connected components of $\mathbb{H}_{ab}$

In this section, we explain that the connected components of $\mathbb{H}_{ab}$ can be parametrized by Hilbert functions H.

Recall that k is an algebraically closed field, that a and b are two relatively prime integers, and that $T_{ab} = \{(t^{-b}, t^a), t \in k^*\}$ is a one dimensional subtorus of $k^* \times k^* =: T$. The Hilbert scheme $\mathbb{H}^l$ parametrizing the zero-dimensional subschemes of $Spec\ k[x,y]$ of length l is connected and by definition $\mathbb{H} = \amalg\ \mathbb{H}^l$ where $\amalg$ stands for a disjoint union. There are actions $T \times \mathbb{H} \to \mathbb{H}$ and $T_{ab} \times \mathbb{H} \to \mathbb{H}$ of the tori on $\mathbb{H}$, which in terms of coordinates are given by $(t_1,t_2).x^\alpha y^\beta = (t_1.x)^\alpha (t_2.y)^\beta$. We denote by $\mathbb{H}_{ab} \subset \mathbb{H}$ the subscheme parametrizing the subschemes invariant under the action of T_{ab}. Then $\mathbb{H}_{ab} = \amalg\ \mathbb{H}^l_{ab}$, where $\mathbb{H}^l_{ab} := \mathbb{H}_{ab} \cap \mathbb{H}^l$. However, the connected components of $\mathbb{H}_{ab}$ are not the naive candidates $\mathbb{H}^l_{ab}$: one still needs to separate $\mathbb{H}^l_{ab}$ into disjoint components by fixing a finer Hilbert function.

To do this, let us characterize the subschemes of $Spec\ k[x,y]$ which lie in $\mathbb{H}_{ab}$ by their ideals. Define the degree d of a monomial by the formula $d(x^\alpha y^\beta) = -b\alpha + a\beta$. If I is an ideal of $k[x,y]$, we let $I_n := I \cap k[x,y]_n$, where $k[x,y]_n$ denotes the vector space generated by the monomials m of degree $d(m) = n$. A subscheme Z is in $\mathbb{H}_{ab}$, if and only if its ideal is quasi-homogeneous with respect to d, ie. $I(Z) = \oplus_{n\in\mathbb{Z}} I(Z)_n$.

By semi-continuity, if a subscheme $Z' \in \mathbb{H}_{ab}$ is a specialization of $Z \in \mathbb{H}_{ab}$, then the codimensions of $I_n(Z)$ and $I_n(Z')$ in $k[x,y]_n$ verify $codim I_n(Z) \geq codim I_n(Z')$. But Z and Z', being in the same connected component of $\mathbb{H}$, have the same length $l = \sum_n codim I_n(Z) = \sum_n codim I_n(Z')$. It follows that the sequence $H(Z) = (\dots, h_{-1}, h_0, h_1, \dots)$, where $h_i = codim I_n(Z)$, is constant on the connected components of $\mathbb{H}_{ab}$ and that $h_n = 0$ for $n >> 0$ and $n << 0$. If $H = (\dots, 0, 0, h_r, \dots, h_s, 0, 0, \dots)$ is any sequence, we denote by $\mathbb{H}_{ab}(H)$ the closed subscheme of $\mathbb{H}_{ab}$ (possibly empty) parametrizing the subschemes Z verifying $H(Z) = H$. By the above, we have $\mathbb{H}_{ab} = \amalg\ \mathbb{H}_{ab}(H)$. This is the decomposition of $\mathbb{H}_{ab}$ into connected components. It is in fact possible to show even more: this is the decomposition into irreducible components of $\mathbb{H}_{ab}$ [E] (though we won't use it in the sequel).

2.2 The possible definitions

In this section, we explain that there are three ways to describe the stratification on $\mathbb{H}_{ab}(H)$. The fact that the definitions coincide is shown in [Y]. We call a cell of this stratification a "Schubert cell" of $\mathbb{H}_{ab}(H)$ as it is a Schubert cell when $\mathbb{H}_{ab}(H)$ is a Grassmannian (section 4.1). Each cell corresponds to a staircase E as follows: to a staircase E corresponds a subscheme $Z(E)$ and the cell $C(E)$ is the unique cell containing the subscheme $Z(E)$.

To be more precise, we recall that a staircase is a subset of $\mathbb{N}^2$ whose complementary is stable by addition of $\mathbb{N}^2$. In this paper, we will identify freely the monomial $x^p y^q$ with the couple (p,q) and therefore the expression "staircase of monomials" will make sense. More generally, we will transpose unscrupulously the definitions between couples of integers and monomials. If E is a staircase, then the vector space I^E generated by the monomials which are not in E is an ideal and conversely, every monomial ideal is an ideal I^E for a unique staircase E. The subscheme $Z(E)$ whose ideal is I^E is in $\mathbb{H}_{ab}(H)$ if and only if E has h_i elements in degree i.

