Logarithmic residues along curves

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Geometry, Topology and Combinatorics of Hyperplane Arrangements and related problems
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Contents

Definitions

We set:

- $(D,0) \subset (\mathbb{C}^n, 0)$ a germ of reduced hypersurface defined by $f \in \mathbb{C}\{\underline{x}\}$, and $\mathscr{O}_D = \mathbb{C}\{\underline{x}\}/(f)$
- $\Theta_{\mathbb{C}^n}$: germs of holomorphic vector fields, $\Omega^q_{\mathbb{C}^n}$: germs of holomorphic q-forms

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- $Q(\mathscr{O}_D)$: total ring of fractions of $\mathscr{O}_D$
- $\Omega^q_D := \dfrac{\Omega^q_{\mathbb{C}^n}}{f \cdot \Omega^q_{\mathbb{C}^n} + \mathrm{d}f \wedge \Omega^{q-1}_{\mathbb{C}^n}}$: Kähler differentials on D

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Definition (K.Saito)

$$\Omega^q(\log D)=\left\{\omega\in\frac{1}{f}\Omega^q_{\mathbb{C}^n};\mathrm{d}f\wedge\omega\in\Omega^{q+1}_{\mathbb{C}^n}\right\}$$

$$\mathrm{Der}(-\log D)=\{\delta\in\Theta_{\mathbb{C}^n};\delta(f)\in(f)\}$$

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A divisor $(D,0)$ is **free** if $\mathrm{Der}(-\log D)$ (or equivalently $\Omega^1(\log D)$) is a free $\mathbb{C}\{\underline{x}\}$-module.

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Proposition (A.Aleksandrov)

Let D be a germ of reduced divisor. The following are equivalent:

- D is a free divisor
- The singular locus of D is Cohen-Macaulay of codimension one in D
- The Jacobian ideal $\mathscr{J}_D \subseteq \mathscr{O}_D$ is maximal Cohen-Macaulay

Logarithmic residues

Proposition (K.Saito)

A meromorphic q-form $\omega \in \frac{1}{f}\Omega^q_{\mathbb{C}^n}$ is logarithmic of and only if there exist $g \in \mathbb{C}\{\underline{x}\}$, which does not induce a zero divisor in $\mathscr{O}_D$, $\xi \in \Omega^{q-1}_{\mathbb{C}^n}$ and $\eta \in \Omega^q_{\mathbb{C}^n}$ such that:

$$g\omega = \frac{\mathrm{d}f}{f} \wedge \xi + \eta \qquad (1)$$

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Definition

Let $\omega \in \Omega^q(\log D)$. The residue of ω is

$$\mathrm{res}(\omega) = \frac{\xi}{g} \in \Omega^{q-1}_D \otimes_{\mathscr{O}_D} Q(\mathscr{O}_D)$$

Properties of $\mathscr{R}_D$

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- (M.Granger, M.Schulze)

$$\mathscr{J}_D^\vee := \mathrm{Hom}_{\mathscr{O}_D}(\mathscr{J}_D, \mathscr{O}_D) = \mathscr{R}_D$$

If moreover D is free, $\mathscr{R}_D^\vee = \mathscr{J}_D$.

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If moreover D is free, $\mathscr{R}_D^\vee = \mathscr{J}_D$.

Theorem (⇒ K.Saito, ⇐ M.Granger, M.Schulze)

D is normal crossing in codimension 1 iff $\mathscr{R}_D = \mathscr{O}_{\widetilde{D}}$

Symmetry of the values along curves

Let $C = \bigcup_{i=1}^{p} C_i$ be a **plane curve germ**, with normalization

$$\mathscr{O}_{\widetilde{C}} = \bigoplus_{i=1}^{p} \mathbb{C}\{t_i\}$$

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It induces a **valuation map** val_i along each C_i, with the convention $\mathrm{val}_i(0) = \infty$.
Let $g \in Q(\mathscr{O}_C)$. The **value** of g is:

$$\mathrm{val}(g) = (\mathrm{val}_1(g), \ldots, \mathrm{val}_p(g)) \in (\mathbb{Z} \cup \{\infty\})^p$$

