UNSTABLE K-MODULES AND THE NILPOTENT FILTRATION

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Abstract. The nilpotent filtration of the category of unstable modules gives rise to the invariants d_0, d_1 of Noetherian unstable algebras and of finitely generated modules in unstable modules over such algebras.

This paper considers the effect of the induction functor $L \otimes_K -$ associated to a flat K-algebra L in unstable algebras on the invariant d_0. The behaviour is analysed by using invariants from commutative algebra, namely the transcendence degree and the depth.

These results are applied to examples from invariant theory, motivated by the work of Henn, Lannes and Schwartz on the invariants d_0, d_1 for group cohomology.

Key words: Steenrod algebra – unstable module – unstable algebra – module – nilpotent filtration – transcendence degree – depth – group cohomology – invariant theory

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Contents

1. Introduction

A graded commutative algebra over a prime field $\mathbb{F}_p$ can be studied by passage to the associated algebraic variety. The presence of an action of the Steenrod algebra which defines an object of the category $\mathscr{K}$ of unstable algebras over the Steenrod algebra gives rise to structure which recovers information on the nilpotency in the algebra; this can be analysed by using the nilpotent filtration of the category $\mathscr{U}$ of unstable modules over the Steenrod algebra, introduced by Schwartz and studied in the work of Henn, Lannes and Schwartz [11, 12].

The nilpotent filtration of the abelian category $\mathscr{U}$ is a descending filtration by thick, localizing subcategories $\mathscr{N}il_n$, $n \in \mathbb{N}$. In applying the theory to an unstable algebra K, one is led naturally to consider modules over K in unstable modules; these objects form an abelian category, K–$\mathscr{U}$. If the underlying algebra of K is finitely generated then the full subcategory of finitely generated modules, K_{fg}–$\mathscr{U}$, is abelian.

The nilpotent filtration leads to the definition of invariants $d_0, d_1 : \text{Obj } \mathscr{U} \rightarrow \mathbb{N}\cup\{\infty\}$. The invariant $d_0 M$ is the least integer t such that M contains no subobject in $\mathscr{N}il_{t+1}$; the invariant $d_1 M$ is defined in terms of the notion of closure associated to the localizing subcategories $\mathscr{N}il_t$. For K a Noetherian unstable algebra and M an object of K_{fg}–$\mathscr{U}$, Henn [9] showed that these invariants are finite: $d_0 M \leq d_1 M < \infty$. For example, if G is a finite group and $H^*(BG)$ denotes singular cohomology with $\mathbb{F}_p$-coefficients, then $d_0 H^*(BG) \leq d_1 H^*(BG) < \infty$ define integer-valued invariants of the finite group G. The calculation of these invariants is a deep and interesting problem.

A morphism $K \rightarrow L$ of unstable algebras induces an induction functor $L \otimes_K - : K$–$\mathscr{U} \rightarrow L$–$\mathscr{U}$. The behaviour of the nilpotent filtration under the induction functor is interesting; in particular, given an object M of K–$\mathscr{U}$, what can be said about the invariants d_0, d_1 of $L \otimes_K M$ in terms of those of M?

We restrict to the Noetherian setting, where K is a Noetherian unstable algebra and L is a finitely generated K-module; furthermore, we suppose that K and L are connected unstable algebras and L is a flat K-module. This hypothesis is fairly restrictive, in that it implies that L is a free, finitely generated K-module, yet it contains cases of interest arising from invariant theory. For example, the theory applies to the inclusion

$$\mathbb{F}_p[x_1, \ldots, x_n]^{\mathfrak{S}_n} \cong \mathbb{F}_p[\sigma_1, \ldots, \sigma_n] \hookrightarrow \mathbb{F}_p[x_1, \ldots, x_n]$$

of the symmetric invariants.

The study of induced modules of the form $L \otimes_K M$ is carried out by considering the injective objects of the category K_{fg}–$\mathscr{U}$. A family $I_{(V,\varphi)}(n)$ of injective cogenerators was defined by Henn, indexed by natural numbers n and pairs (V, φ) corresponding to morphisms of unstable algebras $K \xrightarrow{\varphi} H^*(BV)$, such that $H^*(BV)$ is finitely generated as a K-module. An analysis of these objects, together with techniques developed by Lannes and Zarati in their study of the category $H^*(BV)$–$\mathscr{U}$ leads to the following general result.

Theorem 1. *Let M be an object of K_{fg}–$\mathscr{U}$ which admits an embedding $M \hookrightarrow \bigoplus_{i \in \mathscr{I}} I_{(V_i,\varphi_i)}(n_i)$, then:*

$$d_0 M \leq d_0(L \otimes_K M) \leq d_0 M + \sup_{i \in \mathscr{I}} \{||T_{V_i,\varphi_i} L \otimes_{T_{V_i,\varphi_i} K} \mathbb{F}_p||\}.$$

(Here, $||N||$ denotes the supremum of the dimensions of elements of N and $T_{V,\varphi}$ denotes a component of the T-functor applied to K–$\mathscr{U}$).

To gain meaningful information from this result, it is necessary to have some control over the embedding of M in an injective. Full information is given in terms of the associated primes; weaker bounds are given by considering the values of $\dim V_i$. An upper bound is given by the transcendence degree of M, which is an invariant which can be defined in terms of the action of the T-functor. A lower bound is given by the depth, by applying one of the main results of Bourguiba and Zarati used in their proof of the Landweber-Stong conjecture.

Theorem 2. *Let K be a Noetherian unstable algebra and M be an object of K_{fg}–$\mathscr{U}$, then there exists an embedding in K_{fg}–$\mathscr{U}$*

$$M \hookrightarrow \bigoplus_{i \in \mathscr{I}} I_{(V_i,\varphi_i)}(n_i),$$

where $\mathscr{I}$ is a finite indexing set and

(1) *$n_i \leq d_0 M$, with equality for some i;*
(2) $\mathbf{Depth}_K M \leq \dim V_i \leq \mathbf{TrDeg}_K M$.

In the case of invariant theory, the calculation of the T-functor in terms of other algebras of invariants is straightforward. When this theory is applied to the algebra of symmetric invariants $\mathbb{F}_2[\sigma_1, \ldots, \sigma_n] \hookrightarrow \mathbb{F}_2[x_1, \ldots, x_n]$, one deduces the following result.

Corollary 3. *Let M be an object of $\mathbb{F}_2[\sigma_1, \ldots, \sigma_n]_{fg}$–$\mathscr{U}$ which has depth d. Then there are inequalities:*

$$d_0 M \leq d_0(\mathbb{F}_2[x_1, \ldots, x_n] \otimes_{\mathbb{F}_2[\sigma_1,\ldots,\sigma_n]} M) \leq d_0 M + \frac{1}{2}n(n-1) - \frac{1}{2}d(d-1).$$

This result sheds light upon the motivating example for this study, which is the bound obtained by Henn, Lannes and Schwartz on $d_0 H^*(BG; \mathbb{F}_2)$, for a finite group G. Their work uses the flat base change argument of Quillen, associated to the extension $H^*(BO(n)) = \mathbb{F}_2[\sigma_1, \ldots, \sigma_n] \hookrightarrow \mathbb{F}_2[x_1, \ldots, x_n] = H^*(B\mathbb{F}_2^n)$. The bound is in fact a bound for $d_0(H^*(B\mathbb{F}_2^n) \otimes_{H^*(BO(n)} H^*(BG))$, which is obtained from geometric considerations, using results and methods of Duflot.

Organization of the paper: Section 2 provides a brief review of some of the notions which are essential to the paper, such as Lannes' T-functor. The following section 3 recalls the definition and the fundamental properties of the categories K–$\mathscr{U}$ together with the induction and restriction functors which are the subject of the paper. The nilpotent filtration of the category of unstable modules is reviewed in Section 4; the relation with the notion of locally finite submodule is explained and the behaviour of the filtration in the presence of a module structure is considered. Section 5 continues the foundational part of the paper, with a review of the division functors which are used in the paper, in particular the functors Fix. The foundational part of the paper concludes with Section 6, which covers the necessary material on the injective objects in K–$\mathscr{U}$.

Section 7 explains how the transcendence degree and the depth can be used to give information on the first term of an injective resolution of an object of K–$\mathscr{U}$. This builds on work of Henn, Lannes and Schwartz and of Bourguiba and Zarati. These results are applied in Section 8, which gives the general results on the behaviour of d_0 with respect to induction. Section 8.4 specializes to the case of rings of invariants and Section 9 specializes further to the case of symmetric invariants. Finally, Section 10 explains the original motivation for these considerations, which is related to understanding the invariants d_0, d_1 for group cohomology.

Notation: Throughout the paper, a prime p is fixed and $\mathbb{F}$ denotes the prime field of characteristic p; the category of unstable modules over the mod-p Steenrod algebra $\mathscr{A}$ is written $\mathscr{U}$ and $\mathscr{K}$ denotes the subcategory of unstable algebras.

2. Preliminaries

2.1. **Recollections on unstable modules and algebras.** The reader is referred to the book by Schwartz [20] as a general reference for the categories of unstable modules and unstable algebras.

Definition 2.1.1. An unstable algebra K is :

(1) connected if $K^0 = \mathbb{F}$;
(2) Noetherian if the underlying algebra is finitely generated.

The degree zero part K^0 of an unstable algebra $K \in \text{Obj } \mathscr{K}$ is a p-Boolean algebra (Cf. [20, Section 3.8]). The category $\mathscr{B}$ of p-Boolean is equivalent to the opposite of the category of profinite sets, via the functor Spec, where $\text{Spec}(B)$ is defined as $\text{Hom}_{\mathscr{B}}(B, \mathbb{F}_p)$. The following result is fundamental when passing to connected components of an unstable algebra.

Proposition 2.1.2. *Let B be an object of $\mathscr{B}$ and $\varphi : B \to \mathbb{F}$ be an element of* $\mathrm{Spec}(B)$*, then $\mathbb{F}$ is a flat B-module via φ.*

Let V be a finitely generated elementary abelian p-group. The T-functor $T_V : \mathscr{U} \to \mathscr{U}$ is the left adjoint to the functor $H^*(V) \otimes - : \mathscr{U} \to \mathscr{U}$, where $H^*(V)$ denotes the group cohomology of V with $\mathbb{F}$ coefficients. The functor T_V has remarkable algebraic properties: for instance, it is exact and commutes with tensor products. Moreover, the association $V \mapsto T_V$ is functorial in V. The functor T_V restricts to a functor $T_V : \mathscr{K} \to \mathscr{K}$, which is left adjoint to the functor $H^*(V) \otimes - : \mathscr{K} \to \mathscr{K}$.

2.2. **Categories associated to unstable algebras.** There are two fundamental categories which are associated to an unstable algebra K. Related categories occur in the work of Lam and Rector, which depends on the seminal work of Adams-Wilkerson [1]. This theory was exploited in the current form by Henn, Lannes and Schwartz in [11, 12].

Definition 2.2.1. Let K be an unstable algebra.

(1) The category $\mathscr{S}(K)$ has objects pairs (V, φ), where V is an elementary abelian p-group and $\varphi : K \to H^*(V)$ is a morphism of unstable algebras. A morphism $(V_1, \varphi_1) \to (V_2, \varphi_2)$ is a homomorphism $\alpha : V_1 \to V_2$ such that $\varphi_1 = H^*(\alpha)\varphi_2$.
(2) The category $\mathscr{R}(K)$ is the full subcategory of $\mathscr{S}(K)$ with objects (V, φ) such that $H^*(V)$ is a finitely generated K-module via φ.

Lemma 2.2.2. *A morphism $\alpha : K \to L$ of unstable algebras induces a functor $\mathscr{S}(\alpha) : \mathscr{S}(L) \to \mathscr{S}(K)$. If L is a finitely generated K-module via α, then $\mathscr{S}(\alpha)$ restricts to a functor $\mathscr{R}(\alpha) : \mathscr{R}(L) \to \mathscr{R}(K)$.*

The category $\mathscr{R}(K)$ is a fundamental object in the study of a Noetherian unstable algebra K.