The Gröbner bases point of view. The theory of Gröbner bases associates with every ideal in $k[x,y]$ a monomial ideal called initial ideal. The initial ideal depends on the choice of a monomial order. In our case, we choose to order the monomials by the rule: $m_1 < m_2$ if $(d(m_1), d_y(m_1), d_x(m_1)) < (d(m_2), d_y(m_2), d_x(m_2))$ for the lexicographic order, where d_x and d_y are defined by $d_x(x^\alpha y^\beta) = \alpha$, $d_y(x^\alpha y^\beta) = \beta$. Fixing once and for all a, b and $H = (\dots, 0, 0, h_r, \dots, h_s, 0, \dots)$, we define $C(E)$ to be the locus in $\mathbb{H}_{ab}(H)$ parametrizing the ideals whose initial ideal is I^E. This locus is non empty exactly when E has h_i elements in degree i.
For the reader not familiar with Gröbner bases, we give a characterization which is sufficient for the sequel. Let $m_1, m_2, \dots$ be the monomials which don't belong to E. An ideal of $\mathbb{H}_{ab}(H)$ is in $C(E)$ if, regarding it as a k-vector space, it admits a base $f_1, f_2, \dots$ where $f_i = m_i + R_i$, R_i being a linear combination of monomials strictly smaller than m_i.

Remark 1. *When the product ab is non negative, there is at most one staircase compatible with the Hilbert function H (i.e. such that E has h_i elements in degree i) and $\mathbb{H}_{ab}(H)$ is either empty or is reduced to a unique cell. These cases are not relevant for the incidence problem and we suppose from now on $ab < 0$ and $a > 0$. A Hilbert function compatible with a staircase E will then be non zero only in non negative degrees and we write down $H = (h_0, \dots, h_r, 0, 0, \dots)$ instead of $H = (\dots, 0, 0, h_0, h_1, \dots, h_r, 0, 0, \dots)$.*

The Schubert cells point of view. Let $H = (h_0, \dots, h_r, 0, 0, \dots)$, and $Z \in \mathbb{H}_{ab}(H)$. The vector space $I_n(Z)$ of codimension h_n in $k[x,y]_n$ corresponds to a point p_n in a Grassmannian G_n. So $I(Z) = \oplus I_n(Z)$ corresponds to a point $p = (p_0, \dots, p_r)$ in the product of Grassmannians $G_0 \times G_1 \dots \times G_r$. This set theoretical representation turns out to be a closed embedding $g : \mathbb{H}_{ab}(H) \to G_0 \times G_1 \dots \times G_r$. Let V_i be the subspace generated by the i smallest monomials of $k[x,y]_n$. The flag $\{0\} = V_0 \subset V_1 \dots \subset V_s = k[x,y]_n$ defines by a classical construction Schubert cells on G_n which stratify it. The stratification we consider on $\mathbb{H}_{ab}(H)$ is the stratification whose strata are the locally closed subschemes $g^{-1}(C_0 \times C_1 \times \dots \times C_r)$, where C_i is a Schubert cell on G_i.

The Bialynicki-Birula point of view. Let X be a smooth projective variety over k admitting an action of the torus k^*. Suppose that the action has a finite number of fixed points $x_1, \dots, x_n$. Let T^+_{X,x_i} be the part of the tangent space to x_i in X where the weights of the k^*-action are positive, and let $X_i := \{x \in X, \lim_{t \to 0}(t.x) = x_i\}$. Then a theorem of Bialynicki-Birula asserts that the X_i are a cellular decomposition of X in affine spaces and satisfy $T_{X_i,x_i} = T^+_{X,x_i}$. In our case, fixing two integers p and q with $ap + bq > 0$, the torus k^* acts on $k[x,y]$ by $t.x = t^p.x$ and $t.y = t^q y$. This action induces an action of k^* on $\mathbb{H}_{ab}(H)$. The fixed points of $\mathbb{H}_{ab}(H)$ under k^* are the monomial subschemes $Z(E)$. Applying the Bialynicki-Birula theorem to the action of k^* on $\mathbb{H}_{ab}(H)$, we get a stratification and $C(E)$ is the cell associated with the fixed point $Z(E)$.

2.3 Incidence relations

In this section, we recall the main theorem of [Y] about incidence relations.
The monomials of $k[x, y]$ can be ordered in an infinite sequence $m_0 < m_1 < \dots$ thanks to the monomial order chosen in section 2.2. Let S_E be the function from $\mathbb{N}$ to $\mathbb{N}$ defined by $S_E(k)$ =number of monomials in E smaller or equal to m_k.

Theorem 2. *If $\overline{C(E)} \cap C(F) \neq \emptyset$ then $S_E \geq S_F$.*

Remark 3. *In [Y], the result says $\leq$ instead of $\geq$ as above. The reason is that we have slightly changed the definition of S_E to get a shorter presentation. Moreover, the paper deals with the homogeneous case $a = 1, b = -1$ but the extension is immediate.*

The condition is not sufficient: if E and F are the staircases whose monomial ideals are $I^E = (y^4, xy^2, x^2y, x^5)$ and $I^F = (y^5, xy^2, x^3)$ then $S_E \geq S_F$ but the incidence $\overline{C(E)} \cap C(F) \neq \emptyset$ is not fulfilled([Y]).

3 A necessary condition for weak incidence

3.1 Statement of the result

In this section, we give a necessary condition on two staircases E and F to fulfill the relation $\overline{C(E)} \cap C(F) \neq \emptyset$. The condition will rely on the notions of system of arrows and of dual staircase that we introduce now.