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Definition

A **fractional ideal** $I \subset Q(\mathscr{O}_C)$ is a finite $\mathscr{O}_C$-submodule of $Q(\mathscr{O}_C)$ which contains a non zero divisor.
We set $\mathrm{val}(I) := \{\mathrm{val}(g); g \in I \text{ non zero divisor }\}$.

Semigroup of a curve

Definition

The semigroup of a curve is $\mathrm{val}(\mathscr{O}_C)$. There exists a minimal $\gamma \in \mathbb{N}^p$, called the **conductor**, such that $\gamma + \mathbb{N}^p \subseteq \mathrm{val}(\mathscr{O}_C)$.

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Proposition (E.Kunz)

If C is **irreducible**, $\mathscr{O}_C$ is Gorenstein iff

$$\forall v \in \mathbb{Z}, v \in \mathrm{val}(\mathscr{O}_C) \iff \gamma - v - 1 \notin \mathrm{val}(\mathscr{O}_C) \qquad (2)$$

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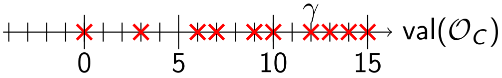
Example: $f(x,y) = x^3 - y^7$



We set:

$$\Delta_i(v, \mathrm{val}(I)) = \{\alpha \in \mathrm{val}(I)\ ;\ \alpha_i = v_i \text{ and } \forall j \neq i, \alpha_j > v_j\}$$

and $\Delta(v, \mathrm{val}(I)) = \bigcup_{i=1}^{p} \Delta_i(v, \mathrm{val}(I))$.

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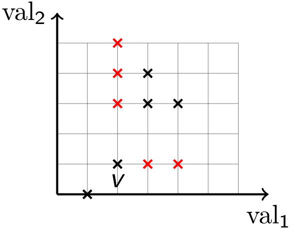
With $p = 2$, we represent in red the set $\Delta(v, \mathrm{val}(I))$, where the crosses denote the values of I.



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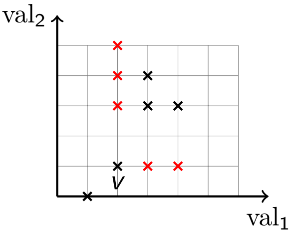

Theorem (F.Delgado de la Mata)

Let C be a curve with reduced ring $\mathscr{O}_C$. Then, $\mathscr{O}_C$ is a Gorenstein ring if and only if

$$\forall v \in \mathbb{Z}^p, v \in \mathrm{val}(\mathscr{O}_C) \iff \Delta(\gamma - v - \underline{1}, \mathrm{val}(\mathscr{O}_C)) = \varnothing \qquad (3)$$

Our purpose is to extend this symmetry to $\mathscr{R}_C$ and $\mathscr{J}_C$:

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Theorem (P.)

Let C be a plane curve. Then:

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Remark. In particular, it means that the values of $\mathscr{R}_C$ are determined by the values of $\mathscr{J}_C$.

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In fact, this symmetry is satisfied for

- any fractional ideal $I \subset Q(\mathscr{O}_C)$ and its dual $I^\vee$
- any Gorenstein curve C, in particular, complete intersection curves

