On canonical bases of a formal \( \mathbb{K} \)-algebra

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Abstract

We study canonical bases of a subalgebra \( \mathbf{A} = \mathbb{K}\llbracket f_1, \cdots, f_s \rrbracket \subseteq \mathbb{K}\llbracket x_1, \cdots, x_n \rrbracket \) over a field \( \mathbb{K} \), then we associate with \( \mathbf{A} \) a fan called the canonical fan of \( \mathbf{A} \). This generalizes the notion of the standard fan of an ideal.

Introduction

Let \( \mathbb{K} \) be a field and let \( f_1, \cdots, f_s \) be nonzero elements of the ring \( \mathbf{F} = \mathbb{K}\llbracket x_1, \cdots, x_n \rrbracket \) of formal power series in \( x_1, \cdots, x_n \) over \( \mathbb{K} \). Let \( \mathbf{A} = \mathbb{K}\llbracket f_1, \cdots, f_s \rrbracket \) be the \( \mathbb{K} \)-algebra generated by \( f_1, \cdots, f_s \). Set \( U = \mathbb{R}^*_+ \) and let \( a \in U^n \). If \( a = (a_1, \cdots, a_n) \), then \( a \) defines a linear form on \( \mathbb{R}^n \) that maps \( \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n \) to the inner product

\[ a \cdot \alpha = \sum_{i=1}^{n} a_i \alpha_i \]

of \( a \) with \( \alpha \). Let \( \underline{x} = (x_1, \cdots, x_n) \) and let \( f = \sum c_\alpha \underline{x}^\alpha \) be a nonzero element of \( \mathbf{F} \). We set \( \text{Supp}(f) = \{ \alpha \mid c_\alpha \neq 0 \} \) and we call it the support of \( f \). We set

\[ \nu(f, a) = \min\{a \cdot \alpha \mid \alpha \in \text{Supp}(f)\} \]

and we call it the \( a \)-valuation of \( f \). We set by convention \( \nu(0, a) = +\infty \). Let

\[ \text{in}(f, a) = \sum_{\alpha \in \text{Supp}(f), a \cdot \alpha = \nu(f, a)} c_\alpha \underline{x}^\alpha. \]

We call \( \text{in}(f, a) \) the \( a \)-initial form of \( f \) (note that \( \text{in}(f, a) \) is a polynomial).

Let the notations be as above, and let \( \prec \) be a well ordering on \( \mathbb{N}^n \). We set \( \exp(f, a) = \max_{\prec} \text{Supp}(\text{in}(f, a)) \) and \( M(f, a) = c_{\exp(f,a)} \underline{x}^{\exp(f,a)} \). We set \( \text{in}(\mathbf{A}, a) = \mathbb{K}\llbracket \text{in}(f, a) \mid f \in \mathbf{A} \setminus \{0\} \rrbracket \). We also set \( M(\mathbf{A}, a) = \mathbb{K}\llbracket M(f, a) \mid f \in \mathbf{A} \setminus \{0\} \rrbracket \). The set \( \{\exp(f, a) \mid f \in \mathbf{A} \setminus \{0\} \} \) is an affine subsemigroup of \( \mathbb{N}^n \). We denote it by \( \exp(\mathbf{A}, a) \). A set \( S \subseteq \mathbf{A} \) is said to be an \( a \)-canonical basis of \( \mathbf{A} \) if \( \exp(\mathbf{A}, a) \) is generated by \( \{\exp(f, a) \mid f \in S\} \).

If \( a \in \mathbb{R}^n \), then \( \text{in}(f, a) \) may not be a polynomial, hence \( \exp(f, a) \) is not well defined. If \( a = (a_1, \cdots, a_n) \) with \( a_{i_1} = \cdots = a_{i_l} = 0 \), then we can avoid this difficulty in completing by the tangent cone order on \( (x_{i_1}, \cdots, x_{i_l}) \). We shall however consider elements in \( U^n \) in order to avoid technical definitions and results.

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This paper has two objectives: we first give an algorithm that calculates, given $a \in U^n$ and a system of generators of $A$, an $a$-canonical basis of $A$, under the assumption that such a basis is finite, and we give criteria for $\exp(A, a)$ to be finitely generated. Then we study the stability of $M(A, a)$ and $\in(A, a)$ when $a$ varies in $U^n$. More precisely, we prove, under some specific conditions, that the sets $\{M(A, a)|a \in U^n\}$ and $\{\in(A, a)|a \in U^n\}$ are finite. Then we prove, given a finitely generated affine semigroup $S$ of $N^n$, that the set $E_S = \{a \in U^n | \exp(A, a) = S\}$ is a union of convex polyhedral cones and the set of $E_S, S$ defines a fan of $U^n$. These results generalize those in [12] for ideals in $\mathbb{K}[x_1, \ldots, x_n]$ and [1], [2] for ideals in the ring of differential operators over $\mathbb{K}$.

The paper is organized as follows. In Section 1 we recall the notion of canonical basis of $A$ with respect to $a \in U^n$, and we give an algorithm that computes an $a$-canonical basis starting with a set of generators of $A$ (see [10] for the case $a = (-1, \cdots, -1)$). In Section 2 we give criteria for an $a$-canonical basis to be finite. In Section 3 we study the possible connections between the set of $M(A, a), a \in U^n$, and we prove a finiteness theorem for the set $M(A)$. In Section 4 we prove the existence of a fan associated with $A$.

1 Computing canonical bases

Let the notations be as in the introduction. In particular $A = \mathbb{K}[f_1, \cdots, f_s]$ is a subalgebra of $F$ generated by $\{f_1, \cdots, f_s\} \subseteq F$ and $<$ is a total well ordering on $N^n$ compatible with sums. Let $a \in U^n$ and consider the total well ordering on $N^n$ defined by:

$$a <_a a' \iff \begin{cases} a \cdot a < a \cdot a' \\ \text{or} \\ a \cdot a = a \cdot a' \text{ and } a < a' \end{cases}$$

The total ordering $<_a$ is compatible with sums in $N^n$. We shall use sometimes the notations $a > \beta$ for $\beta < a$ and $a > a \beta$ for $a < a \alpha$. We have the following:

Lemma 1 There doesn't exist infinite sequences $(\alpha_k)_{k \geq 0}$ such that

$$\alpha_0 > a \alpha_1 > a \cdots > a \alpha_k > a \cdots$$

with $a \cdot a_0 = a \cdot a_k$ for all $k \geq 1$.

Proof. This is a consequence of Dickson’s Lemma, since such a sequence satisfies $\alpha_0 > a \alpha_1 > \cdots > a \alpha_k > \cdots$.

Let $a \in U^n$. Then $a$ defines a graduation on $F : F = \sum_{d \geq 0} F_d$ where $F_d$ is the $\mathbb{K}$-vector space generated by $\mathbb{K}^+ a \cdot a = d$. Let $U_1 = Q^+_1$. If $a \in U_1^n$, then, given two indices $d_1 < d_2$, the set of indices $d$ such that $d_1 < d < d_2$ is clearly finite. In particular we get the following:

Lemma 2 Let $\alpha, \beta \in U_1^n$. If $a > a \beta$, then there doesn't exist infinite sequences $(\alpha_k)_{k \geq 0}$ such that

$$a > a \alpha_0 > a \alpha_1 > a \cdots > a \beta.$$

Proof. The set $\{a \cdot \gamma \in \mathbb{R} | a \cdot a > a \cdot \gamma > a \cdot \beta\}$ is finite, hence the result is a consequence of Lemma 1.
Definition 3 Let \( a \in U^n \) and let \( f = \sum c_\alpha x^\alpha \) be a nonzero element of \( F \). We say that \( f \) is \( a \)-homogeneous if \( f \in F_a \) for some \( d \). This is equivalent to \( \nu(f, a) = \nu(f, a) \) for all \( \alpha \in \text{Supp}(f) \). Every nonzero element \( f = \sum c_\alpha x^\alpha \in F \) decomposes into \( a \)-homogeneous elements, i.e. \( f = \sum_{k \geq d} f_k \), with \( f_d = \text{in}(f, a) \) and for all \( k > d \), if \( f_k \neq 0 \), then \( f_k \in F_k \).

The following result will be used later in the paper.

Definition 4 Let \( a \in U^n \) and let \( H \) be a subalgebra of \( F \). We say that \( H \) is \( a \)-homogeneous if it can be generated by \( a \)-homogeneous elements of \( F \).

Definition 5 Let \( a \in U^n \) and suppose that \( \exp(A, a) \) is finitely generated. Let \( \{g_1, \cdots, g_r\} \subseteq A \). We say that \( \{g_1, \cdots, g_r\} \) is an \( a \)-canonical basis of \( A \) if \( M(A, a) = \mathbb{K}[M(g_1, a), \cdots, M(g_r, a)] \). Clearly \( \{g_1, \cdots, g_r\} \) is an \( a \)-canonical basis of \( A \) if and only if \( \exp(A, a) \) is generated by \( \{\exp(g_1, a), \cdots, \exp(g_r, a)\} \). In this case we write \( \exp(A, a) = \langle \exp(g_1, a), \cdots, \exp(g_r, a) \rangle \).