Definition 4. *An arrow on E is a couple of monomials (P, Q), (P being the origin of the arrow and Q being the end of the arrow), such that P is in E and such that the vector $\overrightarrow{PQ}$ of $\mathbb{N}^2$ is a negative multiple of (a, b) (recall that we have adopted the convention $a > 0$).*
If $f = (P, Q)$, the multiplication by a monomial m of f is the arrow $m.f := (mP, mQ)$.
An arrow $f = (P, Q)$ is shorter that $f' = (P', Q')$ if $\overrightarrow{PQ} = \lambda(a, b)$, $\overrightarrow{P'Q'} = \lambda'(a, b)$, and if the absolute values of λ and λ' verify $|\lambda| \leq |\lambda'|$.
A system of arrows on E is the data for each element $p \in E$ of an arrow (p, f_p) on E such that the set S of arrows parametrized by E satisfies the following conditions:

- $p \neq q \Rightarrow f_p \neq f_q$
- *the arrows are compatible with division, which means that $\forall f = (P, Q) \in S$, $\forall Q'$ monomial of $k[x, y]$ dividing Q, $\exists g \in S$ such that g has end Q' and such that g is shorter than f.*

With a staircase E and a system of arrows S, one can define a new staircase $S(E)$ as follows.

Proposition 5. *Let E be a staircase and S a system of arrows on E. Let $S(E)$ be the set of monomials which are the end of an arrow. Then $S(E)$ is a staircase.*

Proof: We have to verify that if m and Q' are two monomials with $mQ' \in S(E)$ then $Q' \in S(E)$. By definition of $S(E)$, $Q := mQ'$ is the end of an arrow. The compatibility of S with division shows that $Q' \in S(E)$. ■

Definition 6. *If E and F are two staircases, and if S is a system of arrows on E such that $F = S(E)$, we will say that S is a system of arrows from E to F.*

Definition 7. *The box of size $M \times N$ is $B_{M\times N} := \{(x,y) \in \mathbb{N}^2, x < M, y < N\}$. If a staircase E is included in $B_{M\times N}$, the dual of E in the box is by definition the set $E^{\nu}_{MN} := \{(x,y) \in \mathbb{N}^2$ such that $\forall (e_1,e_2) \in E, e_1 + x < M-1$ and $e_2 + y < N-1\}$. In the sequel, we will often write E^{ν} instead of E^{ν}_{MN} for simplicity.*

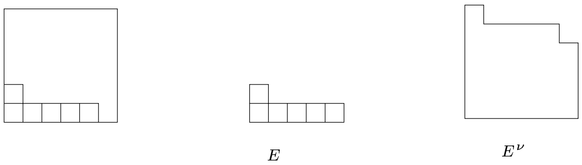


Figure 3

Theorem 8. *Let E and F be two staircases included in a box $B_{M\times N}$. Let E^{ν} and F^{ν} be their respective dual in $B_{M\times N}$. If the incidence $\overline{C(E)} \cap C(F) \neq \emptyset$ is fulfilled, then*

1. *there exists a system of arrows S on E such that $S(E) = F$*
2. *there exists a system of arrows S^{ν} on E^{ν} such that $S^{\nu}(E^{\nu}) = F^{\nu}$*

Proof: see section 3.2 ∎

Remark 9. *One can prove that the existence of S^{ν} does not depend on the choice of the box $B_{M\times N}$ in which we construct the dual staircase E^{ν}.*

The following example shows that conditions 1) and 2) are not equivalent.

Example 10. *Let E and F be the staircases corresponding to the monomial ideals $I^E = (y^4, xy^2, x^2y, x^5)$ and $I^F = (y^5, xy^2, x^3)$. Let $(a,b) = (1,-1)$. There exists a system S on E such that $S(E) = F$ but there is no system of arrows S on $E^{\nu}_{5\ 5}$ such that $S^{\nu}(E^{\nu}_{5\ 5}) = F^{\nu}_{5\ 5}$.*

The set $S := \{(x^3, x^2y), (x^4, y^4), (p,p), p \in E, p \neq x^3, p \neq x^4\}$ is a system of arrows on E such that $S(E) = F$. The dual staircases E^{ν} and F^{ν} in $B_{5\times 5}$ are such that $I^{E^{\nu}} = (y^4, x^3y^3, x^4y, x^5)$ and $I^{F^{\nu}} = (y^5, x^2y^3, x^4)$. Suppose that there exists a system of arrows S^{ν} on E^{ν} such that $S^{\nu}(E^{\nu}) = F^{\nu}$. Because x^2y^3 is in E^{ν} but not in F^{ν}, S^{ν} must contain an arrow f from x^2y^3 to a point $p > x^2y^3$, $p \in F^{\nu}$ ie. $p = xy^4$. The monomial $x^3y^2 \in F^{\nu}$ so there is an arrow f' from a monomial $p \in E^{\nu}$ and $\leq x^3y^2$ to x^3y^2, i.e. $p = x^3y^2$. The compatibility with division and the arrow f' show that all monomials dividing x^3y^2 admit an arrow to themselves. The compatibility with division by y applied to the arrow f ensures the existence of an arrow from xy^3 to itself. It follows that the compatibility with division by x applied to f cannot be satisfied. ∎

3.2 Proof of theorem 8

3.2.1 Proof of point 2

To show that there exists a system S^ν on E^ν such that $S^\nu(E^\nu) = F^\nu$, we will exhibit a system S^c on the complementary E^c of E in $\mathbb{N}^2$ such that $S^c(E^c) = F^c$. So the first step is to explain what is a system on E^c, and to reduce the proof to a construction of such a system S^c. The reduction is given in proposition 14

Definition-Proposition 11. *An arrow on E^c is a couple of monomials (P, Q) such that $P \notin E$ and such that the vector $\overrightarrow{PQ}$ of $\mathbb{N}^2$ is a positive multiple of (a, b). A system of arrows on E^c is the data for each element $p \notin E$ of an arrow (p, f_p) on E^c such that the set S^c of arrows parametrized by E^c satisfy the following conditions:*

- $p \neq q \Rightarrow f_p \neq f_q$
- *the arrows are compatible with multiplication, which means that $\forall f = (P, Q) \in S^c$, $\forall m$ monomial of $k[x, y]$, $\exists g \in S^c$ such that g has end the product mQ and such that g is shorter than f.*

If S^c is a system of arrows on E^c, then the set

$$S^c(E^c) := \{(x, y) \text{ which are the end of an arrow of } S^c\}$$

is the complementary of a staircase F. The system S^c will be called a system of arrows from E^c to F^c.