Proposition 2.2.3. *Let K be a Noetherian unstable algebra of transcendence degree d. The following properties hold.*

(1) *The category $\mathscr{R}(K)$ has a finite skeleton and* $\dim(V) \leq d$ *for each object (V, φ) of $\mathscr{R}(K)$.*
(2) *If $(V, \varphi) \to (W, \psi)$ is a morphism of $\mathscr{R}(K)$, then the underlying morphism $\alpha : V \to W$ is a monomorphism; the morphism is an isomorphism if and only if α is an isomorphism.*

Considerations of base change are simplified by the following observation.

Lemma 2.2.4. *Let $\alpha : K \to L$ be a morphism of Noetherian unstable algebras for which L is a finitely generated K-module. The following is a pullback diagram of categories*

$$\begin{array}{ccc} \mathscr{R}(L) & \xrightarrow{\mathscr{R}(\alpha)} & \mathscr{R}(K) \\ \big\downarrow & & \big\downarrow \\ \mathscr{S}(L) & \xrightarrow[\mathscr{S}(\alpha)]{} & \mathscr{S}(K). \end{array}$$

3. Modules over an unstable algebra

3.1. **The category of K-modules.** Fix an unstable algebra $K \in \mathscr{K}$.

Definition 3.1.1. Let K–$\mathscr{U}$ denote the category of modules in $\mathscr{U}$ over K and K_{fg}–$\mathscr{U}$ denote the full subcategory of modules which are finitely generated as modules over the underlying algebra of K.

Theorem 3.1.2. [14] *The category K–$\mathscr{U}$ is abelian. Moreover it is a Grothendieck category: it contains all small colimits and filtered colimits are exact.*

If K is Noetherian, then K_{fg}–$\mathscr{U}$ is an abelian subcategory of K–$\mathscr{U}$.

Corollary 3.1.3. [7, Théorème II.2] *The category K–$\mathscr{U}$ has injective envelopes.*

Theorem 3.1.4. [17] [14, Théorème 3.1.5] *The category K–$\mathscr{U}$ is locally Noetherian if K is Noetherian.*

Corollary 3.1.5. [7, Théorème IV.2 and Proposition IV.6] *Let K be a Noetherian unstable algebra. There exists a set of indecomposable injectives $\{E_\lambda | \lambda \in \mathscr{L}\}$ such that*

(1) *for any injective J in K–$\mathscr{U}$, there exists a unique set of cardinals $\{a_\lambda | \lambda \in \mathscr{L}\}$ such that*
$$J \cong \bigoplus_{\lambda \in \mathscr{L}} E_\lambda^{\oplus a_\lambda}$$
(2) *any object of K–$\mathscr{U}$ of this form is injective.*

3.2. **Tensor structures, restriction and induction.** The category of unstable modules $\mathscr{U}$ is a tensor abelian category; this structure induces internal and external tensor products on the categories of modules over unstable algebras. These satisfy associativity conditions which can be formulated by the reader.

Proposition 3.2.1. *Let K, L be unstable algebras. The tensor product of $\mathscr{U}$ induces an exterior tensor product K–$\mathscr{U} \times L$–$\mathscr{U} \to (K \otimes L)$–$\mathscr{U}$.*

The internal tensor product on the category K–$\mathscr{U}$ is induced by the tensor product on the category of graded K-modules.

Proposition 3.2.2. *Tensor product over K induces a functor:*
$$\otimes_K : K\text{–}\mathscr{U} \times K\text{–}\mathscr{U} \to K\text{–}\mathscr{U},$$
which gives rise to a symmetric monoidal structure $(K\text{–}\mathscr{U}, \otimes_K, K)$.

A morphism $\alpha : K \to L$ of unstable algebras induces an exact and faithful restriction functor $\alpha^* : L$–$\mathscr{U} \to K$–$\mathscr{U}$.

Proposition 3.2.3. *Let $\alpha : K \to L$ be a morphism of unstable algebras. The restriction functor α^* admits a left adjoint, the induction functor:*
$$L \otimes_K - : K\text{–}\mathscr{U} \to L\text{–}\mathscr{U},$$
which satisfies the following properties:

(1) *induction is right exact and is exact if L is flat as a K-module;*
(2) *the functor $L \otimes_K -$ induces a symmetric monoidal functor*
$$L \otimes_K - : (K\text{–}\mathscr{U}, \otimes_K, K) \to (L\text{–}\mathscr{U}, \otimes_L, L).$$

3.3. **Algebras under an unstable algebra.** The category of K-algebras in $\mathscr{K}$ is denoted by $K{\downarrow}\mathscr{K}$. There is a commutative diagram of forgetful functors:
$$\begin{array}{ccc} K{\downarrow}\mathscr{K} & \longrightarrow & K\text{–}\mathscr{U} \\ \downarrow & & \downarrow \\ \mathscr{K} & \hookrightarrow & \mathscr{U} \end{array}$$

Proposition 3.3.1. *Let K, L be unstable algebras.*

(1) *The exterior tensor product restricts to an exterior tensor product $K{\downarrow}\mathscr{K} \times L{\downarrow}\mathscr{K} \to (K \otimes L){\downarrow}\mathscr{K}$.*

(2) *The tensor product over* K *restricts to a functor*

$$\otimes_K : K\downarrow\mathscr{K} \ \times \ K\downarrow\mathscr{K} \to K\downarrow\mathscr{K},$$

which induces a symmetric monoidal structure $(K\downarrow\mathscr{K}, \otimes_K, K)$.

(3) *If* $\alpha : K \to L$ *is a morphism of unstable algebras, the restriction functor* $\alpha^* : L\text{–}\mathscr{U} \to K\text{–}\mathscr{U}$ *restricts to* $\alpha^* : L\downarrow\mathscr{K} \to K\downarrow\mathscr{K}$, *which has left adjoint* $L\otimes_K - : K\downarrow\mathscr{K} \to L\downarrow\mathscr{K}$.

4. The nilpotent filtration

4.1. **Recollections.** This section reviews the nilpotent filtration of the category $\mathscr{U}$ (Cf. [20, Chapter 6] and [13] for further details). Recall that the object $H^*(V)\otimes J(k)$ is an injective of $\mathscr{U}$, where V is an elementary abelian p-group and $J(k)$ denotes the Brown-Gitler module, which is the injective envelope of $\Sigma^k\mathbb{F}$.

Definition 4.1.1. For s a positive integer, let $\mathscr{N}il_s$ denote the full subcategory of $\mathscr{U}$ with objects which are annihilated by the exact functor

$$\bigoplus_{k=0,V}^{k=s-1} \mathrm{Hom}_{\mathscr{U}}(-, H^*(V)\otimes J(k)).$$

It follows from the definition that $\mathscr{N}il_s$ is a thick, localizing subcategory of $\mathscr{U}$; it is the smallest such subcategory which contains all objects of the form $\Sigma^s M$. There are inclusions of full subcategories

$$\ldots \mathscr{N}il_{s+1} \subset \mathscr{N}il_s \subset \ldots \subset \mathscr{N}il := \mathscr{N}il_1 \subset \mathscr{N}il_0 = \mathscr{U}.$$

The quotient category $\mathscr{U}/\mathscr{N}il_s$ is defined and there is an adjunction

$$l_s : \mathscr{U} \leftrightarrows \mathscr{U}/\mathscr{N}il_s : r_s$$

and the quotient functor l_s is exact. The composite functor $r_s l_s : \mathscr{U} \to \mathscr{U}$ is denoted L_s in [12, Section I.3] and the adjunction unit defines a natural transformation $\lambda_s : 1 \to L_s$. In the terminology of localization of abelian categories [7], the natural transformation λ_s is the localization of $\mathscr{U}$ away from $\mathscr{N}il_s$ and the functor $L_s : \mathscr{U} \to \mathscr{U}$ is the localization functor.

The inclusion $\mathscr{N}il_s \hookrightarrow \mathscr{U}$ admits a right adjoint, $\mathrm{nil}_s : \mathscr{U} \to \mathscr{N}il_s$. The adjunction counit defines a canonical monomorphism $\mathrm{nil}_s M \hookrightarrow M$, for $M \in \mathrm{Obj}\,\mathscr{U}$, and this induces the nilpotent filtration

$$\ldots \subset \mathrm{nil}_{s+1}M \subset \mathrm{nil}_s M \subset \ldots \subset M.$$

Remark 4.1.2. An unstable module M is reduced if it contains no non-trivial suspension; this condition is equivalent to $\mathrm{nil}_1 M = 0$.

Proposition 4.1.3. [13, Propositions 2.2, 2.5] *Let* M, N, Q *be unstable modules and* s *be a natural number.*

(1) *The unstable module* $\mathrm{nil}_s M/\mathrm{nil}_{s+1}M$ *is the* s*-fold suspension of a reduced unstable module,* $R_s M$.
(2) *The functor* T_V *commutes with the functor* $\mathrm{nil}_s M$, *so that there is a natural isomorphism* $T_V(\mathrm{nil}_s M) \cong \mathrm{nil}_s(T_V M)$.
(3) *There is an equality of submodules of* $M\otimes N$

$$\mathrm{nil}_s(M\otimes N) = \Sigma_{i+j=s}\mathrm{nil}_i M \ \otimes \ \mathrm{nil}_j N.$$

(4) *A morphism* $f : M\otimes N \to Q$ *induces a canonical morphism* $\mathrm{nil}_i M \otimes \mathrm{nil}_j N \to \mathrm{nil}_{i+j} Q$.

4.2. **The invariants d_0, d_1.** Henn, Lannes and Schwartz [12] introduce invariants of an unstable module which are defined in terms of the nilpotent filtration.

Definition 4.2.1. For $M \in \mathrm{Obj}\,\mathscr{U}$, the invariants $d_0M, d_1M \in \mathbb{N} \cup \{\infty\}$ are given by:

(1) $d_0M := \inf\{t \mid \lambda_{t+1} : M \to L_{t+1}M$ is a monomorphism$\}$;
(2) $d_1M := \inf\{t \mid \lambda_{t+1} : M \to L_{t+1}M$ is an isomorphism$\}$.

The following Lemma is a formal consequence of the definitions.

Lemma 4.2.2. *For M an unstable module, $d_0M = \inf\{t \mid \mathrm{nil}_{t+1}M = 0\}$.*

Proposition 4.2.3. [12, Proposition I.3.6]

(1) *If $M \hookrightarrow N$ is a monomorphism in $\mathscr{U}$, then $d_0M \leq d_0N$.*
(2) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence, then $d_0M_2 \leq \max\{d_0M_1, d_0M_3\}$.*
(3) *Let M, N be non-trivial unstable modules, then $d_0(M \otimes N) = d_0M + d_0N$.*
(4) *If X is an unstable module of finite total dimension, then $d_0X = ||X||$, where $||X|| = \inf\{t \mid X^{t+1} = 0\}$.*

4.3. **Locally finite unstable modules and the nilpotent filtration.** There is an intimate relationship between the nilpotent filtration of $\mathscr{U}$ by the subcategories $\mathscr{N}il_s$ and locally finite unstable modules. Recall that an unstable module is locally finite if each cyclic submodule is finite; the reader is referred to [20, Chapter 6] and [13, Section 3] for further details.

The inclusion of the full subcategory $\mathscr{U}_{lf}$ of locally finite unstable modules in $\mathscr{U}$ admits a right adjoint $\mathbf{lf} : \mathscr{U} \to \mathscr{U}_{lf}$. This will be considered abusively as a functor $\mathscr{U} \to \mathscr{U}$, via the canonical inclusion. Thus, for M an unstable module, there is a natural monomorphism $\mathbf{lf}M \hookrightarrow M$ and $\mathbf{lf}M$ is the largest locally finite submodule of M.

Notation 4.3.1. For N an unstable module and s a non-negative integer, let $N^{\geq s}$ denote the submodule of elements of degree at least s. Thus, there is a canonical short exact sequence of unstable modules:

$$0 \to N^{\geq s} \to N \to N^{<s} \to 0$$

where the surjection $N \to N^{<s}$ is the unit of the truncation adjunction.

The following result is fundamental.

Proposition 4.3.2. [13, Proposition 2.11] *Let s be a non-negative integer and M be an unstable module.*

(1) *There is a natural isomorphism $\mathrm{nil}_s(\mathbf{lf}M) \cong (\mathbf{lf}M)^{\geq s}$.*
(2) *The canonical inclusion $\mathbf{lf}M \hookrightarrow M$ induces an isomorphism*

$$(\mathbf{lf}M)^s \cong (\mathrm{nil}_sM)^s.$$

Recall from Proposition 4.1.3 that the unstable module $\mathrm{nil}_sM/\mathrm{nil}_{s+1}M$ is the s-fold suspension of a reduced unstable module R_sM.