A finite \( a \)-canonical basis \( \{g_1, \cdots, g_r\} \) of \( A \) is said to be minimal if \( \{M(g_1, a), \cdots, M(g_r, a)\} \) is a minimal set of generators of \( M(A, a) \). It is said to be reduced if the following conditions are satisfied:

i) \( \{g_1, \cdots, g_r\} \) is minimal.

ii) For all \( 1 \leq i \leq r, c_{\exp(g_i, a)} = 1 \).

iii) For all \( 1 \leq i \leq r \), if \( g_i - M(g_i, a) \neq 0 \), then \( x^\alpha \notin \mathbb{K}[M(g_1, a), \cdots, M(g_r, a)] \) for all \( \alpha \in \text{Supp}(g_i - M(g_i, a)) \).

Lemma 6 If \( \exp(A, a) \) is finitely generated and if an \( a \)-reduced canonical basis exists, then it is unique. We write \( e(A, a) \) for its cardinality.

Proof. Let \( F = \{g_1, \cdots, g_r\} \) and \( G = \{g'_1, \cdots, g'_t\} \) be two \( a \)-reduced canonical bases of \( A \). Let \( i = 1 \). Since \( M(g_1, a) \in \mathbb{K}[M(g'_1, a), \cdots, M(g'_t, a)] \), we have \( M(g_1, a) = M(g'_1, a)^{l_1} \cdots M(g'_t, a)^{l_t} \) for some \( l_1, \cdots, l_t \in \mathbb{N} \). Every \( M(g'_i, a), i \in \{1, \cdots, t\}, \) is in \( \mathbb{K}[M(g_1, a), \cdots, M(g_r, a)] \), then the equation above is possible only if \( M(g_i, a) = M(g'_i, a)^{l_i} \) for some \( k_i \in \{1, \cdots, t\} \). This gives an injective map from \( \{M(g_1, a), \cdots, M(g_r, a)\} \) to \( \{M(g'_1, a), \cdots, M(g'_t, a)\} \). We construct in the same way an injective map from \( \{M(g'_1, a), \cdots, M(g'_t, a)\} \) to \( \{M(g_1, a), \cdots, M(g_r, a)\} \). Hence \( r = t \) and both sets are equal. Suppose, without loss of generality, that \( M(g_i, a) = M(g'_i, a) \) for all \( i \in \{1, \cdots, r\} \). If \( g_i \neq g'_i \), then \( M(g_i - g'_i) \in M(A, a) \) because \( g_i - g'_i \in A \). This contradicts iii). \( \square \)

We now recall the division process for algebras in \( F \) (see [10] for the case \( a = (1, \cdots, 1) \) and [5] for \( n = 1 \)).

Theorem 7 Let \( a \in U^n \) and let \( \{F_1, \cdots, F_s\} \subseteq F \). Let \( F \) be a nonzero element of \( F \). There exist \( H \in \mathbb{K}[F_1, \cdots, F_s] \) and \( R \in F \) such that the following conditions hold:

1. \( F = H + R \).

2. Set \( H = \sum \alpha c_\alpha \prod_{i=1}^s F_i^{\alpha_i} \). If \( H \neq 0 \), then \( \exp(F, a) = \max_{\alpha} \{\exp(\prod_{i=1}^s F_i^{\alpha_i}, a), c_\alpha = 0}\} \).

3. Set \( R = \sum \beta b_\beta x^\beta \). If \( R \neq 0 \), then for all \( \beta \in \text{Supp}(R), x^\beta \notin \mathbb{K}[M(F_1, a), \cdots, M(F_s, a)] \) (or equivalently \( \beta \notin \langle \exp(F_1), \cdots, \exp(F_s) \rangle \)). Moreover, \( \nu(R, a) \geq \nu(F, a) \).

We set \( R = R_A(F, \{F_1, \cdots, F_s\}) \) and we say that \( R \) is an \( a \)-remainder of the division of \( F \) with respect to \( \{F_1, \cdots, F_s\} \).

3
Proof. We define the sequences \((F^k)_{k \geq 0}, (h^k)_{k \geq 0}, (r^k)_{k \geq 0}\) in \(F\) by \(F^0 = F, h^0 = r^0 = 0\) and \(\forall k \geq 0:\)

(i) If \(F^k \neq 0\) and if \(M(F^k, a) \in \mathbb{K}[M(F_1, a), \ldots, M(F_s, a)]\), write \(M(F^k, a) = c_a \prod_{i=1}^s M(F_i, a)^{\alpha_i}\). We set

\[
F^{k+1} = F^k - c_a \prod_{i=1}^s F_i^{\alpha_i}, \quad h^{k+1} = h^k + c_a \prod_{i=1}^s F_i^{\alpha_i}, \quad r^{k+1} = r^k
\]

(ii) If \(F^k \neq 0\) and if \(M(F^k, a) \notin \mathbb{K}[M(F_1, a), \ldots, M(F_s, a)]\), we set

\[
F^{k+1} = F^k - M(F^{k+1}, a), \quad h^{k+1} = h^k, \quad r^{k+1} = r^k + M(F^k, a)
\]

in such a way that for all \(k \geq 0\), \(\exp(F^k, a) < \exp(F^{k+1}, a)\) and \(F = F^{k+1} + h^{k+1} + r^{k+1}\). If \(F^l = 0\) for some \(l \geq 1\), we set \(H = h^l\) and \(R = r^l\). We easily verify that \(H, R\) satisfy conditions (1) to (3). Suppose that \(\{F^k \mid k \geq 0\}\) is an infinite set. Note that, by Lemma 1, given \(k \geq 1\), if \(F^k \neq 0\), then there exists \(k_1 > k\) such that \(\nu(F^k, a) < \nu(F^{k_1}, a)\). Hence, there exists a subsequence \((F^{j_l})_{l \geq 1}\) such that \(\nu(F^{j_l}, a) < \nu(F^{j_{l+1}}, a) < \cdots\). In particular, if we set \(G = \lim_{k \to +\infty} F^k, h = \lim_{k \to +\infty} h^k\), and \(R = \lim_{k \to +\infty} r^k\), then \(G = 0, F = H + R\), and \(H, R\) satisfy conditions (1) to (3). This completes the proof. \(\blacksquare\)

Remark 8 The element \(R = R_a(F, \{F_1, \ldots, F_s\})\) of Theorem 7 is not necessarily unique. For example, if \(a = 1\), \(F_1 = x^4, F_2 = x^6 + x^7\), and \(F = x^{12}\), then \(M(F, a) = x^{12} \in \mathbb{K}[M(F_1, a), M(F_2, a)] = \mathbb{K}[t^4, t^6]\). But \(x^{12} = (x^4)^3 = (x^6)^2\). If we divide first by \(F_1\), we get \(F - F_3 = 0\), hence \(F = F_3 + 0\), and \(R_a(F, \{F_1, F_2\}) = 0\). If we divide first by \(F_2\), we get \(F - F_3 = -2x^{13} + x^{14}\). As \(x^{14} = (x^4)^2x^6\), we have \(R_a(F, \{F_1, F_2\}) = -2x^{13} + x^{15}\).

The following Lemma shows that \(R_a(F, \{F_1, \ldots, F_s\})\) becomes unique if we divide by an \(a\)-canonical basis.

Lemma 9 Let \(a \in U^n_1\) and let \(\{F_1, \ldots, F_s\} \subseteq F\). Suppose that \(\{F_1, \ldots, F_s\}\) is an \(a\)-canonical basis of \(\mathbb{K}[F_1, \ldots, F_s]\). If \(F\) is a nonzero element of \(F\), then \(R_a(F, \{F_1, \ldots, F_s\})\) is unique.

Proof. Let the notations be as in Theorem 7 and write \(F = H_1 + R_1 = H_2 + R_2\) where \(H_i, R_i, i = 1, 2\) satisfy conditions (1) to (3) of the theorem. We have \(R_1 - R_2 = H_2 - H_1\). Clearly \(R_1 - R_2 \in \mathbb{K}[F_1, \ldots, F_s]\). If \(R \neq 0\), then, by condition (2), \(M(R, a) \notin \mathbb{K}[M(F_1, a), \ldots, M(F_s, a)]\). This is a contradiction because \(\{F_1, \ldots, F_s\}\) is an \(a\)-canonical basis of \(\mathbb{K}[F_1, \ldots, F_s]\). \(\blacksquare\)

Suppose that \(\{f_1, \ldots, f_s\}\) is an \(a\)-canonical basis of \(A\). If \(M(f_i, a) \in \mathbb{K}[M(f_j, a)\mid j \neq i]\) for some \(1 \leq i \leq s\), then obviously \(\{f_j\mid j \neq i\}\) is also an \(a\)-canonical basis of \(A\). Consequently we can get this way a minimal \(a\)-canonical basis of \(A\). Assume that \(\{f_1, \ldots, f_s\}\) is minimal and let \(1 \leq i \leq s\). Dividing \(f = f_i - M(f_i, a)\) by \(\{f_1, \ldots, f_s\}\), and replacing \(f_i\) by \(M(f_i, a) + R_a(f, \{f_1, \ldots, f_s\})\), we obtain an \(a\)-reduced canonical basis of \(A\).

The next proposition gives a criterion for a finite set of \(A\) to be an \(a\)-canonical basis of \(A\).