The proof is similar to the proof of proposition 5 ∎

The next lemma identifies systems on E^c and systems on E^ν. Let $\mathfrak{S}^\nu$ (resp. $\mathfrak{S}^c$) be the set of systems of arrows on E^ν (resp. on E^c). Let $\varphi : B_{M\times N} \to B_{M\times N}$, $(x, y) \mapsto (M-1-x, N-1-y)$ be the dualizing map. For an arrow $f = (p, q)$ on E^ν, if $q \in B_{M\times N}$, we define $\varphi(f) := (\varphi(p), \varphi(q))$. Let

$$\mathfrak{S}^\nu_{MN} := \{S^\nu \text{ system of arrows on } E^\nu \text{ s.t. } s = (p, q) \in S^\nu \Rightarrow q \in B_{M\times N}\}.$$

If $S^\nu \in \mathfrak{S}^\nu_{MN}$, let $\psi(S^\nu) := \{\varphi(f), f \in S^\nu\} \cup \{(p, p),\ p \in E^c \setminus \varphi(E^\nu)\}$.

Lemma 12. *The map $\psi : \mathfrak{S}^\nu_{MN} \to \mathfrak{S}^c$ is well defined, (ie. $\psi(S^\nu)$ is a system of arrows on E^c) and injective. The image $Im\ \psi$ contains the systems S^c such that: for all monomial $m \notin B_{M\times N}$, the arrow of S^c whose origin is m is the arrow (m, m). Moreover if S^ν and S^c are two systems such that $\psi(S^\nu) = S^c$, then $S^\nu(E^\nu) = (S^c(E^c)^c)^\nu$.*

Proof: it consists in a sequence of easy combinatorial verifications which are left to the reader. ∎

Here is a another criterion to check whether a system S^c on E^c is in $Im\ \psi$.

Lemma 13. *Let E be a staircase included in the box $B_{M\times N}$, S^c be a system of arrows on E^c, and F be the staircase such that $S^c(E^c) = F^c$. If $F \subset B_{M\times N}$, then $S^c \in Im\ \psi$.*

Proof: Let $p_0, \dots, p_t$ be the set of monomials of $k[x,y]_s$. Up to reordering, one can suppose that $p_0 > p_1 > \dots > p_t$. There exist integers l and m such that $k[x,y]_s \backslash B_{M\times N} = \{p_0,\dots,p_l\} \cup \{p_m,\dots,p_t\}$, $l < m$, $-1 \le l \le t$, $0 \le m \le t+1$. By 12, to prove the lemma, we have to show that $\forall i \in \{0,\dots,l-1,l,m,m+1,\dots,t\}$, the arrow (p_i,p_i) is in S^c. If the set $\{p_m,\dots,p_t\}$ is non empty, $p_t \notin B_{M\times N}$ so $p_t \in E^c$. It follows that S^c contains an arrow $(p_t,*)$ by definition of a system on E^c and necessarily $* = p_t$ since $* \le p_t$. Now, if $p_{t-1} \in \{p_m,\dots,p_t\}$, then $p_{t-1} \in E^c$, so there is in S^c an arrow $(p_{t-1},*)$. But distinct arrows have distinct ends so $* = p_{t-1}$. By decreasing induction, one shows that S^c contains $(p_m,p_m),(p_{m+1},p_{m+1}),\dots,(p_t,p_t)$.
Similarly, if $p_0 \notin B_{M\times N}$ then $p_0 \in F^c$. By definition of $S^c(E^c)$, this means that S^c contains an arrow $(*,p_0)$ and the only possibility is (p_0,p_0). If the monomial $p_1 \notin B_{M\times N}$, then p_1 is the end of an arrow $(*,p_1)$ and we must have $(* = p_1)$ because the arrow with origin p_0 is already determined. By induction, S^c contains the arrows $(p_0,p_0),\dots,(p_l,p_l)$.
■

Proposition 14. *To prove the existence of a system S^ν such that $S^\nu(E^\nu) = F^\nu$, it suffices to exhibit a system S^c on E^c such that $S^c(E^c) = F^c$.*

Proof: it is an immediate consequence of the last two lemmas (12) and 13. ■

The goal of the next proposition is to exhibit such a system S^c. The cell $C(E)$ is an affine space as a cell of a Bialynicki-Birula stratification. So, if $\overline{C(E)}$ meets $C(F)$, there exists a morphism f from the affine line $\mathbb{A}^1 = Spec\ k[t]$ to $\overline{C(E)}$ such that the image of the generic point is in $C(E)$ and such that the limit of $f(t)$ when t tends to infinity is $X_\infty \in C(F)$. Let $p_1,\dots,p_s$ be the points in $\mathbb{A}^1$ whose image by f is in $\overline{C(E)} - C(E)$. By the universal property of $C(E)$, the morphism f is defined by a closed subscheme U of $\mathbb{A}^1 - \{p_1,\dots,p_s\} \times Spec\ k[x,y]$. Let $I(t) \subset k[x,y][t]$ be the ideal defining the closure of U in $\mathbb{A}^1 \times Spec\ k[x,y]$. The staircase over the generic point being E, for each monomial $m \notin E$, there exists a quasi-homogeneous element $P \in I(t)$ with initial monomial m. Among all possible P, choose one as follows. Let $d_t(P)$ be the degree of P in t, and $d_{in,t}(P)$ be the degree in t of the initial coefficient of P. Let S_m be the set of P such that $\Delta(P) := d_t(P) - d_{in,t}(P)$ is minimal. For a fixed P, denote by $val(P)$ the greatest monomial of P whose coefficient has degree $d_t(P)$. Now choose a P in S_m such that $val(P)$ is minimum. Call this element $P(m)$ and put $e(m) := val(P(m))$.