Proposition 4.3.3. *Let M be an unstable module and s be a natural number. The reduced unstable module R_sM is trivial if and only if $(\mathbf{lf}(T_VM))^s = 0$ for each elementary abelian p-group V.*

Proof. The theory of nil-localization for unstable modules [11] implies that a reduced unstable module N is trivial if and only if $T_V^0N = 0$, for each V. The functor T_V commutes with R_s and Proposition 4.3.2 implies that $T_V^0R_sM = (\mathbf{lf}(T_VM))^s$. The result follows. □

Corollary 4.3.4. *Let M be an unstable module, then*

$$d_0 M = \inf\{t \mid (\mathbf{lf}(T_V M))^s = 0,\ \forall s \geq t+1,\ V\}.$$

Proof. The invariant $d_0 M$ identifies with $\inf\{t \mid \mathrm{nil}_{t+1} M = 0\}$ by Lemma 4.2.2 and hence, by filtration, with $\inf\{t \mid R_s M = 0, \forall s \geq t+1\}$. The result follows from the identification given by Proposition 4.3.3. □

4.4. **The nilpotent filtration and $K-\mathscr{U}$.** The nilpotent filtration of the category $\mathscr{U}$ behaves well with respect to algebra structures and module structures.

Proposition 4.4.1. *Let K be an unstable algebra, M be an object of K–$\mathscr{U}$ and i, j be natural numbers. The following properties hold:*

1. $\mathrm{nil}_i M$ *is naturally an object of* K–$\mathscr{U}$*;*
2. *the structure morphisms induce morphisms*

$$\begin{array}{rcl} \mathrm{nil}_i K \otimes \mathrm{nil}_j K & \rightarrow & \mathrm{nil}_{i+j} K \\ \mathrm{nil}_i K \otimes \mathrm{nil}_j M & \rightarrow & \mathrm{nil}_{i+j} M; \end{array}$$

3. $K/\mathrm{nil}_1 K$ *has the structure of an unstable algebra in* $K{\downarrow}\mathscr{K}$ *and* $\mathrm{nil}_i M/\mathrm{nil}_{i+1} M$ *is naturally an object of* $(K/\mathrm{nil}_1 K)$–$\mathscr{U}$.

Remark 4.4.2. If K is a Noetherian unstable algebra and M is an object of K_{fg}–$\mathscr{U}$, then each module considered above is finitely generated over its respective algebra.

The behaviour of the Noetherian hypothesis under nil-localization is given by the following result.

Theorem 4.4.3. [12, Corollary 4.10] *Let K be a Noetherian unstable algebra, M be an object of K_{fg}–$\mathscr{U}$ and s be a natural number. The following properties hold:*

1. $L_s M \in \mathrm{Obj}\ K_{fg}$–$\mathscr{U}$*;*
2. $L_s K \in K{\downarrow}\mathscr{K}$ *and* $L_s K \in K_{fg}$–$\mathscr{U}$*; in particular, $L_s K$ is Noetherian;*
3. $L_s M \in \mathrm{Obj}\ (L_s K)_{fg}$–$\mathscr{U}$.

A fundamental problem is to understand the behaviour of the nilpotent filtration under the functor $L \otimes_K - : K$–$\mathscr{U} \rightarrow L$–$\mathscr{U}$, where $\alpha : K \rightarrow L$ is a morphism of unstable algebras.

Lemma 4.4.4. *Let M be an object of K–$\mathscr{U}$ and s be a natural number. The morphism $M \rightarrow L \otimes_K M$ induces a canonical morphism in L–$\mathscr{U}$:*

$$L \otimes_K \mathrm{nil}_s M \rightarrow \mathrm{nil}_s(L \otimes_K M)$$

which is a monomorphism if the unstable algebras K, L are connected and L is a flat K-module.

Remark 4.4.5. The failure of the morphism $L \otimes_K \mathrm{nil}_t M \rightarrow \mathrm{nil}_t(L \otimes_K M)$ to be an isomorphism is an interesting phenomenon. For example, the unstable algebra $L = \mathbb{F}_2[x]$ on a generator of degree one has $\mathrm{nil}_1 L = 0$; for $K = \mathbb{F}_2[x^2] \subset L$, $\mathrm{nil}_1(L \otimes_K L)$ is non-trivial.

5. Division functors

5.1. **Lannes' T-functor and K-modules.** Let K be an unstable algebra and M be an object of K–$\mathscr{U}$. An object (V, φ) of $\mathscr{S}(K)$ induces a morphism of unstable algebras $T_V K \rightarrow \mathbb{F}$, by adjunction, which restricts to a morphism of p-Boolean algebras $T_V^0 K \rightarrow \mathbb{F}$. This allows the passage to connected components:

Notation 5.1.1. For (V, φ) an object of $\mathscr{S}(K)$, denote

1. $T_{V,\varphi} K := T_V K \otimes_{T_V^0 K} \mathbb{F}$;
2. $T_{V,\varphi} M := T_V M \otimes_{T_V^0 K} \mathbb{F} \cong T_V M \otimes_{T_V K} T_{V,\varphi} K$.

The T-functor induces an exact functor between categories of modules over unstable algebras $T_V : K\text{–}\mathscr{U} \to T_V K\text{–}\mathscr{U}$. This functor commutes with tensor products, so that there is a natural isomorphism $T_V(M \otimes_K N) \cong T_V M \otimes_{T_V K} T_V N$.

Proposition 5.1.2. [14, Proposition 1.3.1] *Let K, L be unstable algebras and $\psi : K \to H^*(V) \otimes L$ be a morphism of unstable algebras. For $M \in \mathrm{Obj}\ K\text{–}\mathscr{U}$ and $N \in \mathrm{Obj}\ L\text{–}\mathscr{U}$, the adjunction isomorphism*

$$\mathrm{Hom}_{\mathscr{U}}(M, H^*(V) \otimes N) \cong \mathrm{Hom}_{\mathscr{U}}(T_V M, N)$$

induces an isomorphism

$$\begin{aligned} \mathrm{Hom}_{K\text{–}\mathscr{U}}(M, H^*(V) \otimes N) &\cong \mathrm{Hom}_{T_V K\text{–}\mathscr{U}}(T_V M, N) \\ &\cong \mathrm{Hom}_{L\text{–}\mathscr{U}}(L \otimes_{T_V K} T_V M, N) \end{aligned}$$

where $H^*(V) \otimes N$ is a K-module via ψ and N is a $T_V K$-module via the adjoint $\psi' : T_V K \to L$ to ψ.

Definition 5.1.3. For (V, φ) an object of $\mathscr{S}(K)$, let

$$H^*(V) \otimes^{\varphi} - : T_{V,\varphi} K\text{–}\mathscr{U} \to K\text{–}\mathscr{U}$$

denote the functor which associates to $N \in \mathrm{Obj}\ T_{V,\varphi}K\text{–}\mathscr{U}$ the object $H^*(V) \otimes N$, considered as an object of $K\text{–}\mathscr{U}$ via the morphism $K \to H^*(V) \otimes T_{V,\varphi}K$ adjoint to the projection $T_V K \twoheadrightarrow T_{V,\varphi}K$.

Corollary 5.1.4. *For (V, φ) an object of $\mathscr{S}(K)$, the functor $T_{V,\varphi} : K\text{–}\mathscr{U} \to T_{V,\varphi}K\text{–}\mathscr{U}$ is left adjoint to the functor $H^*(V) \otimes^{\varphi} -$. In particular, for $M \in \mathrm{Obj}\ K\text{–}\mathscr{U}$ and $N \in \mathrm{Obj}\ T_{V,\varphi}K\text{–}\mathscr{U}$, there is an adjunction isomorphism*

$$\mathrm{Hom}_{K\text{–}\mathscr{U}}(M, H^*(V) \otimes^{\varphi} N) \cong \mathrm{Hom}_{T_{V,\varphi}K\text{–}\mathscr{U}}(T_{V,\varphi}M, N).$$

5.2. **Division functors and Fix.** The functor Fix was introduced by Lannes in his work on the Sullivan conjecture [15]; this was extended by Lannes-Zarati to a relative version in [14] and exploited by the same authors in [16]. The proof of the Landweber-Stong conjecture by Bourguiba and Zarati [2] also exploits these functors.

Notation 5.2.1. Let $\iota_V : T_V H^*(V) \to \mathbb{F}$ denote the adjoint to the identity on $H^*(V)$.

Proposition 5.2.2. [15, 14, 16] *The external tensor product functor*

$$H^*(V) \otimes - : \mathscr{U} \to H^*(V)\text{–}\mathscr{U}$$

admits a left adjoint $\mathrm{Fix}_V : H^*(V)\text{–}\mathscr{U} \to \mathscr{U}$, *which is given on $M \in H^*(V)\text{–}\mathscr{U}$ by*

$$\mathrm{Fix}_V M := \mathbb{F} \otimes_{T_V H^*(V)} T_V M$$

where $\mathbb{F}$ is a $T_V H^*(V)$-algebra via ι_V.

Recall the essential properties of the functor Fix_V.

Theorem 5.2.3. [15, 14]

1. *The functor* $\mathrm{Fix}_V : H^*(V)\text{–}\mathscr{U} \to \mathscr{U}$ *is exact.*
2. *For $N \in \mathrm{Obj}\ \mathscr{U}$, there is a natural isomorphism* $\mathrm{Fix}_V(H^*(V) \otimes N) \cong T_V N$.
3. *The functor* Fix_V *commutes with tensor product in the following sense: for M, N two objects of $H^*(V)\text{–}\mathscr{U}$, there is a natural isomorphism*

$$\mathrm{Fix}_V(M \otimes_{H^*(V)} N) \cong \mathrm{Fix}_V M \otimes \mathrm{Fix}_V N.$$

5.3. **Unstable algebras and** Fix**.** The functors Fix behave well with respect to unstable algebra structures:

Proposition 5.3.1. [14, Theorem 0.2] *The functor* $\mathrm{Fix}_V : H^*(V)\text{–}\mathscr{U} \to \mathscr{U}$ *restricts to a functor* $\mathrm{Fix}_V : H^*(V) \downarrow \mathscr{K} \to \mathscr{K}$ *which is right adjoint to the base change induction functor $\mathscr{K} \to H^*(V) \downarrow \mathscr{K}$ induced by $H^*(V) \otimes -$, so that there is an adjunction*

$$H^*(V) \otimes - : \ \mathscr{K} \rightleftarrows \ H^*(V) \downarrow \mathscr{K} : \mathrm{Fix}_V.$$

5.4. **The relation between** Fix **and the locally finite submodule.** The functor Fix_V applied to an object of $H^*(V)_{fg}$–$\mathscr{U}$ can be given an alternative description using the functor $\mathbf{lf} : \mathscr{U} \to \mathscr{U}$.

Notation 5.4.1. Let M be an object of $H^*(V)$–$\mathscr{U}$.

(1) Let (V, ι) denote the object of $\mathscr{R}(H^*(V))$ which corresponds to the identity morphism of $H^*(V)$.
(2) Let $\kappa_M : T_V M \to H^*(V) \otimes \mathrm{Fix}_V M$ denote the morphism which is adjoint to the composite
$$M \to H^*(V) \otimes \mathrm{Fix}_V M \stackrel{\Delta \otimes 1}{\to} H^*(V) \otimes H^*(V) \otimes \mathrm{Fix}_V M,$$
where the first morphism is the adjunction unit and Δ is the diagonal of $H^*(V)$.
(3) Let $\overline{\kappa}_M : T_{V,\iota} M \to H^*(V) \otimes \mathrm{Fix}_V M$ denote the canonical factorization of κ_M. (Cf. [15, Section 4.5]).

The functor $T_{V,\iota} : H^*(V)$–$\mathscr{U} \to H^*(V)$–$\mathscr{U}$ is exact; it is related to the functor Fix by the following result.

Proposition 5.4.2. [15, Proposition 4.5] *Let M be an object of $H^*(V)$–$\mathscr{U}$. Then the morphism*
$$\overline{\kappa}_M : T_{V,\iota} M \to H^*(V) \otimes \mathrm{Fix}_V M$$
is an isomorphism in $H^*(V)$–$\mathscr{U}$.