Proposition 10 Let \(a \in U^n_1\). The set \(\{f_1, \ldots, f_s\} \subseteq A\) is an \(a\)-canonical basis of \(A\) if and only if \(R_a(f, \{f_1, \ldots, f_s\}) = 0\) for all \(f \in A\).
Proof. Suppose that \( \{ f_1, \cdots, f_s \} \) is an \( a \)-canonical basis of \( A \) and let \( f \in A \). Let \( R = R_a(f, \{ f_1, \cdots, f_s \}) \). If \( R \neq 0 \), then \( M(R,a) \notin M(A,a) \). This is a contradiction because \( R \in A \). Conversely, suppose that \( R_a(f, \{ f_1, \cdots, f_s \}) = 0 \) for all \( f \in A \) and let \( F \in A \). If \( M(F,a) \notin \mathbb{K}[M(f_1,a), \cdots, M(f_s,a)] \), then \( M(F,a) \) is a monomial of \( R_a(F, \{ f_1, \cdots, f_s \}) \), which is 0. This is a contradiction. ■

The criterion given in Proposition 10 is not effective since we have to divide infinitely many elements of \( F \). In the following we shall see that it is enough to divide a finite number of elements.

Let \( \{ f_1, \cdots, f_s \} \subseteq F \) and let \( \phi : \mathbb{K}[X_1, \cdots, X_s] \to \mathbb{K}[M(f_1,a), \cdots, M(f_s,a)] \) be the morphism of rings defined by \( \phi(X_i) = M(f_i,a) \) for all \( 1 \leq i \leq s \). We have the following.

**Lemma 11** The ideal \( \text{Ker}(\phi) \) is a binomial ideal, i.e., it can be generated by binomials.

Proof. See [8], Proposition 1.1.9., for example. ■

Let \( S_1, \cdots, S_m \) be a system of generators of \( \text{Ker}(\phi) \), and assume, by Lemma 11, that \( S_1, \cdots, S_m \) are binomials in \( \mathbb{K}[X_1, \cdots, X_s] \). Assume that \( f_1, \cdots, f_s \) are monic with respect to \( <_a \). For all \( 1 \leq i \leq m \), we can write \( S_i(X_1, \cdots, X_s) = X_1^{a_1} \cdots X_s^{a_s} - X_1^{b_1} \cdots X_s^{b_s} \). Let \( S_i = S_i(f_1, \cdots, f_s) = f_1^{a_1} \cdots f_s^{a_s} - f_1^{b_1} \cdots f_s^{b_s} \). We have \( \exp(S_i, a) = \exp(f_1^{a_1} \cdots f_s^{a_s}, a) = \exp(f_1^{b_1} \cdots f_s^{b_s}, a) \). Moreover, for all \( 1 \leq k \leq s \), if \( \max(\alpha_k, \beta_k) > 0 \), then \( \exp(S_i, a) > a \exp(f_1^{b_1} \cdots f_s^{b_s}, a) \) and \( \nu(S_i, a) \geq \nu(f_2, a) \).

**Proposition 12** Let \( a \in U_1^n \) and let \( \{ f_1, \cdots, f_s \} \subseteq A \). With the notations above, the following conditions are equivalent:

1. The set \( \{ f_1, \cdots, f_s \} \) is an \( a \)-canonical basis of \( A \).
2. For all \( i \in \{ 1, \cdots, m \} \), \( R_a(S_i, \{ f_1, \cdots, f_s \}) = 0 \).

Proof. (1) implies (2) by Proposition 10.

(2) \( \implies \) (1): We shall prove that \( R_a(f, \{ f_1, \cdots, f_s \}) = 0 \) for all \( f \in A \). Let \( f \) be a nonzero element of \( A \) and let \( R = R_a(f, \{ f_1, \cdots, f_s \}) \). By Theorem 7, \( f = H + R \) and \( H \in A \), hence \( R \in A \). Consequently, if \( R \neq 0 \), then \( M(R,a) \in M(A,a) \). Write

\[
R = \sum_{\theta} c_{\theta} f_1^{\theta_1} \cdots f_s^{\theta_s}
\]

and let \( \alpha = \min_{\theta, \varphi \neq 0}(\exp(f_1^{\theta_1} \cdots f_s^{\theta_s}, a)) \). Let \( S = \langle \exp(f_1, a), \cdots, \exp(f_s, a) \rangle \). By condition (2) of Theorem 7, \( \exp(R, a) \notin S \). On the other hand, \( \exp(R, a) \geq \alpha S \). Consequently \( \exp(R, a) \geq \alpha \). Let \( \{ \theta_1, \cdots, \theta_l \} \) be the set of elements such that \( \alpha = \exp(f_1^{\theta_1} \cdots f_s^{\theta_s}, a) \) for all \( i \in \{ 1, \cdots, l \} \) (such a set is clearly finite). If \( \sum_{i=1}^{l} c_{\theta_i} M(f_1^{\theta_i} \cdots f_s^{\theta_i}, a) = 0 \), then \( \exp(R, a) \in S \), which is a contradiction. It follows that \( \sum_{i=1}^{l} c_{\theta_i} M(f_1^{\theta_i} \cdots f_s^{\theta_i}, a) = 0 \), and consequently \( \sum_{i=1}^{l} c_{\theta_i} X_1^{\theta_1} \cdots X_s^{\theta_s} \) is an element of \( \text{Ker}(\phi) \). In particular

\[
\sum_{i=1}^{l} c_{\theta_i} X_1^{\theta_1} \cdots X_s^{\theta_s} = \sum_{k=1}^{m} \lambda_k S_k
\]

with \( \lambda_k \in \mathbb{K}[X_1, \cdots, X_s] \) for all \( k \in \{ 1, \cdots, m \} \). Whence

\[
\sum_{i=1}^{l} c_{\theta_i} f_1^{\theta_1} \cdots f_s^{\theta_s} = \sum_{k=1}^{m} \lambda_k (f_1, \cdots, f_s) S_k
\]
with \( \exp(\lambda_k(f_1, \cdots, f_s)S_k, a) >_a \alpha \). From the hypothesis, \( R_a(S_k; \{f_1, \cdots, f_s\}) = 0 \) for all \( k \in \{1, \cdots, m\} \). It follows from Theorem 7 that we can write \( S_k \) as \( S_k = \sum_{j=1}^l c_j^k f_1^{\beta_1^k} \cdots f_s^{\beta_j^k} \) with 

\[
\min_{\beta_k, c_k \neq 0} \exp(f_1^{\beta_1^k} \cdots f_s^{\beta_j^k}, a) = \exp(S_k, a).
\]

Replacing \( \sum_{i=1}^m c_i^k f_1^{\beta_1^k} \cdots f_s^{\beta_j^k} \) by

\[
\sum_{k=1}^m \lambda_k(f_1, \cdots, f_s) \sum_{\beta_k^j} c_{\beta^k} f_1^{\beta_1^k} \cdots f_s^{\beta_j^k}
\]

in the expression of \( R \), we can rewrite \( R = \sum_{\beta_k, a} c_{\beta}^k f_1^{\beta_1^k} \cdots f_s^{\beta_j^k} \) with \( \alpha_1 = \min_{\beta_k, c_k \neq 0} \exp(f_1^{\beta_1^k} \cdots f_s^{\beta_j^k}, a) >_a \alpha \). Then we restart with this expression. We construct this way an infinite sequence \( \exp(R, a) >_a \cdots >_a \alpha_1 >_a \alpha \). This contradicts Lemma 2. \( \blacksquare \)

The characterization given in Proposition 12 suggests an algorithm that constructs, starting with a set of generators of \( A \), an \( a \)-canonical basis of \( A \). More precisely we have the following.

**Algorithm.** Let \( A = \mathbb{K}[f_1, \cdots, f_s] \) and let \( a \in U^a \). Suppose that \( \exp(A) \) is finitely generated and let \( \{S_1, \cdots, S_m\} \) be a set of generators of the map \( \phi \) of Proposition 11. Let \( S_i = S_i(f_1, \cdots, f_s) \) for all \( i \in \{1, \cdots, m\} \).

1. If \( R_a(S_i; \{f_1, \cdots, f_s\}) = 0 \) for all \( i \in \{1, \cdots, m\} \), then \( \{f_1, \cdots, f_s\} \) is an \( a \)-canonical basis of \( A \).

2. If \( R = R_a(S_i; \{f_1, \cdots, f_s\}) \neq 0 \) for some \( i \in \{1, \cdots, m\} \), then we set \( R = f_{s+1} \) and we restart with \( \{f_1, \cdots, f_s, f_{s+1}\} \). Note that in this case, we have \( \langle \exp(f_1, a), \cdots, \exp(f_s, a), \exp(f_{s+1}, a) \rangle \subseteq \langle \exp(f_1, a), \cdots, \exp(f_s, a) \rangle \). Let us prove that this process will stop. By hypothesis, \( \exp(A, a) \) is finitely generated. Let \( \{\gamma_1, \cdots, \gamma_r\} \) be a minimal set of generators of \( \exp(A, a) \) and suppose that \( \nu(\gamma_1, a) \leq \cdots \leq \nu(\gamma_r, a) \). Suppose that the process above gives infinitely many new elements \( f_{s+k}, k \geq 1 \). Then there exists \( t \geq 1 \) such that \( \nu(f_{s+k}, a) > \nu(\gamma_r, a) \) for all \( k > t \). We claim that \( \{f_1, \cdots, f_{s+1}\} \) is an \( a \)-canonical basis of \( A \). Suppose otherwise, then \( \gamma_j \notin \{\exp(f_1, a), \cdots, \exp(f_{s+t}, a)\} \) for some \( j \in \{1, \cdots, r\} \). Let \( f \in A \) such that \( \exp(f, a) = \gamma_j \). As \( A = \mathbb{K}[f_1, \cdots, f_s, f_{s+1}, \cdots, f_{s+t}] \), we can write \( f = \sum_{k=1}^{s+t} c_{\alpha}^k f_1^{\alpha_1^k} \cdots f_{s+t}^{\alpha_k} \). We shall use a similar argument as in Proposition 12. Let \( \alpha_0 = \min_{\alpha, c_{\alpha}^k \neq 0} \exp(f_1^{\alpha_1^k} \cdots f_{s+t}^{\alpha_k}, a) \) and let \( \{\alpha^1, \cdots, \alpha^p\} \) be the set of elements such that \( \alpha_0 = \exp(f_1^{\alpha_1^k} \cdots f_{s+t}^{\alpha_k}, a) \) for all \( k \in \{1, \cdots, p\} \). We have \( \gamma_j >_a \alpha_0 \). If \( \gamma_j >_a \alpha_0 = \exp(f_1^{\alpha_1^k} \cdots f_{s+t}^{\alpha_k}, a) \), then this contradicts the minimality of \( \gamma_j \). If \( \gamma_j >_a \alpha_0 \), then we can write, as in Proposition 12,