Proposition 15. *Let $S^c := \{(m,e(m)),\ m \in E^c\}$. Then S^c is a system of arrows on E^c and $S^c(E^c) = F^c$.*

To prove the proposition, we need two lemmas.

Lemma 16. *Let P be a quasi-homogeneous element of $I(t)$ with $in(P) = m$ and $val(P) = n$. Let Q be a quasi-homogeneous element of $I(t)$ such that $in(Q) < m$ and $val(Q) = n$. Let $b(t)$ be the coefficient of n in P, $d(t)$ be the coefficients of n in Q. Then $R = d(t)P - b(t)Q$ is an element of $I(t)$ with initial term m satisfying $\Delta(R) < \Delta(P)$ or ($\Delta(R) = \Delta(P)$ and $val(R) < val(P)$).*

Proof: the fact that R is in $I(t)$ with initial term m is obvious and it remains to calculate $\Delta(R)$. By construction, $d_t(R) \le d_t(P) + d_t(Q)$ and $d_{in,t}(R) = d_{in,t}(P) + d_t(Q) = d_t(P) - \Delta(P) + d_t(Q)$ so $\Delta(R) \le \Delta(P)$. If $\Delta(R) = \Delta(P)$, then $val(R)$ is the greater monomial of R whose coefficient in t has degree $d_t(P) + d_t(Q)$. By construction, R has no term in n and all monomials greater than n have degree in t strictly less than $d_t(P) + d_t(Q)$. So, the greatest monomial having degree $d_t(P) + d_t(Q)$ is smaller than n, showing $val(R) < val(P)$. ■

Lemma 17. *1) The map $e : E^c \to \mathbb{N}^2$ sending m to $e(m)$ is injective.*
2) $Im\ e = F^c$.
3) The inverse map to $e : E^c \to F^c$ sends a monomial $n \notin F$ to the smallest monomial $m \notin E$ for which there exists a quasi-homogeneous polynomial $P \in I(t)$ with $in(P) = m$ and $val(P) = n$.

Proof: 1) If e was not injective, there would exist two monomials m_1 and m_2 with $m_1 > m_2$ and $val(P(m_1)) = val(P(m_2))$. Then the preceding lemma would construct from $P(m_1)$ and $P(m_2)$ an element $R \in I(t)$ contradicting the minimality of $P(m_1)$.
2) By definition of the flat limit, for any $m \notin E$, the limit $P_\infty(m)$ of $P(m)/t^{d_t(P(m))}$ as $t \to \infty$ is an element of $I(X_\infty)$. It follows by definition of X_∞ that the initial monomial $e(m)$ of $P_\infty(m)$ is not in F. Conversely, we show that an element $n \notin F$ can be written $e(m)$ for some $m \in E^c$. Let S (resp. T) be the set of elements of E^c (resp. of F^c) with the same degree as n. The flatness shows that S and T have the same (finite) cardinal. By the above, e is an injection from S into T, so is a bijection. In particular, $n \in T$ can be written $e(m)$ for some m.
3) Let $n = e(m)$ be a monomial in F^c. The polynomial $P(m) \in I(t)$ verifies $in(P(m)) = m$ and $val(P(m)) = n$. Suppose that there exists $m' < m$ and a quasi-homogeneous polynomial $P' \in I(t)$ such that $in(P') = m'$ and $val(P') = n$. Then lemma 16 would construct from $P(m)$ and P' a polynomial R contradicting the minimality of $P(m)$. ■

Proof of proposition 15. Point 1) of the last lemma (17) shows that each point is the end of at most one arrow. To prove that the set is a system of arrows, the compatibility with multiplication by a monomial m' remains to be shown. Let $f = (m, e(m))$ be an arrow. Then $P(m) \in I(t)$ verifies $in(P(m)) = m$ and $val(P(m)) = e(m)$. The polynomial $m'P(m) \in I(t)$ verifies $in(m'P(m)) = m'm, val(m'P(m)) = m'e(m)$. Then point 3) shows that the arrow g ending at $m'e(m)$ has an origin smaller or equal to $m'm$, i.e. g is shorter than f. So S^c is a system of arrows on E, and $S^c(E^c) = F^c$ by 2). ■

3.2.2 Proof of point 1

It will essentially rely on the proof of point 2 thanks to a notion of duality corresponding to the notion of linkage in a complete intersection, introduced in [PS].