Corollary 5.4.3. *Let M be an object of $H^*(V)_{fg}$–$\mathscr{U}$. The following properties hold:*

(1) *the morphism $\overline{\kappa}_M$ induces a natural isomorphism $\mathbf{lf} T_{V,\iota} M \cong \mathrm{Fix}_V M$;*
(2) *the morphism $T_{V,\iota} M \twoheadrightarrow \mathrm{Fix}_V M$ is a retract of the inclusion $\mathbf{lf} T_{V,\iota} M \hookrightarrow T_{V,\iota} M$.*
(3) *If K is an object of $H^*(V) \downarrow \mathscr{K}$ which belongs to $H^*(V)_{fg}$–$\mathscr{U}$, then the isomorphism $\mathbf{lf} T_{V,\iota} K \cong \mathrm{Fix}_V K$ is an isomorphism of unstable algebras.*

Proof. There is a natural isomorphism $\mathbf{lf}(\mathrm{Fix}_V M) \cong \mathbf{lf}(H^*(V) \otimes \mathrm{Fix}_V M)$, since $\mathrm{Fix}_V M$ is a finite unstable module and $H^*(V)$ is nil-closed. Hence the first statement follows from Proposition 5.4.2. The statement concerning the retraction is a straightforward verification and the final statement concerning the unstable algebra structures is clear. □

5.5. **Structure results.**

Notation 5.5.1. [14] Let c_V denote the element of $H^*(V)$ defined as follows:

$(p = 2)$ $c_V := \prod_{u \in H^1(V) \backslash \{0\}} u$;
$(p \text{ odd})$ $c_V := \prod_{u \in H^1(V) \backslash \{0\}} \beta u$.

Thus, for $p = 2$, c_V corresponds to the top Dickson invariant. The work of Lannes-Zarati on algebraic Smith theory analyzes the localization obtained by inverting the class c_V. Their results include the following:

Theorem 5.5.2. *(Cf.* [14, Théorème 2.1]*) Let M be an object of $H^*(V)_{fg}$–$\mathscr{U}$. Then the unstable module $\mathrm{Fix}_V M$ is finite and the adjunction morphism induces an isomorphism*
$$M[c_V^{-1}] \to H^*(V)[c_V^{-1}] \otimes \mathrm{Fix}_V M.$$

This result will be applied via the following Corollary.

Corollary 5.5.3. *Let M be an object of $H^*(V)_{fg}$–$\mathscr{U}$ such that the underlying $H^*(V)$-module is free. Then the adjunction morphism induces an embedding in the category $H^*(V)$–$\mathscr{U}$*
$$M \hookrightarrow H^*(V) \otimes \mathrm{Fix}_V M$$

and $\mathrm{Fix}_V M$ *is a finite unstable module. Moreover the total dimension of* $\mathrm{Fix}_V M$ *is equal to the rank of* M *as an* $H^*(V)$*-module.*

Proof. Theorem 5.5.2 implies that the kernel of the adjunction morphism $M \to H^*(V) \otimes \mathrm{Fix}_V M$ is c_V-torsion. The hypothesis that M is free as an $H^*(V)$-module implies that it is c_V-torsion free, whence the first statement. The remainder of the result is straightforward. □

6. Injective objects

6.1. **Families of injectives in K–$\mathscr{U}$.** There are analogues in the category K–$\mathscr{U}$ the Brown-Gitler modules:

Definition 6.1.1. [14] Let $J_K(n)$ be the object of K–$\mathscr{U}$ which is determined (up to isomorphism) by the natural isomorphism:

$$\mathrm{Hom}_{K\text{–}\mathscr{U}}(M, J_K(n)) \cong (M^n)^*.$$

The object $J_K(n)$ is trivial in degrees strictly greater than n and has finite total dimension if K is finite dimensional in each degree, for example if K is Noetherian.

Theorem 6.1.2. *The set of objects* $\{J_K(n) \mid n \geq 0\}$ *forms a set of injective cogenerators of* K–$\mathscr{U}$*.*

The study of the injective objects in the category K–$\mathscr{U}$ for K an unstable algebra was initiated by Henn in [10], which is primarily concerned with the injective objects which lie in K_{fg}–$\mathscr{U}$, and extended by Meyer in her thesis [17].

The following definition uses the functor $H^*(V) \otimes^{\varphi} -$ introduced in Definition 5.1.3.

Definition 6.1.3. [10] Let (V, φ) be an object of the category $\mathscr{S}(K)$ and n be a natural number. The object $I_{(V,\varphi)}(n)$ of K–$\mathscr{U}$

$$I_{(V,\varphi)}(n) := H^*(V) \otimes^{\varphi} J_{T_{V,\varphi}(K)}(n).$$

Example 6.1.4. The object $I_{(V,\varphi)}(0)$ is the unstable algebra $H^*(V)$, which is considered as a K-module via φ. This object is sometimes denoted by $H^*(V)(\varphi)$.

The following result is a formal consequence of the adjunction properties of T_V, as stated in Corollary 5.1.4; it implies that the object $I_{(V,\varphi)}(n)$ is injective.

Proposition 6.1.5. [10, Proposition 1.7] *Let* $\tilde{\varphi} : K \to H^*(V)$ *be a morphism of unstable algebras and* J *be an injective of* $T_{V,\varphi}(K)$*, then* $H^*(V) \otimes^{\varphi} J$ *is an injective object of* K–$\mathscr{U}$*.*

Remark 6.1.6. In general, the injective objects $I_{(V,\varphi)}(n)$ are not indecomposable. See [17] for the classification of the indecomposable injectives of K–$\mathscr{U}$ in the case that K is Noetherian.

The following result gives the defining property of such injectives.

Lemma 6.1.7. *let* M *be an object of* K–$\mathscr{U}$ *and* (V, φ) *be an object of* $\mathscr{R}(K)$*. There is an isomorphism*

$$\mathrm{Hom}_{K\text{–}\mathscr{U}}(M, I_{(V,\varphi)}(n)) \cong (T_{V,\varphi} M)^{n*}.$$

Example 6.1.8. Let N be an object of $T_{V,\varphi}(K)$–$\mathscr{U}$ which has finite total dimension. Then there exists an embedding $N \hookrightarrow \bigoplus_j J_{T_{V,\varphi} K}(n_j)$ into a finite direct sum of injectives in $T_{V,\varphi}(K)$–$\mathscr{U}$ which induces an embedding

$$H^*(V) \otimes^{\varphi} N \hookrightarrow \bigoplus_j I_{(V,\varphi)}(n_j)$$

in K–$\mathscr{U}$.

In particular, taking $N = \Sigma^n \mathbb{F}$, which is naturally an object over the connected unstable algebra $T_{V,\varphi}K$, the above induces an embedding in K–$\mathscr{U}$:

$$\Sigma^n H^*(V) \hookrightarrow I_{(V,\varphi)}(n).$$

The following result provides a useful analysis of the structure of the injective $I_{(V,\varphi)}(n)$.

Proposition 6.1.9. *Let K be a Noetherian unstable algebra and (V,φ) be an object of $\mathscr{S}(K)$, then the injective $I_{(V,\varphi)}(n)$ admits a finite filtration with filtration quotients of the form $\Sigma^k H^*(V)(\varphi)$, where k is an integer $0 \leq k \leq n$.*

Proof. The filtration $J_{T_{V,\varphi}K}(n)$ by the degree is finite; the hypothesis that K is Noetherian serves to ensure that $T_{V,\varphi}K$ is finite-dimensional in each degree, so that $J_{T_{V,\varphi}K}(n)$ has finite total dimension. The filtration refines to give the required filtration with quotients as stated. □

6.2. **Injectives in K_{fg}–$\mathscr{U}$.** Throughout this section, let K be a Noetherian unstable algebra. The following result forms the foundations of the theory.

Theorem 6.2.1. [10, Theorem 1.9] *Let K be a Noetherian unstable algebra and M be an object of K_{fg}–$\mathscr{U}$.*

1. *If (V,φ) is an object of $\mathscr{R}(K)$, then $I_{(V,\varphi)}(n)$ lies in K_{fg}–$\mathscr{U}$.*
2. *There exists a finite set of pairs $\{((V_j,\varphi_j),n_j) \mid j \in \mathscr{J}\}$, where (V_j,φ_j) is an object of $\mathscr{R}(K)$ and n_j is a natural number, and an embedding in K–$\mathscr{U}$:*

$$M \hookrightarrow \bigoplus_{j\in\mathscr{J}} I_{(V_j,\varphi_j)}(n_j).$$

In particular, M embeds in an injective object of K–$\mathscr{U}$ which is finitely generated as a K-module.

This result implies the existence of injective resolutions of finite type.

Corollary 6.2.2. *Let K be a Noetherian unstable algebra and let M be an unstable module in K_{fg}–$\mathscr{U}$. There exists an injective resolution of M in K–$\mathscr{U}$ in which each term is a finite direct sum of injective objects of the form $I_{(V_l,\varphi_l)}(n_l)$, for objects $(V_l,\varphi_l) \in \mathrm{Obj}\ \mathscr{R}(K)$ and natural numbers n_l.*

An important result is the following, which reflects the unimodular nature of the category $\mathscr{R}(K)$.

Proposition 6.2.3. *Let $I_{(V_i,\varphi_i)}(n_i)$ be injectives of K_{fg}–$\mathscr{U}$, for $i \in \{1,2\}$. If $\dim V_1 < \dim V_2$, then*

$$\mathrm{Hom}_{K\text{–}\mathscr{U}}(I_{(V_1,\varphi_1)}(n_1), I_{(V_2,\varphi_2)}(n_2)) = 0.$$

Proof. By adjunction and Proposition 6.1.9, the proof reduces to showing that

$$T_{V_2,\varphi_2}(H^*(V_1)(\varphi_1)) = 0.$$

It is a standard calculation to show that

$$T_{V_2,\varphi_2}(H^*(V_1)(\varphi_1)) \cong H^*(V) \otimes \mathbb{F}^{\mathrm{Hom}_{\mathscr{R}(K)}((V_2,\varphi_2),(V_1,\varphi_1))}.$$

Proposition 2.2.3 shows that $\mathrm{Hom}_{\mathscr{R}(K)}((V_2,\varphi_2),(V_1,\varphi_1)) = \emptyset$, since $\dim V_1 < \dim V_2$. This completes the proof. □

7. Bounds on injective resolutions

Injective envelopes exist in the category K–$\mathscr{U}$ and, if M is an object of K_{fg}–$\mathscr{U}$, the injective envelope will be of the form

$$\bigoplus_{\lambda \in \mathscr{L}} E_\lambda^{b_\lambda}$$

where $\{E_\lambda | \lambda \in \mathscr{L}\}$ is a set of representatives of the isomorphism classes of indecomposable injectives in K_{fg}–$\mathscr{U}$, the b_λ are finite natural numbers which are non-zero for only finitely many λ. (Cf. Corollary 3.1.5).

The aim of this section is to provide information on the injective envelope of an object of K_{fg}–$\mathscr{U}$ in terms of invariants from commutative algebra. For the purposes of this paper, it is sufficient to work with embeddings into injectives of the form $\bigoplus_{i \in \mathscr{I}} I_{(V_i, \varphi_i)}(n_i)$, where $(V_i, \varphi_i) \in \mathscr{R}(K)$. It should be noted, however, that the injective objects $I_{(V,\varphi)}(n)$ are not indecomposable in general, so that the resolutions constructed using Corollary 6.2.2 will usually not be minimal.

7.1. Transcendence degree for unstable algebras.

The results of Henn, Lannes and Schwartz [11, Part II] show that the transcendence degree of a Noetherian unstable algebra can be interpreted in terms of the unstable algebra structure.

Theorem 7.1.1. [11, Section 7] *The transcendence degree* $\mathbf{TrDeg}(K)$ *of* K *is*

$$\mathbf{TrDeg}(K) = \sup\{\dim V | (V, \varphi) \in \mathscr{R}(K)\}.$$

The Dickson invariants play a fundamental rôle in the theory of Noetherian unstable algebras, by the work of Henn, Lannes and Schwartz. The main result is given in the appendix by Lannes to the paper of Bourguiba and Zarati [2], whose notation is adopted here.