\[
f = \sum_k \lambda_k(f_1, \cdots, f_t)S_k + f^1
\]

with \( f^1 \in A \), and \( \alpha_0 <_a \min(\exp(f^1, a), \exp(\sum_k \lambda_k(f_1, \cdots, f_t)S_k, a)) \). By Theorem 7, for all \( k \), there exist \( H_k, R_k \in A \) such that \( S_k = H_k + R_k \) with \( \exp(S_k, a) = \exp(F_k, a) \) if \( F_k \neq 0 \), and either \( R_k = 0 \) or \( R_k = f_{s+t} \) for some \( l > s \), in particular \( \nu(R_k) > \nu(\gamma_r, a) \). As in Proposition 12 we can rewrite \( f \) as:

\[
f = \sum_{\alpha} c_{\alpha} f_1^{\alpha_1} \cdots f_t^{\alpha_t} + \sum_k R_k
\]
and if \( \alpha_1 = \min_{\alpha, r \neq 0} \exp(f_{1}^{\alpha_1} \cdots f_{r}^{\alpha_r}, a) \), then \( \alpha_0 <_a \alpha_1 <_a \exp(f, a) \). Moreover, \( \nu(R_b) > \nu(\gamma_r) > \nu(f) \). Then we restart with this new expression. We construct this way an infinite increasing sequence \( \alpha_0 <_a \alpha_1 <_a \cdots <_a \exp(f, a) \), which is a contradiction.

2 Finiteness criterion of canonical bases

Let the notations be as in Section 1. In particular \( A = \mathbb{K}[f_1, \cdots, f_s] \) with \( \{f_1, \cdots, f_s\} \subseteq F \). Let \( a \in U^n \). In the following we shall give some criteria for \( \exp(A, a) \) to be finitely generated (or equivalently \( M(A, a) \) is a finitely generated \( \mathbb{K} \)-algebra). Note first that \( F \) being an \( A \)-module, the quotient \( F/A \) is an \( A \)-module, which is also a \( \mathbb{K} \)-vector space. The same holds for \( F/M(A, a) \) for all \( a \in U^n \). With these notations we have the following.

Lemma 13 Let \( a \in U^n \) and suppose that \( \exp(A, a) \) is finitely generated. Let \( \{f_1, \cdots, f_r\} \) be an \( a \)-reduced canonical basis of \( A \). The map

\[
\theta : F/A \rightarrow F/M(A, a)
\]

defined by \( \theta(f) = R_a(f, \{f_1, \cdots, f_s\}) \) is an isomorphism of \( \mathbb{K} \)-vector spaces.

Proof. It is easy to verify that \( \theta \) is well defined and also that it is a linear map. Clearly \( \theta \) is surjective, and it follows from Proposition 10 that \( \operatorname{Ker}(\theta) = 0 \). This proves our assertion. ■

Let the notations be as above, and let \( a \in U^n \). Let \( i \in \{1, \cdots, n\} \), and define \( S_i = \exp_i(A, a) \subseteq \exp(A, a) \) to be the set of \( s \in \mathbb{N} \) such that \( x_i^s \in M(A, a) \). Clearly \( S_i \) is a semigroup of \( \mathbb{N} \). With these notations we have the following result which will be used in Section 3.

Proposition 14 Let \( a \in U^n \). With the notations of Proposition 15, If \( \operatorname{rk}_{\mathbb{K}} F/M(A, a) < +\infty \), then the following conditions hold.

1. For all \( i \in \{1, \cdots, n\} \), \( S_i \) is a numerical semigroup of \( \mathbb{N} \), i.e. \( \mathbb{N} \setminus S_i \) is a finite set.
2. \( S_1 \times \cdots \times S_n \subseteq \exp(A, a) \).
3. \( \sum_{i=1}^{n} e(S_i) \leq e(A, a) \leq \sum_{i=1}^{n} e(S_i) + \prod_{i=1}^{n} \gamma_i \), where \( e(S_i) \) (resp. \( \gamma_i \)) denotes the cardinality of the minimal set of generators of \( S_i \) (resp. its conductor).

Proof. 1. If \( \operatorname{rk}_{\mathbb{K}} F/M(A, a) < +\infty \) then \( \operatorname{Card}(\mathbb{N} \setminus \exp(A, a)) < +\infty \) (where \( \operatorname{Card} \) stands for the cardinality). But \( \operatorname{Card}(\mathbb{N} \setminus S_i) \leq \operatorname{Card}(\mathbb{N} \setminus \exp(A, a)) \) for all \( i \in \{1, \cdots, n\} \), hence \( \operatorname{Card}(\mathbb{N} \setminus S_i) < +\infty \).

2. Obvious.

3. Every minimal generator of \( S_i \) is also a minimal generator of \( \exp(A, a) \), and by 2., if \( \alpha \geq \prod_{i=1}^{n} \gamma_i \) pairwise coordinates, then \( x^\alpha \) is not a minimal generator of \( \exp(A, a) \). Moreover, if \( m = x_1^{s_1} \cdots x_n^{s_n} \) is a minimal generator of \( \exp(A, a) \) with \( s_1 \notin S_1 \), for example, then for all \( s \geq 0 \), \( x_1^{s_1+\gamma_1+s} \cdots x_n^{s_n} = x_1^{\gamma_1+s+m} \), and \( x_1^{\gamma_1+s} \in S_1 \). Hence at most \( \gamma_1 \) minimal generators of \( \exp(A, a) \) are of the form \( x_1^{s_1+\gamma_1+s} \cdots x_n^{s_n} \), \( s \geq 0 \). This finishes the proof. ■

As a corollary we get the following.

Proposition 15 With the notations above, if \( \operatorname{rk}_{\mathbb{K}} F/M(A, a) < +\infty \) for some \( a \in U^n \) then \( \operatorname{rk}_{\mathbb{K}} F/M(A, a) < +\infty \) for all \( a \in U^n \). In particular \( \exp(A, a) \) is finitely generated for all \( a \in U^n \).
Proof. This follows from Lemma 13 and Proposition 14. ■

Next we recall a finiteness criterion given in [13].

**Proposition 16** (see [13], Proposition 4.7. and 4.9.) Let \( A \) be as above and let \( a \in U^n \). If for all \( i \in \{1, \ldots, n\} \), there exists \( \alpha_i \in \mathbb{N} \setminus \{0\} \) such that \( x_i^{\alpha_i} \in M(A, a) \), then \( \exp(A, a) \) is finitely generated.

Proof. The hypothesis is equivalent to saying that \( F \) is integral over \( M(A, a) \) (see [13], 4.9.). The result follows by applying [6], Proposition 7.8. to the inequalities \( \mathbb{K} \subseteq M(A, a) \subseteq F \). ■

Let the notations be as above, and suppose that \( F \) is integral over \( M(A, a) \). For all \( i \in \{1, \ldots, n\} \), let \( \alpha_i \in \mathbb{N} \setminus \{0\} \) such that \( x_i^{\alpha_i} \in M(A, a) \), then the cone \( C = \{ \sum_{i=1}^{n} \lambda_i \alpha_i \mid \lambda_i \in \mathbb{R}_+ \} \) is nothing but \( \mathbb{R}_+^n \), and every element of \( \exp(A, a) \) is in this cone. Thinking of this approach, Proposition 16 can be adapted to a more general setting. Recall first the following lemma.

**Lemma 17** (see [7], for example) Let \( \alpha_1, \ldots, \alpha_s \) be a subset of \( \mathbb{N}^n \) and let \( C = \{ \sum_{i=1}^{n} \lambda_i \alpha_i \mid \lambda_i \in \mathbb{R}_+ \} \) be the cone generated by \( \alpha_1, \ldots, \alpha_s \). There exists a subset \( \{ \beta_1, \ldots, \beta_r \} \) of \( C \cap \mathbb{N}^n \) such that for all \( \beta \in C \cap \mathbb{N}^n \), there exist \( \lambda_1, \ldots, \lambda_s \in \mathbb{N} \), and \( j \in \{1, \ldots, r\} \), such that \( \beta = \sum_{i=1}^{s} \lambda_i \alpha_i + \beta_j \).