Definition 18. *Let Z be a zero-dimensional scheme included in the complete intersection $Y = (x^M, y^N)$. The scheme Z^ν defined by the ideal $(I(Y) : I(Z))$ is called the scheme obtained by linkage in the complete intersection (x^M, y^N). Let $\mathbb{H}(Y)$ be the reduced subscheme of $\mathbb{H}$ parametrizing the subschemes included in Y.*

Proposition 19. *The morphism $\varphi : \mathbb{H}(Y) \to \mathbb{H}(Y)$, $Z \mapsto Z^\nu$ is well defined and sends $C(E)$ to $C(E^\nu_{MN})$. Moreover $Z^{\nu\nu} = Z$.*

Proof: if Y is a 0-dimensional Gorenstein scheme, and if $Z \subset Y$, then the ideal $(I(Y) : I(Z))$ defines a scheme of degree $deg(Y) - deg(Z)$. A complete intersection being Gorenstein, the degree of Z^ν is constant on the connected components of $\mathbb{H}(Y)$ and the morphism is well defined.
Let $Z \in C(E)$, let $in(I(Z^\nu))$ be the set of initial monomials of the elements of $I(Z^\nu)$, and let $e = x^{e_1}y^{e_2} \in in(I(Z^\nu))$. The inclusion $in(I(Z^\nu))in(I(Z)) \subset in(I(Z)I(Z^\nu)) \subset in(x^M, y^N)$ shows that: $\forall(\alpha,\beta) \notin E$, $e_1 + \alpha \geq M$ or $e_2 + \beta \geq N$. In other words, $in(I(Z^\nu)) \subset (E^\nu_{MN})^c$. The complementary sets $in(I(Z^\nu))^c$ and E^ν_{MN} have the same cardinal $MN - card(E)$ so $in(I(Z^\nu)) = (E^\nu_{MN})^c$. This shows that φ sends $C(E)$ to $C(E^\nu_{MN})$. Finally, we have by definition of the dual, $Z^{\nu\nu} \subset Z$. But Z and $Z^{\nu\nu}$ have the same degree so they are equal. ■

Corollary 20. *Let E and F be two staircases of the same cardinal n. Then $\overline{C(E)} \cap C(F) \neq \emptyset \Rightarrow \overline{C(E^\nu_{nn})} \cap C(F^\nu_{nn}) \neq \emptyset$.*

Proof: apply the dualizing morphism φ of the last proposition with $M = N = n$. ■

Now, we come to the proof of point 1 of theorem 8. If $\overline{C(E)} \cap C(F) \neq \emptyset$, then $\overline{C(E^\nu_{nn})} \cap C(F^\nu_{nn}) \neq \emptyset$. Then, point 2 states that there exists a system of arrows S on $(E^\nu_{nn})^\nu_{nn} = E$ such that $S(E) = (F^\nu_{nn})^\nu_{nn} = F$. ■

4 Comparison to the known results

4.1 Grassmannians

In the case $H = (1, 2, 3, \ldots, k-1, k, k+1-n, 0, 0, \ldots, 0)$, $\mathbb{H}_{1,-1}(H)$ is isomorphic to the Grassmannian $G(n, k+1) = G(n, k[x,y]_k)$. The isomorphism consists in associating $I = \oplus I_d$ with the vector space I_k and the inverse isomorphism associates I_k with the ideal $I = I_k \oplus k[x,y]_{k+1} \oplus k[x,y]_{k+2} \oplus \ldots$. A non increasing sequence of non negative integers $(p_1, \ldots, p_n)$ with $p_1 \leq k+1-n$ defines a staircase $E(p_1, \ldots, p_n)$ which contains the monomials of degree at most $k-1$ and the monomials of degree k which are not in $\{x^{n-i+p_i}y^{k-n+i-p_i}, 1 \leq i \leq n\}$. Let V_i be the vector space generated by the i smallest monomials of degree k. The Schubert cell $C(p_1, \ldots, p_n)$ on $G(n, k[x,y]_k)$ is by definition $C(p_1, \ldots, p_n) := \{W \in G(n, k[x,y]_k), dim(W \cap V_j) = i \text{ if } k+1-n+i-p_i \leq j < k+1-n+i-p_{i+1}\}$. By construction, the Schubert cells $C(E(p_1, \ldots, p_n))$ and $C(p_1, \ldots, p_n)$ correspond under the above isomorphism. A classical result on Grassmannians asserts that the closure of a cell $C(p_1, \ldots, p_n)$ is the union of the cells $C(q_1, \ldots, q_n)$ for which $q_i \geq p_i$ for all i. The following result says that in the particular case of Grassmannians, the two points of our necessary condition coincide, and they coincide with the known incidence condition on Grassmannians.

Proposition 21. *The four following conditions are equivalent:*

1. $\overline{C(p_1, \ldots, p_n)} \cap C(q_1, \ldots, q_n) \neq \emptyset$
2. *$q_i \geq p_i$ for all i.*
3. *There exists a system S on $E(p_1, \ldots, p_n)$ such that $S(E(p_1, \ldots, p_n)) = E(q_1, \ldots, q_n)$*