Namely, let P_n denote the polynomial subalgebra of $H^*(\mathbb{F}^n)$ and let D_n denote the Dickson invariants $D_n := P_n^{GL_n(\mathbb{F})}$, which is a polynomial algebra $\mathbb{F}[c_1, \ldots, c_n]$. For s a natural number, let $D_{n,s}$ denote the unstable algebra $\Phi^s(D_n)$, where $\Phi : \mathscr{U} \to \mathscr{U}$ is the functor defined in [20, Section 1.7]. The algebra $D_{n,s}$ identifies with the subalgebra of D_n on the p^s-th powers of the elements of D_n.

Theorem 7.1.2. [2, Theorem A.1] *Let K be a Noetherian unstable algebra of transcendence degree n. Then*

1. *there exists a natural number s and a monomorphism of unstable algebras $\iota : D_{n,s} \to K$ for which K is a finitely generated $D_{n,s}$-module;*
2. *if $\iota' : D_{n,t} \to K$ is a morphism of unstable algebras such that K is a finitely generated $D_{n,t}$-module, then ι' is injective and there exists an integer $u \geq s, t$ such that the restrictions of ι, ι' to $D_{n,u}$ coincide.*

Notation 7.1.3. For $\iota : D_{n,s} \hookrightarrow K$ a monomorphism of unstable algebras for which K is a finitely generated $D_{n,s}$-module (so that $\mathbf{TrDeg}(K) = n$), let ω_ι denote the image of the generator $c_n^{p^s}$.

The localization inverting the multiplicative set generated by ω_ι is a standard technique (Cf. the paper on algebraic Smith theory, [14]).

Lemma 7.1.4. *Let (V, φ) be an object of $\mathscr{R}(K)$ with $\dim V < \mathbf{TrDeg}(K)$ and n be a natural number. Then*

$$I_{(V,\varphi)}(n)[\omega_\iota^{-1}] = 0.$$

Proof. By Proposition 6.1.9 and the exactness of the localization functor $(-)[\omega_\iota^{-1}]$, it suffices to prove that $\Sigma^k H^*(V)(\varphi)[\omega_\iota^{-1}] = 0$. Moreover, the suspension functor commutes with localization, so it suffices to treat the case $k = 0$. The hypothesis on V implies that $\varphi(\omega_\iota) = 0$, by Theorem 7.1.2, hence the result follows. □

7.2. **Transcendence degree for modules.** There is a notion of transcendence degree of an object of K–$\mathscr{U}$, defined in terms of the structure of the underlying unstable module, which is implicit in the work of Henn, Lannes and Schwartz, [12]. For the purposes of this paper, it is sufficient to use the following notion, defined in terms of the structure in K–$\mathscr{U}$.

Definition 7.2.1. Let K be a Noetherian unstable algebra.

(1) [10, Definition 2.8] The T-support of M is the set of objects of $\mathscr{R}(K)$:
$$T\text{–Supp}(M) := \{(V,\varphi) \in \mathrm{Obj}\ \mathscr{R}(K) | T_{V,\varphi}M \neq 0\}.$$
(2) Let M be an object of K_{fg}–$\mathscr{U}$; the K–$\mathscr{U}$-transcendence degree of M is
$$\mathbf{TrDeg}_{K\text{–}\mathscr{U}} M := \sup\{\dim V | (V,\varphi) \in T\text{–Supp}(M)\}.$$

This notion of K–$\mathscr{U}$-transcendence degree behaves well under restriction and induction.

Proposition 7.2.2. *Let $\alpha : K \to L$ be a morphism of Noetherian unstable algebras for which L is a finitely generated K-module and M, N be objects of K_{fg}–$\mathscr{U}$, L_{fg}–$\mathscr{U}$ respectively. Then the following equalities hold:*

(1) $\mathbf{TrDeg}_{K\text{–}\mathscr{U}}(\alpha^* N) = \mathbf{TrDeg}_{L\text{–}\mathscr{U}} N$;
(2) $\mathbf{TrDeg}_{L\text{–}\mathscr{U}}(L \otimes_K M) = \mathbf{TrDeg}_{K\text{–}\mathscr{U}} M$.

Proof. The first statement follows by a straightforward argument using the identification
$$T_{V,\varphi}(\alpha^* N) = \bigoplus_{(V,\psi) \in \mathscr{R}(\alpha)^{-1}(V,\varphi)} T_{V,\psi} N,$$
for $(V,\varphi) \in \mathscr{R}(K)$.

For the second statement, consider $(V,\psi) \in \mathscr{R}(L)$ and its image $\mathscr{R}(\alpha)(V,\psi) = (V,\psi\alpha)$. There is an identification
$$T_{V,\psi}(L \otimes_K M) \cong T_{V,\psi} L \otimes_{T_{V,\psi\alpha} K} T_{V,\psi\alpha} M.$$
Hence, if $T_{V,\psi}(L \otimes_K M)$ is non-trivial, so is $T_{V,\psi\alpha} M$. For the converse, the commutative triangle
$$\begin{array}{ccc} K & \xrightarrow{\alpha} & L \\ {\scriptstyle \psi\alpha}\downarrow & \swarrow{\scriptstyle \psi} & \\ H^*(V) & & \end{array}$$
induces morphisms of (finitely generated) Boolean algebras

$$T^0_{V,\psi\alpha} K \to T^0_{V,\psi} L \to \mathbb{F} \qquad (1)$$

by adjunction, with non-trivial composite. Hence the unit provides a section $\mathbb{F} \xrightarrow{\eta} T^0_{V,\psi\alpha} K$ to the composite in the category of Boolean algebras.

Suppose that $T_{V,\psi\alpha} M \neq 0$ and let d be the minimal degree in which $T_{V,\psi\alpha} M$ is non-trivial, so that there is an isomorphism:
$$T^d_{V,\psi}(L \otimes_K M) \cong T^0_{V,\psi} L \otimes_{T^0_{V,\psi\alpha} K} T^d_{V,\psi\alpha} M.$$
The morphisms (1) induce a morphism
$$T^0_{V,\psi} L \otimes_{T^0_{V,\psi\alpha} K} T^d_{V,\psi\alpha} M \to T^d_{V,\psi\alpha} M$$
which admits a section induced by η and is therefore surjective. It follows that $T^d_{V,\psi}(L \otimes_K M)$ is non-trivial, which concludes the proof. □

Definition 7.2.3. Let M be an object of K_{fg}–$\mathscr{U}$. The annihilator ideal $\mathrm{Ann}_K(M)$ is the graded ideal $\{k \in K \mid km = 0,\ \forall \in M\}$.

The following Lemma is standard.

Lemma 7.2.4. *The annihilator ideal* $\mathrm{Ann}_K M$ *is closed under the action of the Steenrod algebra, hence* $\overline{K} := K/\mathrm{Ann}_K M$ *is naturally a Noetherian unstable algebra and the canonical projection* $\alpha : K \to \overline{K}$ *is a morphism of unstable algebras.*

Moreover, $M \cong \alpha^* \overline{M}$ *where* $\overline{M}$ *is the object of* $\overline{K}_{fg}$–$\mathscr{U}$ *induced by* M*.*

Definition 7.2.5. For M an object of K_{fg}–$\mathscr{U}$, let $\mathbf{TrDeg}_K M$ denote the transcendence degree of the Noetherian algebra $\overline{K} := K/\mathrm{Ann}_K M$.

Proposition 7.2.6. *Let* M *be an object of* K_{fg}–$\mathscr{U}$*. There is an equality*

$$\mathbf{TrDeg}_{K\text{–}\mathscr{U}} M = \mathbf{TrDeg}_K M.$$

Proof. Let $\overline{K}$ and $\overline{M}$ be as in Lemma 7.2.4, so that M identifies with $\alpha^*\overline{M}$. Proposition 7.2.2 implies that $\mathbf{TrDeg}_{K\text{–}\mathscr{U}} M = \mathbf{TrDeg}_{\overline{K}\text{–}\mathscr{U}} \overline{M}$.

The results of Henn, Lannes and Schwartz recalled in Theorem 7.1.1 imply the inequality

$$\mathbf{TrDeg}_{\overline{K}\text{–}\mathscr{U}} \overline{M} \leq \mathbf{TrDeg} \overline{K} = \mathbf{TrDeg}_K M,$$

hence it suffices to prove the reverse inequality.

Consider the element ω_ι defined in Notation 7.1.3 with respect to a chosen embedding $\iota : D_{n,s} \hookrightarrow \overline{K}$. The fact that M is a Noetherian module and the definition of $\overline{K}$ implies that the localization $\overline{M}[\omega_\iota^{-1}] \neq 0$ is non-trivial.

There exists an embedding in $\overline{K}$–$\mathscr{U}$ of the form

$$\overline{M} \hookrightarrow \bigoplus_{i \in \mathscr{I}} I_{(V_i, \varphi_i)}(a_i)$$

such that, for each $i \in \mathscr{I}$, the component $\overline{M} \hookrightarrow I_{(V_i,\varphi_i)}(a_i)$ is non-trivial. The transcendence degree of $\overline{K}$ is n, hence $\dim V_i \leq n$, for each n.

By the exactness of localization, this implies that there exists an $i \in \mathscr{I}$ for which $I_{(V_i,\varphi_i)}(a_i)[\omega_\iota^{-1}] \neq 0$. Lemma 7.1.4 implies that, for this i, $\dim V_i = n$. It follows that $\mathbf{TrDeg}_{\overline{K}\text{–}\mathscr{U}} \overline{M}$ is equal to n, as required. □

7.3. **Depth.** Bourguiba and Zarati showed the fundamental rôle of the invariant depth in studying modules over unstable algebras. Namely, a key ingredient in Bourguiba and Zarati's proof of the Landweber-Stong conjecture is [2, Proposition 3.2.1], which gives information on the injective envelope of an object of $H^*(V)_{fg}$–$\mathscr{U}$ in terms of its depth.

This result implies the following:

Proposition 7.3.1. *Let* $0 \neq N \hookrightarrow M$ *be a monomorphism of* K_{fg}–$\mathscr{U}$*, then*

$$\mathbf{TrDeg}_K N = \mathbf{TrDeg}_{K\text{–}\mathscr{U}} N \geq \mathbf{Depth}_K M.$$

Proof. The equality $\mathbf{TrDeg}_K N = \mathbf{TrDeg}_{K\text{–}\mathscr{U}} N$ is given by Proposition 7.2.6. Hence it suffices to show that $\mathbf{TrDeg}_{K\text{–}\mathscr{U}} N \geq \mathbf{Depth}_K M$.

The result can be reduced to the case $K = H^*(V)$ by the standard argument, as in [2, Section 4]. Namely, there exists a monomorphism $D_{n,s} \hookrightarrow K$, for s a natural number and n the transcendence degree of K. The inclusion $D_{n,s} \hookrightarrow H^*(\mathbb{F}^n)$ is faithfully flat and hence there is an induced monomorphism

$$0 \neq H^*(\mathbb{F}^n) \otimes_{D_{n,s}} N \hookrightarrow H^*(\mathbb{F}^n) \otimes_{D_{n,s}} M.$$

Moreover, $\mathbf{TrDeg}_{H^*(\mathbb{F}^n)\text{–}\mathscr{U}}(H^*(\mathbb{F}^n) \otimes_{D_{n,s}} N) = \mathbf{TrDeg}_{K\text{–}\mathscr{U}} N$, by applying Proposition 7.2.2 to the inclusions $H^*(\mathbb{F}^n) \hookleftarrow D_{n,s} \hookrightarrow K$ and

$$\mathbf{Depth}_K M = \mathbf{Depth}_{H^*(\mathbb{F}^n)}(H^*(\mathbb{F}^n) \otimes_{D_{n,s}} M),$$

as in the proof of [2, Proposition 2.4.3]. Hence, it suffices to prove the result for $K = H^*(\mathbb{F}^n)$.