Proof. Let \( P = \{ \sum_{i=1}^{s} \lambda_i a_i \mid 0 \leq \lambda_i < 1 \text{ for all } 1 \leq i \leq s \} \), and let \( \{ \beta_1, \ldots, \beta_r \} = P \cap \mathbb{N}^n \). Let \( \beta \in C \cap \mathbb{N}^n \). We have

\[
\beta - \sum_{i=1}^{s} [\lambda_i] \alpha_i = \sum_{i=1}^{s} \{\lambda_i\} \alpha_i \in P
\]

where \([x]\) denotes the integer part of \( x \), and \( \{x\} \) its fractional part. As \( \sum_{i=1}^{s} \{\lambda_i\} \alpha_i \in P \cap \mathbb{N}^n \), this finishes the proof. ■

**Proposition 18** Let \( A = \mathbb{K}[f_1, \ldots, f_s] \) be as above and let \( a \in U^n \). For all \( i \in \{1, \ldots, s\} \), let \( M(f_i, a) = x^{\alpha_i} \). Let \( C = \{ \sum_{i=1}^{s} \lambda_i \alpha_i \mid \lambda_i \in \mathbb{R}_+ \} \) be the cone generated by \( \{\alpha_1, \ldots, \alpha_s\} \), and let \( F_C = \{ f \in F \mid \text{Supp}(f) \subseteq C \} \) be the ring of polynomials whose supports are in \( C \). If \( M(A, a) \subseteq F_C \), then \( \exp(A, a) \) is finitely generated.

Proof. Let the notations be as in Lemma 17. In particular \( \{\beta_1, \ldots, \beta_r\} = P \cap \mathbb{N}^n \). It follows from the proof of the same lemma that \( F_C = \mathbb{K}[x^{\beta_1}, \ldots, x^{\beta_r}, x^{\alpha_1}, \ldots, x^{\alpha_s}] \), and if \( B = \mathbb{K}[x^{\alpha_1}, \ldots, x^{\alpha_s}] \), then for all \( x^\beta \in F_C, x^\beta \in x^{\beta_i}B \) for some \( i \in \{1, \ldots, r\} \). In particular \( F_C \) is a finite \( M(A, a) \)-module, since it is generated by \( \{x^{\beta_1}, \ldots, x^{\beta_r}\} \), hence (by [6], Proposition 5.1. and Corollary 5.2., for example) it is integral over \( M(A, a) \), and the result follows by applying [6], Proposition 7.8. to the inequalities \( \mathbb{K} \subseteq M(A, a) \subseteq F_C \). ■

The following lemma gives a criteria for \( \exp(A, a) \), \( a \in U^n \), to be in a fixed cone.

**Lemma 19** Let \( A = \mathbb{K}[f_1, \ldots, f_s] \) be as above and let \( a \in U^n \). For all \( i \in \{1, \ldots, s\} \), let \( M(f_i, a) = x^{\alpha_i} \). Let \( C \) be the cone generated by \( \{x^{\alpha_1}, \ldots, x^{\alpha_s}\} \). If \( \text{Supp}(f_i, a) \subseteq C \) for all \( i \in \{1, \ldots, s\} \) then \( \exp(A, a) \subseteq C \). In particular, by Proposition 18, \( \exp(A, a) \) is finitely generated.

Proof. Let \( f \in A \), and write \( f = \sum_{\alpha} c_\alpha f_1^{\alpha_1} \cdots f_s^{\alpha_s} \). For all \( \alpha \), \( \text{Supp}(f_1^{\alpha_1} \cdots f_s^{\alpha_s}) \subseteq C \), then \( \exp(f, a) \subseteq C \). This finishes the proof. ■

In the following we shall give some relevant examples (the well ordering here is the lexicographical order).

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Example 20  Let $A = \mathbb{K}[x+y,xy]$ and let $a = (2,1)$, then $M(x+y,a) = x$, and $M(xy,a) = xy$, hence $\{x+y,xy\}$ is the $a$-reduced canonical basis of $A$. If $C$ is the cone generated by $(1,0),(1,1)$ then $\text{Supp}(x+y) \notin C$. This proves that the converse of Lemma 19 is not true.

Example 21  (see [13]) Let $A = \mathbb{K}[x+y,xy,xy^2]$ and let $a = (2,1)$. We have $M(x+y,a) = x, M(xy,a) = xy,$ and $M(xy^2,a) = xy^2$. The kernel of the map:

$$\phi : \mathbb{K}[X,Y,Z] \to \mathbb{K}[x,y], \phi(X) = x, \phi(Y) = xy, \phi(Z) = xy^2$$

is generated by $S_1 = XZ - Y^2$. We have $S = (x+y)xy^2 - x^2y^2 = -xy^3 - R_a(-xy^3, \{x+y,xy,xy^2\}$). Hence $xy^3$ is a new element of the a-canonical basis of $A$. If we restart with the representation $A = \mathbb{K}[x+y,xy,xy^2,xy^3]$, then a new element, $xy^4$, will be added to the a-canonical basis of $A$. In fact, $xy^3$ is an element of the minimal reduced a-canonical basis of $A$ for all $n \geq 1$. In particular the a-canonical basis of $A$ is infinite. In fact, we can verify that $\text{exp}(A,a)$ is not finitely generated for all $a \in U^n$. Note that if $M(x+y,a) = x$ (resp. $M(x+y,a) = y$) then $(0,1)$ (resp. $(1,0)$) is not in the cone $C$ generated by $(1,0),(1,1),(1,2)$ (resp. $(0,1),(1,1),(1,2)$).

Example 22  Let $A = \mathbb{K}[x,xy+y^2,xy,xy^2]$. If $a = (1,2)$, then $M(x,a) = x, M(xy+y^2,a) = y^2$, and $M(xy^2,a) = x^2y$. The cone $C$ generated by $(1,0),(0,2),(2,1)$ is $\mathbb{R}_+^2$, and the hypotheses of Lemma 19 are satisfied, hence $\text{exp}(A,a)$ is finitely generated. We can verify that $\{x,xy+y^2,xy^2\}$ is an a-reduced canonical basis. In particular $\text{exp}(A,a) = \langle (1,0),(0,2),(2,0)\rangle$.

Let $a = (2,1)$. We have $M(x,a) = x, M(xy+y^2,a) = xy$, and $M(xy^2,a) = x^2y$. We can verify that $xy^n$ is an element of the minimal reduced a-canonical basis of $A$ for all $n \geq 4$. In particular the a-canonical basis of $A$ is infinite. Note that in this case, the hypotheses of Lemma 19 are not satisfied since $\text{Supp}(xy+y^2)$ is not in the cone generated by $(1,0),(1,1),(2,1)$.

Example 23  Let $A = \mathbb{K}[x,xy+y^2,xy,xy^2]$. For all $a \in U^n$, $M(x,a) = x, M(xy,a) = y^3, M(xy^2,a) = x^2y$, and $M(xy+y^2,a)$ is either $xy$ or $y^2$, depending on $(1,0) \succ a, (0,1)$ or $(1,0) \succ a, (1,0)$.

- If $(0,1) \succ a, (1,0)$, then we can verify that $\{x,xy+y^2,xy^2\}$ is an a-canonical basis of $A$. In particular $\text{exp}(A,a) = \langle (1,0),(0,2),(1,3),(2,1) \rangle$, and $\mathbb{N}^2 \setminus \text{exp}(A,a) = \{(0,1), (1,1)\}$.

- If $(1,0) \succ a, (0,1)$, then we can verify that $\{x,xy+y^2,xy^2,xy^3,xy^4,xy^5\}$ is an a-canonical basis of $A$. In particular $\text{exp}(A,a) = \langle (1,0),(1,1),(1,2),(0,3),(0,4),(0,5) \rangle$, and $\mathbb{N}^2 \setminus \text{exp}(A,a) = \{(0,1),(0,2)\}$.

Example 24  1. (see [5]) Suppose that $n = 1$, i.e. $A = \mathbb{K}[f_1(t),\ldots,f_s(t)] \subseteq \mathbb{K}[t]$. The parametrization $(X_1 = f_1(t),\ldots,X_s = f_s(t))$ represents the expansion of a curve in $\mathbb{K}^s$ near the origin. In this case, $\text{exp}(A,a)$ does not depend on $a \in U^n$. If $s = 2$ and $\mathbb{N} \setminus \text{exp}(A,a) < +\infty$, then $\text{exp}(A,a)$ is a free numerical semigroup (see [4] for the definition), and the arithmetic of this semigroup contains a lot of information about the singularity of the curve at the origin.

2. Let $f(X_1,\ldots,X_n,Y) \in \mathbb{K}[X_1,\ldots,X_n,Y]$ and suppose that $f$ has a parametrization of the form $X_1 = A^1,\ldots,X_n = A^n, Y = X^1,\ldots,X^n \in \mathbb{K}[t_1,\ldots,t_n]$ (for instance, this is true if $f$ is a quasi-ordinary polynomial, i.e. the $Y$-derivative $f_Y$ of $f$ with its $Y$-derivative $f_Y$ is of the form $X_1^{n_1} \cdots X_n^{n_s}(c + \phi(X_1,\ldots,X_s))$ with $c \in \mathbb{K}^*$ and $\phi(0,\ldots,0) = 0$). We have $\mathbb{K}[X_1,\ldots,X_n][Y] \simeq \mathbb{K}[t_1,\ldots,t_n]$. In this case, $(e_1,0,\ldots,0),\ldots,(0,\ldots,0,e_n)$ belong to $\text{exp}(A,a)$ for all $a \in U^n$. Moreover, $\text{exp}(A,a)$ is a free finitely generated affine semigroup in the sense of [3].
3 A finiteness Theorem

Let the notations be as in Section 1. In particular \( A = \mathbb{K}[f_1, \ldots, f_s] \) with \( \{f_1, \ldots, f_s\} \subseteq F \). The aim of this section is to prove, under some additional hypotheses, that the set \( M(A) = \{M(A, a) \mid a \in U^n\} \) (resp. the set \( I(A, a) = \{\text{in}(A, a), a \in U^n\} \)) is finite. Recall that, given a nonzero element \( f \in F \) and \( a \in U^n \), we have \( M(f, a) = c_{\text{exp}(f, a)} \alpha^{\text{exp}(f, a)} \). In particular, for all \( a \in U^n \), we have \( M(A, a) = \mathbb{K}[M(f_1, a), \ldots, M(f_s, a)] \) if and only if \( \exp(A, a) = (\exp(f_1, a), \ldots, \exp(f_s, a)) \).