4. There exists a system S^ν on $E(p_1,\dots,p_n)^\nu$ such that $S^\nu(E(p_1,\dots,p_n)^\nu) = E(q_1,\dots,q_n)^\nu$

Proof: $1 \Leftrightarrow 2$ is a classical result on Grassmannians.
$1 \Rightarrow 3$ and $1 \Rightarrow 4$ is exactly theorem 8.
$4 \Rightarrow 1$. If 4) is true, we have a system S^ν on E^ν, which can be identified with a system S^c on E^c (thanks to lemma 12) such that $S^c(E(p_1,\dots,p_n)^c) = E(q_1,\dots,q_n)^c$. For each arrow f of S^c in degree k with origin $o(f)$ and end $g(f)$, let $e(f) := o(f)+tg(f) \in k[x,y,t]$ and denote by V_k the $k[t]$-module generated by $\{e(f),\ f \in S^c \text{ of degree } k\}$. Let $I(t) \subset k[x,y][t]$ be the ideal defined by $I(t) := V_k \oplus k[x,y]_{k+1}[t] \oplus k[x,y]_{k+2}[t] \oplus \dots$. The scheme whose ideal is $\lim_{t\to\infty} I(t) = I^{E(q_1,\dots,q_n)}$ is in $C(E(p_1,\dots,p_n)) \cap C(E(q_1,\dots,q_n))$.
$3 \Rightarrow 4$. We have $4 \Rightarrow 3$. Using the duality of proposition 19 to exchange the roles of $E(p_1,\dots,p_n)$ and $E(p_1,\dots,p_n)^\nu$, we have $3 \Rightarrow 4$. ■

4.2 Comparison to Yameogo's result

The next result asserts that any of the two conditions of theorem 8 implies strictly the condition of theorem 2.

Proposition 22. *Let S be a system of arrows on E. Then $S_{S(E)} \leq S_E$. Moreover, there exist staircases E and F such that:*

- $S_E \geq S_F$
- *there is no system of arrows S on E verifying $F = S(E)$*

Let S^ν be a system of arrows on E^ν. Then $S_{S^\nu(E^\nu_{MN})^\nu_{MN}} \leq S_E$. Moreover, there exist staircases E and F such that:

- $S_E \geq S_F$
- *there is no system of arrows S^ν on E^ν_{MN} verifying $F^\nu_{MN} = S^\nu(E^\nu_{MN})$*

Proof: The passage from E to S_E can be described dynamically as follows. First, you suppress all the arrows which go from a point P to itself. Then, if there is an arrow f_1 from a monomial P to a monomial Q and an arrow f_2 from Q to a monomial R, you suppress the arrows f_1 and f_2 and you replace them by the arrow from P to R. After a finite number of such operations, you come to a finite set of arrows $g_1,\dots,g_s$ with distinct ends such that the origins of the arrows belong to E and such that the ends of the arrows do not belong to E. Now suppress in $E =: E_1$ the origin of g_1 and replace it by the end of g_1. You obtain in this way a subset E_2 of $\mathbb{N}^2$. Replacing in E_2 the origin of g_2 by the end of g_2, a subset E_3 of $\mathbb{N}^2$ is obtained. Continuing in this way, you finally reach the subset $E_{s+1} = S(E)$. We have by construction $S_{S(E)} = S_{E_{s+1}} \leq S_{E_s} \leq \dots \leq S_{E_2} \leq S_{E_1} = S_E$. The same reasoning shows that the inequality $S_{S^\nu(E^\nu_{MN})^\nu_{MN}} \leq S_E$ always holds.
For the second part of the proposition, we consider the staircases E and F whose monomial ideals are $I^E = (y^4, xy^2, x^2y, x^5)$ and $I^F = (y^5, xy^2, x^3)$. They verify $S_E \geq S_F$ but example 10 shows that there is no system of arrows S^ν on $E^\nu_{5\ 5}$ verifying $F^\nu_{5\ 5} = S^\nu(E^\nu_{5\ 5})$. The dual example $I^E = (y^4, x^3y^3, x^4y, x^5)$ and $I^F = (y^5, x^2y^3, x^4)$ verifies $S_E \geq S_F$, but there is no system S on E such that $F = S(E)$. ■

5 An example

For a small Hilbert function, it is easy to show that the necessary condition given by our theorem is in fact sufficient. It follows that we can solve the weak incidence problem on $\mathbb{H}_{ab}(H)$ for small H. In this section, we illustrate this fact in the case $(a, b) = (1, -1)$ and $H = (1, 2, 3, 2, 1, 0, 0, \ldots)$.

To solve the problem, we will use sufficient incidence conditions given by $[Y2]$. We could deal without these conditions, but we use them to shorten the proofs.
There are nine possible staircases corresponding to the Hilbert function H, all of them drawn and named in the figure 4. In the figure, a square whose position is (i, j)

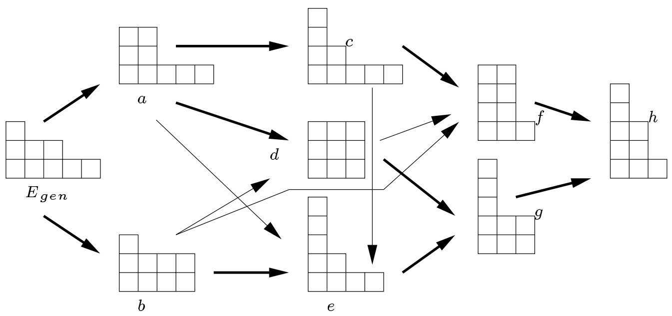


Figure 4

relative to the square at the left bottom represents the monomial $x^i y^j$. We know that $\overline{C(E_1)} \cap C(E_2) \neq \emptyset \Rightarrow S_{E_1} \geq S_{E_2}$. It follows that the only possible relations are