The results of Bourguiba and Zarati, notably [2, Proposition 3.2.1], imply that there exists an embedding in K–$\mathscr{U}$ of the form:

$$M \hookrightarrow \bigoplus_{i\in\mathscr{I}} I_{(V_i,\varphi_i)}(m_i)$$

in which $\dim V_i \geq \mathbf{Depth}_K M$, for each $i \in \mathscr{I}$. Hence there exists a monomorphism in K_{fg}–$\mathscr{U}$ of the form $N \hookrightarrow \bigoplus_{i\in\mathscr{I}} I_{(V_i,\varphi_i)}(m_i)$ and therefore a non-trivial morphism $N \rightarrow I_{(V_i,\varphi_i)}(m_i)$, for some $i \in \mathscr{I}$. It follows that $(V_i, \varphi_i) \in T$–Supp(N), so that $\mathbf{TrDeg}_{K\text{–}\mathscr{U}} N \geq \dim V_i \geq \mathbf{Depth}_K M$, as required. □

7.4. **The invariant d_0.** The previous results concern which objects of $\mathscr{R}(K)$ index injectives in the first term of an injective resolution of an object of K_{fg}–$\mathscr{U}$. It is also necessary to have an understanding of the invariant $d_0 M$ in terms of the beginning of the resolution.

Proposition 7.4.1. *The integer d_0M coincides with the least integer t for which M admits an embedding of the form $M \hookrightarrow \bigoplus_{i\in\mathscr{I}} I_{(V_i,\varphi_i)}(a_i)$ in K_{fg}–$\mathscr{U}$, where $a_i \leq t$, for each i and $(V_i, \varphi_i) \in \mathscr{R}(K)$.*

Proof. The result is proved by an induction upon d_0M. The initial step is the proof that an object of K_{fg}–$\mathscr{U}$ which is reduced embeds in an injective of the from $\bigoplus_i I_{(V_i,\varphi_i)}(0)$ (which is reduced). The inductive step is a straightforward generalization of the initial step using the horseshoe lemma; the details are left to the reader. □

7.5. **Conclusion.** The previous results combine to give the following result:

Theorem 7.5.1. *Let K be a Noetherian unstable algebra and M be an object of K_{fg}–$\mathscr{U}$, then there exists an embedding in K_{fg}–$\mathscr{U}$*

$$M \hookrightarrow \bigoplus_{i\in\mathscr{I}} I_{(V_i,\varphi_i)}(n_i),$$

where
$\mathscr{I}$ is a finite indexing set and

1. *$n_i \leq d_0M$, with equality for some i;*
2. *$(V_i, \varphi_i) \in \mathrm{Obj}\ \mathscr{R}(K)$ and*

$$\mathbf{Depth}_K M \leq \dim V_i \leq \mathbf{TrDeg}_K M.$$

Proof. Consider some embedding $M \hookrightarrow \bigoplus_{i\in\mathscr{I}} I_{(V_i,\varphi_i)}(n_i)$ as provided by Proposition 7.4.1, so that $n_i \leq d_0M$, for each $i \in \mathscr{I}$. This embedding factors across the inclusion of the direct sum of the factors indexed by i such that $\dim V_i \leq \mathbf{TrDeg}_K M = \mathbf{TrDeg}_{K\text{–}\mathscr{U}} M$, by the definition of $\mathbf{TrDeg}_{K\text{–}\mathscr{U}} M$.

Finally consider the projection onto the factors with $\dim V_i \geq \mathbf{Depth}_K M$ and let N be the kernel of this projection. By construction, the unstable module $N \in \mathrm{Obj}\ K_{fg}$–$\mathscr{U}$ embeds in a finite direct sum of injectives of the form $I_{(W,\psi)}(n)$, with $\dim W < \mathbf{Depth}_K M$. This implies that $\mathbf{TrDeg}_{K\text{–}\mathscr{U}} N < \mathbf{Depth}_K M$. Hence, Proposition 7.3.1 implies that $N = 0$, which concludes the proof. □

8. The nilpotent filtration and induction

8.1. **General results.** In order to simplify the theory, the following restriction will be placed on the base-change morphism.

Hypothesis 8.1.1. Let $\alpha : K \rightarrow L$ be a morphism of unstable algebras where K, L are connected, Noetherian unstable algebras such that α makes L a flat, finitely generated K-module.

Remark 8.1.2.

(1) The restriction to considering connected unstable algebras is not significant, since one can reduced to this situtation by passage to connected components.
(2) The hypothesis implies that L is a finitely generated free K-module (Cf. [18, Proposition A.1.5]). In particular, L is a faithfully flat K-module.

Lemma 8.1.3. *Let $\alpha : K \rightarrow L$ be a morphism of unstable algebras which satisfies Hypothesis 8.1.1 and let M be an object of K_{fg}–$\mathscr{U}$. There is an exact sequence in K_{fg}–$\mathscr{U}$:*

$$0 \rightarrow M \rightarrow L \otimes_K M \rightarrow L \otimes_K L \otimes_K M.$$

In particular $d_0 M \leq d_0(L \otimes_K M)$ and $d_1 M \leq \max\{d_1(L \otimes_K M), d_0(L \otimes_K L \otimes_K M)\}$.

Proof. The exact sequence is provided by faithfully flat descent. The inequalities follow from Proposition 4.2.3. □

The hypothesis that L is K-flat implies that an embedding in K_{fg}–$\mathscr{U}$ supplied by Theorem 7.5.1 induces a monomorphism in L_{fg}–$\mathscr{U}$

$$L \otimes_K M \hookrightarrow \bigoplus_{i \in \mathscr{I}} L \otimes_K I_{(V_i, \varphi_i)}(n_i).$$

Lemma 8.1.4. *Let (V, φ) be an object of $\mathscr{R}(K)$ and n be a natural number, then*

$$d_0(L \otimes_K I_{(V,\varphi)}(n)) = n + d_0(L \otimes_K H^*(V)(\varphi)),$$

where $H^*(V)(\varphi) \in K_{fg}$–$\mathscr{U}$ is the object $H^*(V)$ considered as a K-module via φ.

Proof. The result follows from the basic properties of the invariant d_0 recalled in Proposition 4.2.3. There is an inclusion $\Sigma^n(L \otimes_K H^*(V)(\varphi)) \hookrightarrow L \otimes_K I_{(V,\varphi)}(n)$, which implies that $d_0(L \otimes_K I_{(V,\varphi)}(n)) \geq n + d_0(L \otimes_K H^*(V)(\varphi))$. For the reverse inequality, Proposition 6.1.9 provides a filtration of $I_{(V,\varphi)}(n)$ with filtration quotients of the form $\Sigma^k H^*(V)(\varphi)$, with $0 \leq k \leq n$; again the result follows from Proposition 4.2.3. □

As a consequence, one obtains the weak general bound given by:

Proposition 8.1.5. *Let $M \hookrightarrow \bigoplus_{i \in \mathscr{I}} I_{(V_i, \varphi_i)}(n_i)$ be a monomorphism in K_{fg}–$\mathscr{U}$, where $0 \leq n_i \leq d_0 M$, then*

$$d_0(L \otimes_K M) \leq d_0 M + \max_{i \in \mathscr{I}} \{d_0(L \otimes_K H^*(V_i)(\varphi_i))\}.$$

Proof. The result follows from Lemma 8.1.4, using Proposition 4.2.3 to provide the inequality. □

8.2. **Base change for $H^*(V)(\varphi)$.** Proposition 8.1.5 reduces considerations to the study of $d_0(L \otimes_K H^*(V)(\varphi))$, for (V, φ) an object of $\mathscr{R}(K)$.

Lemma 8.2.1. *The object $L \otimes_K H^*(V)(\varphi)$ is an unstable algebra which belongs to $H^*(V){\downarrow}\mathscr{K}$ and defines an object of $H^*(V)_{fg}$–$\mathscr{U}$ with underlying $H^*(V)$-module which is free.*

Proof. Straightforward; the hypothesis upon $K \hookrightarrow L$ implies that L is a finitely generated free K-module, which implies the final statement. □

Lemma 8.2.2. *The adjunction unit induces a monomorphism in $H^*(V){\downarrow}\mathscr{K}$:*

$$L \otimes_K H^*(V)(\varphi) \hookrightarrow H^*(V) \otimes \mathrm{Fix}_V(L \otimes_K H^*(V)(\varphi)).$$

Proof. The existence of the morphism follows from the adjunction properties of Fix_V when considering the category $H^*(V){\downarrow}\mathscr{K}$ (Cf. Proposition 5.3.1). To prove that the morphism is injective, it suffices to show injectivity in the category $H^*(V)$–$\mathscr{U}$. This follows from Corollary 5.5.3, using Lemma 8.2.1 which shows that $L \otimes_K H^*(V)(\varphi)$ satisfies the hypotheses. □

The unstable algebra $\mathrm{Fix}_V(L \otimes_K H^*(V)(\varphi))$ has total finite dimension; recall that $||X||$ denotes $\inf\{t \mid X^{t+1} = 0\}$.

Proposition 8.2.3. *Suppose that $\alpha : K \to L$ satisfies hypothesis 8.1.1 and (V, φ) is an object of $\mathscr{R}(K)$, then*

$$d_0(L \otimes_K H^*(V)(\varphi)) = ||\mathrm{Fix}_V(L \otimes_K H^*(V)(\varphi))||.$$

Proof. Lemma 8.2.2 implies that there is an inequality $d_0(L \otimes_K H^*(V)(\varphi)) \leq ||\mathrm{Fix}_V(L \otimes_K H^*(V)(\varphi))||$, hence it remains to establish the reverse inequality.

Corollary 5.4.3 identifies $\mathrm{Fix}_V(L \otimes_K H^*(V)(\varphi))$ with $\mathbf{lf}T_{V,\iota}(L \otimes_K H^*(V)(\varphi))$, hence $\mathbf{lf}T_V(L \otimes_K H^*(V)(\varphi))$ is non-zero in degree $||\mathrm{Fix}_V(L \otimes_K H^*(V)(\varphi))||$. The result follows from Corollary 4.3.4, which expresses the invariant d_0 in terms of $\mathbf{lf}$ and the T-functor. □

This result lends itself to explicit calculation using the identification of the functor Fix, via Proposition 5.2.2.

Lemma 8.2.4. *There is an isomorphism*

$$\mathrm{Fix}_V(L \otimes_K H^*(V)(\varphi)) \cong T_V L \otimes_{T_V K} \mathbb{F} \cong T_{V,\varphi} L \otimes_{T_{V,\varphi} K} \mathbb{F}$$

where $\mathbb{F}$ is a $T_V K$-module via the adjoint to φ, $T_V K \to \mathbb{F}$.

Proof. Proposition 5.2.2 identifies $\mathrm{Fix}_V(L \otimes_K H^*(V)(\varphi))$ with $\mathbb{F} \otimes_{T_V H^*(V)} \otimes T_V(L \otimes_K H^*(V)(\varphi))$, where $\mathbb{F}$ is a $T_V H^*(V)$-module via $\iota : T_V H^*(V) \to \mathbb{F}$. The first isomorphism follows by properties of the tensor product and the identification of the composite $T_V K \stackrel{T_V \varphi}{\to} T_V H^*(V) \to \mathbb{F}$ with φ, which is formal. The second isomorphism follows by passage to the connected component $T_{V,\varphi} K$. □

Combining Proposition 8.1.5, Proposition 8.2.3 and Lemma 8.2.4, one obtains the following result.

Corollary 8.2.5. *Let $M \hookrightarrow \bigoplus_{i \in \mathscr{I}} I_{(V_i, \varphi_i)}(a_i)$ be a monomorphism in K_{fg}-$\mathscr{U}$, where $0 \leq a_i \leq d_0 M$, then*

$$d_0(L \otimes_K M) \leq d_0 M + \max_{i \in \mathscr{I}} \{||T_{V_i,\varphi_i} L \otimes_{T_{V_i,\varphi_i} K} \mathbb{F}||\}.$$

8.3. **Generalities on calculating Fix.** The following result of Dwyer and Wilkerson [5] illustrates that the calculation of the object Fix as a graded vector space (which is sufficient for the current purposes) is accessible.

Proposition 8.3.1. [5, Proposition 4.1] *Let K be an unstable algebra and M be an object of K–$\mathscr{U}$ which is free as a K-module. Then, for any $(V, \varphi) \in \mathscr{S}(K)$, the unstable module $T_{V,\varphi} M$ in $T_{V,\varphi} K$–$\mathscr{U}$ is free as an $T_{V,\varphi} K$-module.*

This result applies, in particular, to the unstable algebra L, considered as an object of K–$\mathscr{U}$, under Hypothesis 8.1.1.