We shall first consider the case when \( A = \mathbb{K}[f] \), then we prove some preliminary results which will also be used later in the paper. We start with the following definition:

**Definition 25** Let \( f = \sum_{\alpha} c_{\alpha} x^\alpha \) be a nonzero element of \( \mathbb{K}[x_1, \ldots, x_n] \) and let \( E = \bigcup_{\alpha \in \text{Supp}(f)} (\alpha + \mathbb{N}^n) \). We have \( E + \mathbb{N}^n \subseteq E \), and consequently, by Dickson’s Lemma, there exists a unique finite set of \( E \), say \( \{\alpha_1, \ldots, \alpha_r\} \), such that \( E = \bigcup_{i=1}^r \alpha_i + \mathbb{N}^n \). We set \( N(f) = \{\alpha_1, \ldots, \alpha_r\} \), and \( f_N = \sum_{i=1}^r c_{\alpha_i} x^\alpha \).

**Lemma 26** Let \( f \) be a nonzero element of \( \mathbb{K}[x_1, \ldots, x_n] \). The set \( \{M(f, a), a \in U^n\} \) (resp. \( \{\text{in}(f, a), a \in U^n\} \)) is finite.

Proof. Write \( f = \sum_{\alpha} c_{\alpha} x^\alpha \) and let the notations be as in Definition 25. Given \( a \in U^n \), it follows from the definition of \( N(f) \) that for all \( \alpha \in \text{Supp}(f) \), there exists \( \alpha_k \in E \) and \( \beta \in \mathbb{N}^n \) such that \( \alpha = \alpha_k + \beta \). In particular \( \nu(\alpha, a) \geq \nu(\alpha_k, a) \) with inequality if \( \alpha \neq \alpha_k \). Whence \( \text{in}(f, a) = \text{in}(f_N, a) \) and \( \exp(f, a) = \exp(f_N, a) \). As \( f_N \) is a polynomial, the result follows immediately.

**Lemma 27** Let \( f = \sum_{\alpha} c_{\alpha} x^\alpha \) be a nonzero element of \( \mathbb{K}[x_1, \ldots, x_n] \) and let \( a \in U^n \). There exists \( b \in U^n \) such that \( \text{in}(f, a) = \text{in}(f, b) \) and \( \exp(f, a) = \exp(f, b) \).

Proof. Let the notations be as in Definition 25. In particular \( f_N = \sum_{i=1}^r c_{\alpha_i} x^\alpha \). Let \( S = \text{Supp}(\text{in}(f, a)) \) and let \( d \) be the greatest number of elements of \( S \) which are linearly independent, and assume, without loss of generality, that \( S = \{\alpha_1, \ldots, \alpha_d, \ldots, \alpha_l\} \), where \( \alpha_1, \ldots, \alpha_d \) are linearly independent. We have \( \nu = \nu(f, a) = \nu(\alpha_1, a) = \cdots = \nu(\alpha_l, a) < \nu(\alpha_k, a) \) for all \( k \geq l + 1 \). Let \( 0 < t < \min_{k \geq l+1} (\nu(\alpha_k, a) - \nu) \). We need to prove the existence of \( \epsilon \in \mathbb{R}^n_+ \) such that the following conditions hold.

1. \( a + \epsilon \in U^n_+ \).
2. \( \epsilon \cdot (\alpha_1 - \alpha_j) = 0 \) for all \( j \in \{2, \ldots, d\} \) (hence \( \nu(\alpha_1, a + \epsilon) = \cdots = \nu(\alpha_l, a + \epsilon) \)).
3. \( 0 < \epsilon \cdot (\alpha_1 - \alpha_k) < t \) for all \( k \geq l + 1 \) (hence \( \nu(\alpha_1, a + \epsilon) < \nu(\alpha_k, a + \epsilon) \) for all \( k \geq l + 1 \)).

If \( d = 1 \), then \( d = l = 1 \), and \( \text{in}(f, a) = c_{\alpha_1} x^{\alpha_1} \). In this case, the existence of such an \( \epsilon \) is obvious. Suppose that \( 0 < d \leq n \), and let \( L : \mathbb{R}^n \to \mathbb{R}^{d-1}, L(X) = (X \cdot (\alpha_1 - \alpha_2), \ldots, X \cdot (\alpha_1 - \alpha_d)) \). As the dimension of \( \text{Ker}(L) \) is \( > 1 \), there exists \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \text{Ker}(L) \cap \mathbb{R}^n_+ \), relatively small, such that \( b = a + \epsilon \in U^n_+ \), and we can choose \( \epsilon \) such that condition 3. is satisfied. It follows that \( \text{in}(f, a) = \text{in}(f, b) \) and \( \exp(f, a) = \exp(f, b) \). This proves our assertion.

Next we prove that if \( M(A, a) \neq M(A, b) \), then \( M(A, a) \not\subseteq M(A, b) \).

**Lemma 28** Let \( a \neq b \) be two elements of \( U^n \). Assume that \( M(A, a) \) is finitely generated and let \( \{g_1, \ldots, g_r\} \) be an \( a \)-reduced canonical basis of \( A \). If \( M(A, a) = M(A, b) \), then \( \{g_1, \ldots, g_r\} \) is also a \( b \)-reduced canonical basis of \( A \).
Proof. Let \( i \in \{1, \ldots , r\} \) and write \( g_i = M(g_i, a) + \sum \gamma c_\gamma x_\gamma \) where for all \( \beta \), if \( c_\beta \neq 0 \), then \( x_\beta \notin M(A, a) \). Since \( M(A, a) = M(A, b) \), \( x_\beta \notin M(A, b) \) for all \( \beta \) such that \( c_\beta \neq 0 \). This implies that \( M(g_i, a) = M(g_i, b) \). Hence \( \{g_1, \ldots , g_r\} \) is a \( b \)-canonical basis of \( A \), and the same argument shows that this basis is also reduced. \( \blacksquare \)

**Lemma 29** Let \( a \neq b \) be two elements of \( U^n \) and assume that \( M(A, b) \) is finitely generated. If \( M(A, a) \neq M(A, b) \), then \( M(A, a) \not\subset M(A, b) \).

Proof. Assume that \( M(A, a) \subset M(A, b) \) and let \( \{g_1, \ldots , g_r\} \) be a \( b \)-reduced canonical basis of \( A \). By hypothesis, there is \( 1 \leq i \leq r \) such that \( M(g_i, b) \not\subset M(A, a) \). Write \( g_i = M(g_i, b) + \sum \gamma c_\gamma x_\gamma \). For all \( \beta \), if \( c_\beta \neq 0 \), then \( x_\beta \notin M(A, b) \), hence \( x_\beta \notin M(A, a) \). This implies that for all \( \alpha \in \text{Supp}(g_i) \), \( \alpha \notin \exp(A, a) \). This is a contradiction because \( g_i \in A \). \( \blacksquare \)

In the following we prove that \( \{M(A, a) \mid a \in U^n\} = \{M(A, a) \mid a \in U^n_1\} \).

**Lemma 30** If \( a \in U^n \setminus U^n_1 \) and \( M(A, a) \) is finitely generated, then there exists \( b \in U^n_1 \) such that \( M(A, a) = M(A, b) \).

Proof. Let \( \{g_1, \ldots , g_r\} \) be an \( a \)-reduced canonical basis of \( A \). As \( r < +\infty \), we can find, using Lemma 27, \( b \in U^n_1 \) such that for all \( i \in \{1, \ldots , r\} \), \( M(g_i, a) = M(g_i, b) \). It follows that \( M(A, a) \subseteq M(A, b) \), whence, by Lemma 29, \( M(A, a) = M(A, b) \). \( \blacksquare \)

The following Lemma generalizes to initial forms the result of Lemma 29. We first recall the following.

**Definition 31** Let \( B \) be a subalgebra of \( F \) and let \( a \in U^n \). We say that \( B \) is an \( a \)-homogeneous algebra if \( B = \mathbb{K}[G_1, \ldots , G_k, \ldots] \) where \( G_i \) is \( a \)-homogeneous for all \( i \geq 1 \). With the notations above, given \( a \in U^n \), \( \text{in}(A, a) \) is an \( a \)-homogeneous algebra.

**Lemma 32** Let \( a \in U^n_1 \) and \( B = \mathbb{K}[G_1, \ldots , G_s] \) be an \( a \)-homogeneous algebra, where we suppose that \( G_i \) is \( a \)-homogeneous for all \( i \in \{1, \ldots , s\} \). Let \( F \) be a non zero element of \( F \) and let \( F = \sum_i F_{d_i} d_1 < d_2 < \cdots \) be the decomposition of \( F \) into \( a \)-homogeneous components. If \( F \in B \) then \( F_{d_i} \in B \) for all \( i \geq 1 \).