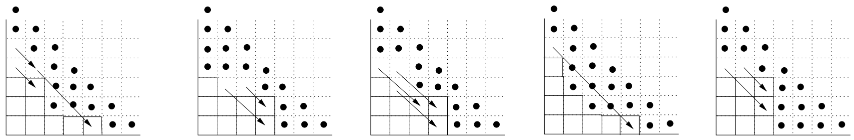
$$\{(E_{gen}, *); (a, c); (a, d); (a, e); (a, f); (a, g); (a, h); (b, d); (b, e); (b, f); (b, g);$$

$$(b, h); (c, e); (c, f); (c, g); (c, h); (d, f); (d, g); (d, h); (e, g); (e, h); (f, h); (g, h)\}$$

where $*$ denotes any staircase and where we have adopted the notation (E_1, E_2) for the incidence relation $\overline{C(E_1)} \cap C(E_2) \neq \emptyset$.
In the figure, a thick arrow between two staircases E_1 and E_2 means that the relation $\overline{C(E_1)} \supset C(E_2)$ occurs and these strong incidence relations come from [Y2]. It follows that if two staircases E_1 and E_2 are linked by a sequence of thick arrows, then the relation (E_1, E_2) holds. The cases then remaining from the above list are:

$$\{(a, e); (b, d); (b, f); (c, e); (c, g); (d, f)\}$$

Example 10 shows that (c, g) does not hold. The five remaining cases are compatible with our necessary condition. In particular, there are systems of arrows on the dual staircases $a^\nu, b^\nu, b^\nu, c^\nu, d^\nu$ coming from the condition 2) of theorem 8. We interpret these systems as systems on a^c, b^c, b^c, c^c, d^c thanks to lemma 12. Figure 5 shows the systems



Arrow Systems(A point symbolizes an arrow from a monomial to itself)

Figure 5

obtained in this way. To exhibit ideals "compatible" with these systems, consider the sub-$k[t]$-modules of $k[x,y,t]$ given by their generators: $T_3 =< y^3 + t^2xy^2 + tx^3, x^2y >$, $T_4 =< y^4 + t^2xy^3, xy^3 + tx^4, x^2y^2, x^3y >$, $P_{\geq 5} =< x^\alpha y^\beta,\ \alpha+\beta \geq 5 >$. Consider the ideal $T := T_3 \oplus T_4 \oplus P_{\geq 5}$ of $k[x,y,t]$. The quotient $k[x,y,t]/T$ is flat over $k[t]$. The associated universal morphism sends the generic point to a point in $C(a)$ and can be extended by sending ∞ to the point whose ideal is the monomial ideal with staircase e. This shows that (a,e) holds.
Similarly, if $U_3 =< y^3, xy^2 + tx^2y + t^2x^3 >$, $U_4 =< y^4, xy^3, x^2y^2 + tx^3y, x^4 >$, $V_3 =< y^3 + tx^2y, xy^2 + tx^3 >$, $V_4 =< y^4, xy^3 + x^3y, x^2y^2, x^4 >$, $W_3 =< xy^2, x^2y >$, $W_4 =< y^4+tx^4, xy^3, x^2y^2, x^3y >$, $Z_3 =< y^3+txy^2+t^2x^2y, x^3 >$, $Z_4 =< y^4, xy^3+tx^2y^2, x^3y, x^4 >$, $U := U_3 \oplus U_4 \oplus P_{\geq 5}$, $V := V_3 \oplus V_4 \oplus P_{\geq 5}$, $W := W_3 \oplus W_4 \oplus P_{\geq 5}$, $Z := Z_3 \oplus Z_4 \oplus P_{\geq 5}$, the ideals U, V, W, Z show respectively that the relations $(b,d), (b,f), (c,e), (d,f)$ are true. The diagram sums up our results: the relation (E,F) holds if and only if the staircases E and F are linked by a thin arrow or by a sequence of thick arrows.

Bibliography:

[BB]: Bialynicki-Birula, A: Some theorems on actions of algebraic groups . *Ann. of Math. 98, 480-497, (1973)*
[BB1]: Bialynicki-Birula, A: Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bulletin de l'académie Polonaise des sciences, Serie des Sciences math. astr. et phys. 24, #9, 667-674, (1976)*
[Br]: Brion, M.: Equivariant Chow groups for torus actions, *Transformations Groups 2, 225–267, (1997)*
[E]: Evain, L.: Irreducibility of the equivariant punctual Hilbert schemes, *En préparation*
[ES]: Ellingsrud, G. and Stromme, S.: On the homology of the Hilbert scheme of points in the plane, *Invent. Math. 87 (1987), no. 2, 343–352*
[GH]: Griffiths, P. and Harris, J.: Principles of algebraic geometry, *J. Wiley, New York (1977)*
[Go]: Göttsche, L.: Betti numbers for the Hilbert function strata of the punctual Hilbert scheme in two variables. *Manuscripta Math. 66, 253-259 (1990)*
[Gr]: Granger, JM.: Géométrie des schémas de Hilbert ponctuels, *Mem. Soc. Math. Fr, Nouv. ser. 7-12(1982-83)*
[I]: Iarrobino, A: Punctual Hilbert Scheme. *Memoirs of AMS, vol. 10, #188, (1977)*
[M]: Manivel, L.: Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence, *Cours specialises de la SMF, 3, (1999)*
[PS]: Peskine C., Szpiro L.: Liaison des variétés algébriques, *Invent. Math. 26, (1974),*

271-302
[Y]: Yameogo, J: Décomposition cellulaire de variétés paramétrant des idéaux homogènes de $\boldsymbol{C}[[x,y]]$. Incidence des cellules I. *Compositio Math. 90, #1, 81–98, (1994)*
[Y2]: Yameogo, J: Décomposition cellulaire de variétés paramétrant des idéaux homogènes de $\boldsymbol{C}[[x,y]]$. Incidence des cellules II. *J.reine angew. Math. 450, 123-137, (1994)*