Corollary 8.3.2. *Let (V, φ) be an object of $\mathscr{R}(K)$. Each connected component of the unstable algebra $T_{V,\varphi} L$ is a free, finitely generated module over $T_{V,\varphi} K$.*

Proof. Proposition 8.3.1 implies that $T_{V,\varphi} L$ is a free, finitely generated module over $T_{V,\varphi} K$. Each connected component is a $T_{V,\varphi} K$-module, which is flat, since it is projective. The algebras and modules are graded and connected, hence the module is free. □

8.4. **Examples associated to rings of invariants.** To maintain clarity of exposition, the prime p is taken to be 2, so that $\mathbb{F}$ is the field with two elements. Hence, $H^*(V)$ is the symmetric algebra on the dual $V^\sharp$ of V.

Let G be a finite subgroup of $\mathrm{Aut}(V)$, then G acts on the right on $V^\sharp$ and hence acts by morphisms of unstable algebras upon $H^*(V)$ on the right. The ring of invariants $H^*(V)^G$ is a Noetherian unstable algebra (Cf. [18]).

We are interested in the case where $H^*(V)^G$ is a polynomial algebra. Dwyer-Wilkerson [5] attribute the following result to Serre; it is an equi-characteristic analogue of the Shephard-Todd, Chevalley theorem [18, Theorem 7.1.4]. Recall that a subgroup G of $\mathrm{Aut}(V)$ is generated by pseudoreflections if it is generated by the pseudoreflections which it contains. (A pseudoreflection is an element of finite order such that the image of $w - 1_V$ has rank one).

Proposition 8.4.1. *If the algebra $H^*(V)^G$ is polynomial, then G is generated by pseudoreflections.*

Throughout this section, the following Hypothesis is imposed.

Hypothesis 8.4.2. Let G be a finite subgroup of $\mathrm{Aut}(V)$ for which $H^*(V)^G$ is a polynomial algebra.

The hypothesis has important consequences; for instance, the following result is standard.

Proposition 8.4.3. [18, Corollary 4.5.4, Theorem 5.4.1] *Suppose that the group $G \subset \mathrm{Aut}(V)$ satisfies hypothesis 8.4.2 and that the polynomial generators have degrees d_i, $1 \leq i \leq n$, where $n = \dim V$. Then the following properties are satisfied:*

(1) $\prod_{i=1}^n d_i = |G|$.
(2) *The algebra $H^*(V) \otimes_{H^*(V)^G} \mathbb{F}$ is a Poincaré duality algebra of dimension $(\Sigma_{i=1}^n d_i) - n$.*

Remark 8.4.4. The terminology *algebra of coinvariants* is used for $H^*(V) \otimes_{H^*(V)^G} \mathbb{F}$ in invariant theory (Cf. [18] for example).

The calculation of the T-functor on rings of invariants is straightforward (Cf. [20, Proposition 3.9.8] or [5, Proof of 1.4]).

Proposition 8.4.5.

(1) *The category $\mathscr{R}(H^*(V)^G)$ has objects (W, φ), for W an $\mathbb{F}_2$-vector space and φ ranging ranges over the set of G-orbits $G\backslash\mathrm{Mono}(W, V)$.*
(2) *The component $T_{W,\varphi}(H^*(V)^G)$ is isomorphic to the algebra of invariants $H^*(V)^{G_\varphi}$, where G_φ denotes the pointwise stabilizer in G of a representative in $\mathrm{Mono}(W, V)$ of the orbit φ.*
(3) *The unstable algebra $T_{W,\varphi}(H^*(V))$ is isomorphic to the unstable algebra $\prod_{[G_\varphi\backslash G]} H^*(V)$, where $[G_\varphi\backslash G]$ denotes the set of cosets.*
(4) *The monomorphism of unstable algebras $T_{W,\varphi}(H^*(V)^G) \hookrightarrow T_{W,\varphi}(H^*(V))$ has component indexed by the coset $G_\varphi g$ given by the monomorphism of unstable algebras*

$$H^*(V)^{G_\varphi} \hookrightarrow H^*(V) \xrightarrow{.g} H^*(V)$$

where $.g$ is induced by the right action of $\mathrm{Aut}(V)$.

Dwyer and Wilkerson [5] prove that the T-functor preserves finitely generated polynomial algebras (after passage to connected components). In particular, [5, Theorem 1.4] shows that the rings of invariants $H^*(V)^{G_\varphi}$ are also polynomial algebras, under the given hypothesis upon G.

Corollary 8.4.6. *Suppose that the group G satisfies hypothesis 8.4.2 and let (W, φ) be an object of $\mathscr{R}(H^*(V)^G)$, then*

$$d_0(H^*(V) \otimes_{H^*(V)^G} H^*(W)) = ||H^*(V) \otimes_{H^*(V)^{G_\varphi}} \mathbb{F}||$$

where the morphism of unstable algebras $H^*(V)^{G_\varphi} \to H^*(W)$ is induced by a monomorphism $W \hookrightarrow V$ in the orbit φ.

Proof. The Corollary is proved by applying Proposition 8.2.3 and Lemma 8.2.4 in the context of the invariant rings, using Proposition 8.4.5 to reduce to the consideration of a single component. □

Example 8.4.7. There are two examples of groups $G \subset \mathrm{Aut}(V)$ which are of particular interest, namely the case $G = \mathrm{Aut}(V)$, which corresponds to the Dickson invariants, and the case $G = \mathfrak{S}_n$, where $n = \dim V$ and $\mathfrak{S}_n$ acts by permuting a choice of basis. In both these examples, the associated rings of invariants are polynomial.

9. Symmetric invariants and induction

This section specializes the considerations of the section 8.4 to the case where the group G is the symmetric group. Throughout the section, the prime p is taken to be two; the monomorphism of unstable algebras $K \hookrightarrow L$ is taken to be the inclusion of the algebra of invariants

$$H^*(V)^{\mathfrak{S}_n} \hookrightarrow H^*(V)$$

where $V = \mathbb{F}^n$ for a fixed positive integer n, with $\mathfrak{S}_n$ acting upon $\mathbb{F}^n$ by permuting a given basis $\{y_1, \ldots, y_n\}$.

9.1. **Pointwise stabilizers of subspaces.**

The purpose of this section is to provide an analysis of the pointwise stabilizer of a subspace $W \leq V$. The main result of the section is Proposition 9.1.5. The precision provided by Lemma 9.1.4 is important.

Let $V = \mathbb{F}^n$ be as above, with chosen basis $\{y_1, \ldots, y_n\}$, considered as a set B_n. There is a bijection between vectors of V and subsets of B_n, since $\mathbb{F}$ is the field with two elements. Under this bijection, a subset $T \subset B_n$ defines a vector v_T and the addition of the vector space corresponds to the law

$$v_{T_1} + v_{T_2} = v_{T_1 \cup T_2 \backslash T_1 \cap T_2}.$$

Lemma 9.1.1. *The isotropy group of the vector v_T is the subgroup $\mathrm{Aut}(T) \times \mathrm{Aut}(T') \subset \mathfrak{S}_n$, where T' denotes the complement of T in B_n.*

Proof. Straightforward. □

The pointwise stabilizer in $\mathfrak{S}_n$ of a subspace $W \subset V$ is the intersection of the isotropy groups of a basis of the space W. A basis of a vector subspace W of dimension d is defined by a set $\{T_1, \ldots, T_d\}$ of subsets of B_n (which satisfies a suitable condition which corresponds to linear independence).

Definition 9.1.2. For $\{T_1, \ldots, T_d\}$ a set of subsets of B_n, let $\mathscr{T}$ denote the subset of non-empty subsets of B_n which are of the form

$$\bigcap_{i=1}^{d} T_i^{\pm}$$

where $T_i^{\pm}$ denotes either T_i or its complement T_i'.

Lemma 9.1.3.

(1) *If X, Y are distinct elements of $\mathscr{T}$, then $X \cap Y = \emptyset$.*
(2) $\bigcup_{X \in \mathscr{T}} X = B_n$.

(3) *The set $\mathscr{T}$ depends only upon the vector space which is generated by the vectors v_{T_i}.*

Proof. The first and second statements are straightforward. To prove the third statement it is sufficient to check that replacing the pair $\{T_1, T_2\}$ by $\{T_1, (T_1 \cap T_2') \cup (T_1' \cap T_2)\}$ does not change $\mathscr{T}$. This is straightforward. □

Hence, the above construction associates to the subspace W a partition of the set B_n, corresponding to a set $\mathscr{T}_W$ of subsets of B_n.

Lemma 9.1.4. *The set $\mathscr{T}_W$ has cardinality at least* $\dim W$.

Proof. The proof is by induction upon $\dim W$. Consider $U \subset W$ of codimension one; the sets of $\mathscr{T}_W$ are of the form $X \cap T$ or $X \cap T'$, where T represents a vector of $W \backslash U$ and X is an element of $\mathscr{T}_U$. For each X, at least one of these is non-trivial, hence $|\mathscr{T}_W| \geq |\mathscr{T}_U|$. Thus, it remains only to consider the case when $|\mathscr{T}_U| = \dim U$. In this case, it is straightforward to check that $U = \langle v_X \mid X \in \mathscr{T}_U \rangle$. Suppose that there is equality $|\mathscr{T}_W| = |\mathscr{T}_U|$; then T is the disjoint union of the sets X in $\mathscr{T}_U$ such that $T \cap X \neq \emptyset$. This implies that the vector v_T lies in W, which contradicts the fact that U is of codimension one. Hence $|\mathscr{T}_W| > |\mathscr{T}_U| = \dim U$, which proves the result. □

A monomorphism of $\mathbb{F}_2$-vector spaces, $c : W \hookrightarrow V$, is fixed by the left action of $\mathfrak{S}_n$ if and only if the subspace $c(W)$ is stabilized pointwise by the action of $\mathfrak{S}_n$ upon V.

Proposition 9.1.5. *The pointwise stabilizer $\mathfrak{S}_W \subset \mathfrak{S}_n$ of the subspace $W \leq V$ is the subgroup*

$$\mathfrak{S}_W \cong \prod_{X \in \mathscr{T}_W} \mathrm{Aut}(X) \subset \mathfrak{S}_n.$$

Proof. The result follows from Lemma 9.1.1 by an analysis of the intersections of the isotropy groups associated to a basis. The details are left to the reader. □

9.2. **Invariants for products of symmetric groups.** The invariants and the coinvariants corresponding to the action of $\mathfrak{S}_n$ upon $V = \mathbb{F}^n$ are well understood. The ring of invariants $H^*(V)^{\mathfrak{S}_n}$ is a polynomial algebra on the elementary symmetric functions $\sigma_i(x_1, \ldots, x_n)$, $1 \leq i \leq n$, where $\{x_j\}$ denotes the dual basis for $V^\sharp$ associated to the basis $\{y_j\}$. In particular $|\sigma_i| = i$ and the algebra $\mathbb{F}[x_j \mid 1 \leq j \leq n] \otimes_{\mathbb{F}[\sigma_j \mid 1 \leq j \leq n]} \mathbb{F}$ is a Poincaré duality algebra of dimension $\Sigma_{j=1}^n (j-1) = \frac{1}{2}n(n-1)$, by Proposition 8.4.3.

These considerations can be generalized to subgroups of the symmetric group which are of the form appearing in Proposition 9.1.5. There is a Künneth formula for the calculation of rings of invariants: if G_i is a subgroup of $\mathrm{Aut}(V_i)$, for $i \in \{1, 2\}$ then $G_1 \times G_2$ is a subgroup of $\mathrm{Aut}(V_1 \oplus V_2)$ and there are isomorphisms

$$H^*(V_1 \oplus V_2)^{G_1 \times G_2} \cong (H^*(V_1) \otimes H^*(V_2))^{G_1 \times G_2} \cong H^*(V_1)^{G_1} \otimes H^*(V_2)^{G_2}.$$

This formula extends by induction to arbitrary direct sums of vector spaces V_i and groups $G_i \subset \mathrm{Aut}(V_i)$.

Notation 9.2.1. If A is a Poincaré duality algebra over $\mathbb{F}$ (see [18, Section 5.4]), write $||A||$ for the dimension of A, which is the degree of the fundamental class.