Proof. Write \( F = \sum c_\alpha \prod_{i=1}^s G_i^{a_\alpha} \) and let \( d = \min_{\alpha \nu} (\prod_{i=1}^s G_i, a) \). Let also \( G = \sum_{d=\nu} (\prod_{i=1}^s G_i, a) c_\alpha \prod_{i=1}^s G_i^{a_\alpha} \). We have \( d_1 \geq d \) and \( d_1 > d \) if and only if \( G = 0 \). In this case we remove \( G \) and we restart with \( F - G \). Finally we may assume that \( d = d_1 \), in particular \( F_{d_1} = G \in B \). Now we restart with \( F - F_{d_1} \). Our assertion follows by an easy induction argument. \( \blacksquare \)

**Lemma 33** Let \( a \neq b \) be two elements of \( U^n \) and assume that \( M(A, a) \) is finitely generated. If \( \text{in}(A, a) \neq \text{in}(A, b) \), then \( \text{in}(A, a) \not\subset \text{in}(A, b) \).

Proof. Suppose that \( \text{in}(A, a) \subseteq \text{in}(A, b) \), and let us prove that \( \text{in}(A, a) = \text{in}(A, b) \). As \( \exp(A, c) = \exp(\text{in}(A, c)) \) for all \( c \in U^n \), it follows that \( \exp(A, a) \subseteq \exp(A, b) \), hence, by Lemma 29, \( \exp(A, a) = \exp(A, b) \). Let \( \{g_1, \ldots , g_r\} \) be an \( a \)-reduced canonical basis of \( A \). Then \( \{g_1, \ldots , g_r\} \) is also a \( b \)-reduced canonical basis of \( A \). In particular \( \text{in}(A, a) = \mathbb{K}[\text{in}(g_1, a), \ldots , \text{in}(g_r, a)] \) (resp. \( \text{in}(A, b) = \mathbb{K}[\text{in}(g_1, b), \ldots , \text{in}(g_r, b)] \)). Let us prove that for all \( i \in \{1, \ldots , r\} \), \( \text{in}(g_i, a) = \text{in}(g_i, b) \). Let \( G = \text{in}(g_i, a) \), and let \( G = G_{d_{t}} + \cdots + G_{d_1}, d_1 < \cdots < d_t \), be the decomposition of \( G \) into \( b \)-homogeneous elements. If \( t > 1 \) then \( G_{d_k} \in \text{in}(A, b) \) for all \( k \in \{2, \ldots , t\} \), hence \( \exp(G_{d_k}, b) \in \exp(A, b) \). But \( \text{Supp}(G_{d_2} + \cdots + G_{d_t}) \cap \exp(A, b) = \emptyset \). This is a contradiction. Hence \( t = 1 \) and \( \text{in}(g_i, a) = G_{d_1} = \text{in}(g_i, b) \). This finishes the proof. \( \blacksquare \)

We can now state and prove the finiteness theorem. We shall first prove the following proposition.
**Proposition 34** Let \( \{f_1, \cdots, f_s\} \) be a set of nonzero elements of \( F \) and let \( A = \mathbb{K}[f_1, \cdots, f_s] \). Assume that for all \( a \in U^n \), \( M(A, a) \) is finitely generated. If \( M(A) = \{M(A, a)|a \in U^n\} \) is infinite, then for all \( n \in \mathbb{N} \setminus \{0\} \), there exists an infinite set \( E_n \) such that for all \( a \in E_n \), \( e(A, a) \geq n \).

Proof. By Lemma 26, the hypothesis implies that \( A \) cannot be generated by only one element. On the other hand, by the same lemma there is an infinite set \( E_1 = \{a_1, a_2, \cdots \} \) in \( U^n \) such that for all \( 1 \leq k \leq s \) and for all \( a \in E_1 \), \( M(f_k, a) = m_k \), where \( m_k \) is a nonzero monomial of \( f_k \). Let \( J_1 = \mathbb{K}[m_1, \cdots, m_s] \); \( J_1 \subseteq M(A, a) \) for all \( a \in E_1 \). Now \( J_1 \neq M(A, a_1) \) (otherwise, \( M(A, a_1) \subseteq M(A, a_2) \), for example. This contradicts Lemme 29). Let \( m \in M(A, a_1) \setminus J_1 \) and let \( f \in A \) such that \( M(f, a_1) = m \). We set \( f_{s+1} = R_{a_1}(f, \{f_1, \cdots, f_s\}) \). Then for all \( \beta \in \text{Supp}(f_{s+1}), \beta \notin J_1 \), and by lemma 26, there is monomial \( m_{s+1} \) of \( f_{s+1} \) and an infinite subset \( E_2 \subseteq E_1 \) such that for all \( a \in E_2, M(f_{s+1}, a) = m_{s+1} \). Let \( J_2 = \mathbb{K}[m_1, \cdots, m_s, m_{s+1}] \); \( J_1 \subseteq J_2 \). The same process applied to \( \{f_1, \cdots, f_{s+1}\} \), \( J_2 \) and \( E_2 \) implies the existence of \( m_{s+2} \notin J_2 \), \( f_{s+2} \in A \), and an infinite subset \( E_3 \subseteq E_2 \) such that for all \( a \in E_3, M(f_{s+2}, a) = m_{s+2} \). We get this way a strictly increasing sequence \( J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots \subseteq J_n \) and for all \( i \), and for all \( a \in E_i \), \( J_i \subseteq M(A, a) \), in particular \( M(A, a) \) is generated by at least \( i \) elements. This proves our assertion. ■

Next we prove that if \( M(A) \) is finite then so is \( I(A) \).

**Lemma 35** Let the notations be as in Proposition 34. If \( M(A) \) is a finite set, then \( I(A) \) is also a finite set.

Proof. Suppose that \( I(A) \) is an infinite set. Then there exists an infinite subset \( E_1 = \{a_1, a_2, \cdots \} \) such that for all \( k \geq 1 \), \( M(A, a_k) = M(A, a_1) \). Let \( \{g_1, \cdots, g_s\} \) be an \( a_1 \)-canonical basis of \( A \). Then \( \{g_1, \cdots, g_s\} \) is an \( a_k \)-canonical basis of \( A \) for all \( k \geq 2 \). In particular \( \text{in}(A, a_k) = \mathbb{K}[\text{in}(g_1, a_k), \cdots, \text{in}(g_s, a_k)] \) for all \( k \geq 1 \). This contradicts Lemma 26. ■

**Theorem 36** Let \( \{f_1, \cdots, f_s\} \) be a set of nonzero elements of \( F \) and let \( A = \mathbb{K}[f_1, \cdots, f_s] \). Assume that there exists \( n_0 \in \mathbb{N} \) such that \( e(A, a) \leq n_0 \) for all \( a \in U^n \). The set \( M(A) = \{M(A, a)|a \in U^n\} \) (hence \( \text{exp}(A) = \{\text{exp}(A, a)|a \in U^n\} \)) is finite. In particular the set \( I(A) = \{\text{in}(A, a)|a \in U^n\} \) is finite.

Proof. By Lemma 35, we only need to prove that \( M(A) \) is a finite set, but this follows immediately from Proposition 34. ■

**Definition 37** With the notations above, suppose that \( M(A) \) is a finite set. The set \( \{g_1, \cdots, g_r\} \) of \( A \) which is an \( a \)-canonical basis of \( A \) for all \( a \in U^n \), is called the universal canonical basis of \( A \).

**Remark 38** 1. We do not have a proof for Theorems 36 for general subalgebras. We think however that the result of this theorem is true.

2. If \( A = \mathbb{K}[f_1, \cdots, f_s] \) is a subalgebra of \( P = \mathbb{K}[x_1, \cdots, x_n] \), then Theorem 36 remains valid when we vary \( a \in \mathbb{R}^n_+ \) (in [9] the author uses a similar argument as in Proposition 34, however, the argument does not seem to suffice without an extra hypothesis). Note that in this case, \( \text{in}(f, a) \) is a polynomial for all \( a \in \mathbb{R}^n_+ \) and for all \( f \in P \).

As a corollary of Proposition 15 we get the following.

**Lemma 39** With the notations above, if \( d = \text{rank}_K F/M(A, a) < +\infty \) for some \( a \in U^n \), then there exists \( n_0 \in \mathbb{N} \) such that \( e(A, a) \leq n_0 \) for all \( a \in U^n \). In particular \( M(A) \) is a finite set.
Proof. Let \( \mathbb{F}/\mathbb{M}(\mathbb{A}, a) = d \) for all \( a \in U^n \). Let the notations be as in Proposition 14. For all \( i \in \{1, \cdots, n\} \), \( \text{Card}(\mathbb{N} \setminus S_i) \leq d \), hence \( \gamma_i \), the conductor of \( S_i \), is also bounded, say by \( D \in \mathbb{N} \). On the other hand, if \( m(S_i) \) denotes the smallest nonzero element of \( S_i \), then \( e(S_i) \leq m(S_i) \leq \gamma_i \) (see [4], for example). By condition 3. of Proposition 14, \( e(\mathbb{A}, a) \leq \sum_{i=1}^{n} e(S_i) + D^n \leq nD + D^n \). This proves the first assertion. The last assertion results from Theorem 36. \( \blacksquare \)

4 The Newton fan

Let the notations and hypotheses be as in Sections 1 to 3. In particular, \( \{f_1, \cdots, f_s\} \) is a set of nonzero elements of \( \mathbb{F} \) and \( \mathbb{A} = \mathbb{K}[f_1, \cdots, f_s] \). Assume that for all \( a \in U^n \), \( \exp(\mathbb{A}, a) \) is finitely generated, and that \( \mathbb{M}(\mathbb{A}) = \{\mathbb{M}(\mathbb{A}, a) \mid a \in U^n\} \) is a finite set. We have the following.