Remark 9.2.2. If A is an unstable algebra over the Steenrod algebra which is a Poincaré duality algebra, the two usages of $||A||$ which have been introduced coincide, so this notation is consistent when applied to rings of invariants.

Lemma 9.2.3. *Let $G_i \subset \mathrm{Aut}(V_i)$, $1 \leq i \leq N$ be subgroups, where V_i is an elementary abelian p-group for each i. Then the following properties hold.*

(1) *The ring of invariants is polynomial.*
(2) *The Poincaré duality algebra* $H^*(\bigoplus_i V_i) \otimes_{H^*(\bigoplus_i V_i)^{\times_i G_i}} \mathbb{F}$ *has dimension*

$$||H^*(\bigoplus_i V_i) \otimes_{H^*(\bigoplus_i V_i)^{\times_i G_i}} \mathbb{F}|| = \Sigma_{i=1}^N ||H^*(V_i) \otimes_{H^*(V_i)^{G_i}} \mathbb{F}.||$$

Proof. Straightforward. □

Proposition 9.1.5 identifies the pointwise stabilizer $\mathfrak{S}_W$ of $W \leq \mathbb{F}^n$. The previous results therefore imply the following.

Proposition 9.2.4. *Let* $W \leq V = \mathbb{F}^n$ *be a subspace. The Poincaré duality algebra* $H^*(V) \otimes_{H^*(V)^{\mathfrak{S}_W}} \mathbb{F}$ *has dimension:*

$$||H^*(V) \otimes_{H^*(V)^{\mathfrak{S}_W}} \mathbb{F}|| = \Sigma_{X \in \mathscr{T}_W} \frac{1}{2}|X|(|X|-1).$$

For the application, it is necessary to have a weak quantitative understanding of the sum above in terms of the number of elements $D = |\mathscr{T}_W|$ of the set $\mathscr{T}_W$.

Notation 9.2.5. Choose a bijection between the elements of $\mathscr{T}_W$ and $\{1, \ldots, D\}$ and write X_j for the element of $\mathscr{T}_W$ indexed by $j \in \{1, \ldots, D\}$. The cardinality $|X_j|$ will be written m_j.

There is an equality

$$\Sigma_{j=1}^D \frac{1}{2} m_j(m_j - 1) = \frac{1}{2} n(n-1) - \Sigma_{1 \leq j < k \leq D}\ m_j m_k,$$

where $n = \Sigma_{j=1}^D m_j$. The natural numbers m_j are non-zero, hence $n \geq D$ with equality if and only if $m_j = 1$ for each j.

Lemma 9.2.6. *There is an inequality*

$$\Sigma_{j=1}^D \frac{1}{2} m_j(m_j - 1) \leq \frac{1}{2} n(n-1) - \frac{1}{2} D(D-1)$$

with equality if and only if $m_j = 1$ *for each* j*, when the expression equals zero.*

Proof. Straightforward. □

9.3. **Application to the invariant** d_0**.** Corollary 8.4.6 expresses

$$d_0(H^*(V) \otimes_{H^*(V)^G} H^*(W)),$$

for (W, φ) an object of $\mathscr{R}(H^*(V)^G)$, in terms of the dimension of the appropriate Poincaré duality algebra. This result applies in the case $V = \mathbb{F}^n$, $G = \mathfrak{S}_n$ where φ is represented by a monomorphism $W \hookrightarrow \mathbb{F}^n$. By abuse of notation, we identify W with its image in $\mathbb{F}^n$, so that the isotropy group G_φ can be identified with $\mathfrak{S}_W$.

Corollary 9.3.1. *Let* (W, φ) *be an object of* $\mathscr{R}(H^*(\mathbb{F}^n)^{\mathfrak{S}_n})$*. There is an inequality:*

$$d_0(H^*(\mathbb{F}^n) \otimes_{H^*(\mathbb{F}^n)^{\mathfrak{S}_n}} H^*(W)) \leq \frac{1}{2} n(n-1) - \frac{1}{2} \dim W (\dim W\ - 1).$$

Proof. The result follows from Corollary 8.4.6 by applying Proposition 9.2.4 together with the inequality provided by Lemma 9.2.6, using Lemma 9.1.4 which implies that $\dim W \leq D = |\mathscr{T}_W|$. □

Corollary 9.3.2. *Let* $M \hookrightarrow \bigoplus_{i \in \mathscr{I}} I_{(W_i, \varphi_i)}(a_i)$ *be a monomorphism in* $H^*(\mathbb{F}^n)^{\mathfrak{S}_n}_{fg}$–$\mathscr{U}$*, where* $0 \leq a_i \leq d_0 M$*, and* $d \leq \dim W_i \leq \mathbf{TrDeg}_{\mathscr{U}} M$*. Then*

$$d_0 M \leq d_0(H^*(\mathbb{F}^n) \otimes_{H^*(\mathbb{F}^n)^{\mathfrak{S}_n}} M) \leq d_0 M + \frac{1}{2} n(n-1) - \frac{1}{2} d(d-1).$$

Proof. The result is a consequence of Corollary 9.3.1 and Proposition 8.1.5. □

10. Group cohomology with $\mathbb{F}_2$-coefficients

The cohomology of a finite group with coefficients in a prime field $\mathbb{F}_p$ is an unstable algebra, which is Noetherian by the fundamental result of Venkov [21] and, by an algebraic approach, Evens [6]. The group cohomology is therefore amenable to study as a module over the Steenrod algebra; in particular the invariants $d_0 H^*(BG; \mathbb{F}_p)$ and $d_1 H^*(BG; \mathbb{F}_p)$ are defined and are finite integers.

Henn, Lannes and Schwartz used the methods of Quillen [19] together with results of Duflot on smooth toral actions [4] to provide an upper bound on d_0 and d_1 [12, Theorem 0.5]. The calculation of this bound involves an induction argument, which can lead to this bound from being far from sharp, even when the bound derived from Duflot is sharp. The algebraic considerations of this paper shed some light on this.

These results have analogues for the equivariant cohomology of a finite G CW-complex. For simplicity of presentation, this is not considered here.

10.1. Methods of Quillen and Henn-Lannes-Schwartz.

Quillen [19] provided an understanding of the algebraic variety corresponding to $H^*(BG; \mathbb{F}_p)$. These results have an elegant interpretation in terms of algebras over the Steenrod algebra, due to Lannes, using the T-functor (see [15] and [12]).

Notation 10.1.1. In this section, the group cohomology $H^*(BG; \mathbb{F}_p)$ is denoted simply by $H^*(BG)$; in particular, the cohomology $H^*(V)$ of an elementary abelian p-group V is written as $H^*(BV)$.

There are two fundamental results on the structure of the ring $H^*(BG)$ which are relevant here, due to Quillen and Duflot respectively.

Theorem 10.1.2. *Let G be a finite group.*

(1) [19] *The transcendence degree of the unstable algebra $H^*(BG)$ is equal to the maximal rank of an elementary abelian p-subgroup of G.*
(2) [3] *The depth of the algebra $H^*(BG)$ is at least the rank of the maximal elementary abelian p-subgroup of the centre of G.*

At the prime two, rather than considering an embedding of G into a unitary group, it is usual to consider an embedding in an orthogonal group. (Recall that the orthogonal group $O(n)$ is a compact Lie group of dimension $\frac{1}{2}n(n-1)$).

Let G be a finite group and choose an embedding $G \hookrightarrow O(n)$ in an orthogonal group. Let T be a maximal torus, which is of rank n, and let V be the 2-torus in T of elements of order 2.

There are induced morphism of Noetherian unstable algebras

$$\begin{array}{ccc} H^*(BO(n)) & \longrightarrow & H^*(BV) \\ \downarrow & & \\ H^*(BG), & & \end{array}$$

where cohomology is taken with $\mathbb{F}_2$ coefficients. The algebra $H^*(BV)$ is the symmetric algebra and $H^*(BO(n))$ identifies with the symmetric invariants, via the canonical inclusion. The methods of Venkov imply that $H^*(BG)$ is a finitely generated module over the algebra $H^*(BO(n))$.

There is a morphism of fibrations

$$\begin{array}{ccccc} O(n)/V & \longrightarrow & EG \times_G (O(n)/V) & \longrightarrow & BG \\ \| & & \downarrow & & \downarrow \\ O(n)/V & \longrightarrow & BV & \longrightarrow & BO(n). \end{array}$$

The right hand square induces a morphism of unstable algebras

$$H^*(BV) \otimes_{H^*(BO(n))} H^*(BG) \to H^*(EG \times_G (O(n)/V)),$$

which an isomorphism, by the arguments of Quillen [19]. (Cf. Henn, Lannes and Schwartz [12, Section II.2], especially Example 2.2 and Proposition 2.4).

Under these hypotheses, writing F for $O(n)/V$, and letting X be a G-space, the diagram of projections

$$F \times F \times X \underset{p_2}{\overset{p_1}{\rightrightarrows}} F \times X \longrightarrow X$$

induces an equalizer diagram in the category of unstable algebras

$$H^*_G(X) \longrightarrow H^*_G(F \times X) \rightrightarrows H^*_G(F \times F \times X),$$

where H^*_G denotes Borel cohomology $H^*(EG \times_G -)$. The equalizer corresponds to that given by faithfully flat descent, as in Lemma 8.1.3.

Henn, Lannes and Schwartz use the equalizer diagram to obtain information on the invariants $d_0 H^*_G(X)$, $d_2 H^*_G(X)$ by considering the invariants d_0, d_1 for the algebras $H^*_G(F \times X)$ and $H^*_G(F \times F \times X)$ which are accessible by the geometric techniques of Duflot [4].

In the case $X = *$, this gives $d_0 H^*(BG) \leq \dim O(n) = \frac{1}{2}n(n-1)$; via the inequality

$$d_0(H^*(BV) \otimes_{H^*(BO(n)} H^*(BG)) \leq \dim O(n)$$

by using the isomorphism of unstable algebras $H^*(BV) \otimes_{H^*(BO(n))} H^*(BG) \cong H^*(EG \times_G (O(n)/V))$. The latter unstable algebra is isomorphic (as an unstable algebra) to the Borel cohomology algebra $H^*_V(G \backslash O(n))$ which is analysed by the methods of Duflot for smooth toral actions.

10.2. **Application of the induction results.** The algebraic result Corollary 9.3.2 can be applied to give information on the relationship between the invariants d_0 of the unstable algebras $H^*(BG)$ and $H^*(EG \times_G (O(n)/V)) \cong H^*(BV) \otimes_{H^*(BO(n))} H^*(BG)$, using the notation of the previous section.

Corollary 10.2.1. *Let G be a finite group equipped with an embedding $G \hookrightarrow O(n)$ and d be the depth of $H^*(BG)$. There are inequalities*

$$d_0 H^*(BG) \leq d_0(H^*(BV) \otimes_{H^*(BO(n))} H^*(BG)) \leq d_0 H^*(BG) + \frac{1}{2}n(n-1) - \frac{1}{2}d(d-1).$$

Proof. The result is a restatement of Corollary 9.3.2, using the depth via Theorem 7.5.1. □

Remark 10.2.2. An explicit lower bound for the depth in terms of the structure of the group G is given by the Theorem of Duflot [3] which is recalled in Theorem 10.1.2.

Example 10.2.3. There are two fundamental examples which shed light on the content of Corollary 10.2.1.

(1) Take $G = V$ the chosen 2-torus of $O(n)$. Then $H^*(BV)$ is nil-closed, so $d_0 H^*(BV) = 0$, and the depth is equal to the Krull dimension which is n. The previous result corresponds to the fact that $H^*(BV) \otimes_{H^*(BO(n))} H^*(BV)$ is reduced, which can be seen directly by algebraic methods. In this case, the bound provided by Henn, Lannes and Schwartz using the methods of Duflot is as far from being sharp as possible.
(2) Let G be the symmetric group $\mathfrak{S}_n$, which embeds in $O(n)$. It is known from the work of Gunawardena, Lannes and Zarati [8] that $H^*(B\mathfrak{S}_n)$ is nil-closed when coefficients are taken in $\mathbb{F}_2$, in particular the invariant d_0 is trivial. The inequality provided by the Theorem therefore recovers the bound for

$d_0(H^*(BV) \otimes_{H^*(BO(n))} H^*(B\mathfrak{S}_n))$ which is given by Henn, Lannes and Schwartz

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