**Theorem 40** There exists a partition \( \mathcal{P} \) of \( U^n \) into convex rational polyhedral cones such that for all \( \sigma \in \mathcal{P} \), \( \exp(\mathbb{A}, a) \) and \( \text{in}(\mathbb{A}, a) \) do not depend on \( a \in \sigma \).

In order to prove Theorem 40 we start by fixing some notations. Let \( S \) be a finitely generated affine semigroup of \( \mathbb{N}^n \) and let

\[
E_S = \{a \in U^n \mid \exp(\mathbb{A}, a) = S\}
\]

Let \( a \in E_S \) and let \( \{g_1, \cdots, g_r\} \) be the \( a \)-reduced canonical basis of \( \mathbb{A} \). By Lemma 28, \( \{g_1, \cdots, g_r\} \) is also the \( b \)-reduced canonical basis of \( \mathbb{A} \) for all \( b \in E_S \). Denote by \( \sim \) the equivalence relation on \( U^n \) defined from \( \{g_1, \cdots, g_r\} \) by

\[
a \sim b \iff \text{in}(g_i, a) = \text{in}(g_i, b) \quad \text{for all} \quad i \in \{1, \cdots, r\},
\]

**Proposition 41** \( \sim \) defines on \( U^n \) a finite partition, denoted \( \mathcal{P}_S \), into convex rational polyhedral cones and \( E_S \) is a union of a part of these cones.

Proof. Let \( c, d \in U^n \) such that \( c \sim d \) and let \( e \in [c, d] \). Let \( \theta \in [0, 1] \) such that \( e = \theta c + (1 - \theta)d \). We have \( e \in U^n \), and \( \text{in}(g(e), e) = \text{in}(g(e), c) = \text{in}(g(e), d) \) by an immediate verification. Moreover, \( c \sim t \cdot c \) for all \( c \in U^n \) and \( t > 0 \). Therefore the equivalence classes are convex rational polyhedral cones (the rationality results from Lemma 30). On the other hand, if \( c \sim d \) and \( c \in E_S \), then \( d \in E_S \). This proves that \( E_S \) is a union of classes of \( \sim \), the number of classes being finite by Theorem 36. \( \blacksquare \)

**Proof of Theorem 40** We define \( \mathcal{P} \) in the following way: for each finitely generated affine semigroup \( S \) we consider the restriction \( \mathcal{P}_S \) on \( E_S \) of the above partition. Then \( \mathcal{P} \) is the finite union of the \( \mathcal{P}_S \)'s. On each cone of the partition, \( \text{in}(\mathbb{A}, a) \) and \( \exp(\mathbb{A}, a) \) are fixed. Conversely assume that \( \text{in}(\mathbb{A}, a) \) and \( \exp(\mathbb{A}, a) \) are fixed and let \( b \) such that \( \text{in}(\mathbb{A}, a) = \text{in}(\mathbb{A}, b) \). By Lemma 33, the \( a \)-reduced canonical basis \( \{g_1, \cdots, g_r\} \) of \( \mathbb{A} \) is also the \( b \)-reduced canonical basis of \( \mathbb{A} \). Moreover, \( \text{in}(g_i, a) = \text{in}(g_i, b) \) and \( \exp(g_i, a) = \exp(g_i, b) \) for all \( i \in \{1, \cdots, r\} \). This ends the proof of the theorem except for the convexity of \( E_S \) proved below.

**Lemma 42** \( E_S \) is a convex set: if \( a \neq b \in E_S \), then \( [a, b] \subseteq E_S \)

Proof. Let \( a, b \in E_S \) and let \( \lambda \in [0, 1] \). Let \( \{g_1, \cdots, g_r\} \) be the \( a \) (whence the \( b \)) reduced canonical basis of \( \mathbb{A} \). Let \( i \in \{1, \cdots, r\} \) and set \( M = \text{in}(g_i, a) \). Write \( M = M_1 + \cdots + M_t \) where \( M_k \) is \( b \)-homogeneous for all \( k \in \{1, \cdots, t\} \) and \( \nu(M_1, b) > \nu(M_k, b) \) for all \( k \in \{2, \cdots, t\} \). We have
\[ \nu(g_i, a) = \nu(M_1, a) = \nu(M_k, a) \text{ and } \nu(M_1, b) < \nu(M_k, b) \text{ for all } k \in \{2, \ldots, t\}. \]

This implies that \( \nu(g_i, \theta a + (1 - \theta)b) = \nu(M_1, \theta a + (1 - \theta)b) < \nu(M_k, \theta a + (1 - \theta)b) \) for all \( k \in \{2, \ldots, t\} \), hence \( \exp(g_i, \theta a + (1 - \theta)b) = \exp(g_i, a) = \exp(g_i, b) \). In particular \( \exp(A, a) \subseteq \exp(A, \theta a + (1 - \theta)b) \).

By Lemma 29 we get the equality. This proves that \( \exp(A, \theta a + (1 - \theta)b) = S. \) ■

**Definition 43** \( \mathcal{P} \) is called the standard fan of \( A \).

Next we shall characterize open cones with maximal dimension. Let \( f \) be a nonzero element of \( K[x_1, \ldots, x_n] \) and let \( a \in U^n \). We say that \( \text{in}(f, a) \) is multihomogeneous if \( \text{in}(f, a) \) is \( b \)-homogeneous for all \( b \in U^n \). This is equivalent to saying that \( \text{in}(f, a) \) is a monomial.

**Definition 44** Let \( a \in U^n \). We say that \( \text{in}(A, a) \) is a multihomogeneous algebra if it is generated by multihomogeneous elements. Note that, by Lemma 32, If \( g \in \text{in}(A, a) \), then every monomial of \( g \) is also in \( \text{in}(A, a) \).

**Lemma 45** Let \( a \in U^n \) and let \( \{g_1, \ldots, g_r\} \) be the \( a \)-reduced canonical basis of \( A \). The ideal \( \text{in}(A, a) \) is a multihomogeneous algebra if and only if \( \text{in}(g_i, a) \) is a monomial for all \( i \in \{1, \ldots, r\} \).

Proof. We only need to prove the if part. Let \( i \in \{1, \ldots, r\} \) and write \( \text{in}(g_i, a) = M_1 + \cdots + M_t \) with \( \exp(g_i, a) = \exp(M_1, a) \). Assume that \( t > 1 \). Since \( \text{in}(A, a) \) is multihomogeneous, we have \( M_i \in \text{in}(A, a) \) for all \( i \in \{2, \ldots, t\} \). But \( \{g_1, \ldots, g_r\} \) is reduced. This is a contradiction. Hence \( t = 1 \) and \( \text{in}(g_i, a) \) is a monomial. ■

**Proposition 46** The set of \( a \in U^n \) for which \( \text{in}(A, a) \) is multihomogeneous defines the open cones of dimension \( n \) of \( \mathcal{P} \).

Proof. Let \( a \in U^n \) and suppose that \( \text{in}(A, a) \) is multihomogeneous. Let \( \sigma \) be the cone corresponding to \( a \) and let \( \{g_1, \ldots, g_r\} \) be the \( a \)-minimal reduced canonical basis of \( A \). For all \( i \in \{1, \ldots, r\} \), \( \text{in}(g_i, a) \) is a monomial, hence there exists \( \epsilon > 0 \) such that \( \text{in}(g_i, b) = \text{in}(g_i, a) \) for all \( b \in B(a, \epsilon) \) (where \( B(a, \epsilon) \) is the ball centered at \( a \) of ray \( \epsilon \)). This proves, by Lemma 29 and Lemma 33, that \( \{g_1, \ldots, g_r\} \) is also the \( b \)-minimal reduced canonical basis of \( A \) for all \( b \in B(a, \epsilon) \). Hence \( B(a, \epsilon) \subseteq \sigma \).

Conversely, if \( a \) is in an open cone \( \sigma \) of dimension \( n \) of \( \mathcal{P} \), then for all \( b \) in a neighbourhood of \( a \), the \( a \)-minimal reduced canonical base \( \{g_1, \ldots, g_r\} \) of \( A \) is also the \( b \)-minimal reduced canonical basis of \( A \). This implies that \( \text{in}(g_i, a) \) is a monomial. ■

**Example 47** Let \( A \) be as in Example 23. Then \( A = \mathbb{K}[x, xy + y^2, y^3, x^2y] \), and \( \exp(A, a) \) is either \langle (1,0), (1,1), (1,2), (0,3), (0,4), (0,5) \rangle \) or \langle (1,0), (0,2), (0,3), (2,1) \rangle \), depending on \( (1,0) > a \) \((0,1) > a \). The fan associated with \( A \) is \( C_1 \cup C_2 \cup C_3 \) where \( C_1 \) is the open cone generated by \( (1,0), (1,1) \) (with \( \text{in}(A, a) = M(A, a) = \mathbb{K}[x, xy + y^2, y^3, y^4, y^5] \) for all \( a \in C_1 \)), \( C_2 \) is the open cone generated by \( (0,1), (1,1) \) (with \( \text{in}(A, a) = M(A, a) = \mathbb{K}[x, y^2, y^3, x^2y] \) for all \( a \in C_2 \)), and \( C_3 \) is the line generated by \( (1,1) \) (with \( \text{in}(A, a) = \mathbb{K}[x, xy + y^2, xy^2, y^3, y^4, y^5] \) and \( M(A, a) = \mathbb{K}[x, xy, xy^2, y^3, y^4, y^5] \) for all \( a \in C_3 \)).

**References**