Example: $f = (x^2 - y^3)(x^4 - y^3)$

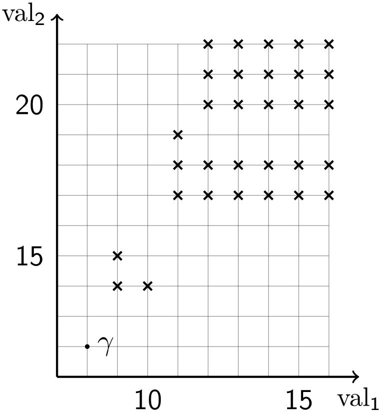

Figure : Values of $\mathscr{J}_C$

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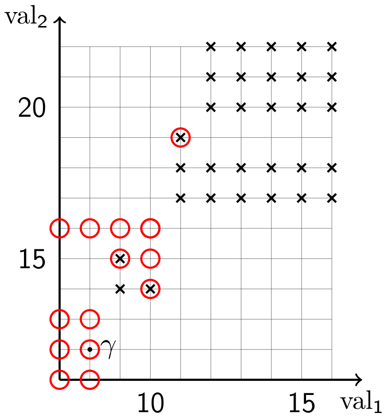

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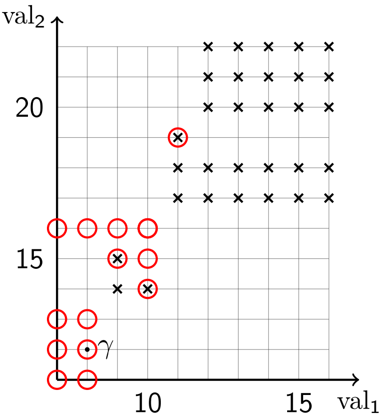

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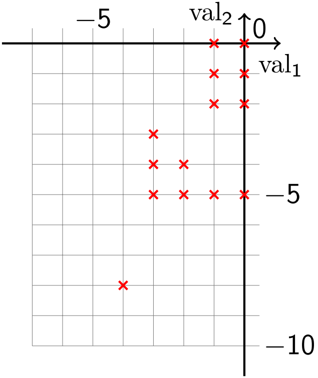

Figure : Values of $\mathscr{R}_C$

Idea of the proof

- The implication $v \in \mathrm{val}(\mathscr{R}_C) \Rightarrow \Delta(\gamma - v - \underline{1}, \mathrm{val}(\mathscr{J}_C)) = \varnothing$ is easy thanks to a result of F.Delgado.

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$$\mathscr{V} = \{v \in \mathbb{Z}^p ; \Delta(\gamma - v - \underline{1}, \mathscr{J}_C) = \varnothing\}$$

Then $\mathrm{val}(\mathscr{R}_C) \subseteq \mathscr{V}$. We have to prove that this is in fact an equality.

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  Then $\mathrm{val}(\mathscr{R}_C) \subseteq \mathscr{V}$. We have to prove that this is in fact an equality.
- We assume that this inclusion is strict. The existence of an "intruder" $w \in \mathscr{V} \setminus \mathrm{val}(\mathscr{R}_C)$ implies some combinatorial and numerical consequences which leads to a contradiction.

Stratification by logarithmic residues

We set:

- $\mu = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f'_x, f'_y)$ (Milnor number)
- $\tau = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f, f'_x, f'_y)$ (Tjurina number)
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Definition

We define the **stratification** of S by the logarithmic residues by:

$$S = \bigcup_{\mathscr{V} \subseteq \mathbb{Z}^p} S_{\mathscr{V}}$$

where $s \in S_{\mathscr{V}}$ if and only if $\mathrm{val}(\mathscr{R}_s) = \mathscr{V}$.

Properties of the stratification by logarithmic residues

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The duality $\mathscr{J}_C^\vee = \mathscr{R}_C$ implies

$$\dim_{\mathbb{C}} \mathscr{R}_C / \mathscr{O}_{\widetilde{C}} = \tau - \delta$$

Consequence: The stratification by logarithmic residues is finer than the stratification by τ.

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Consequence: The stratification by logarithmic residues is finer than the stratification by τ.

Remark. If τ is constant on $S' \subseteq S$ then a generating family of $\mathscr{R}_{s_0}$ with $s_0 \in S'$ gives rise to a generating family $(\rho_1(s), \rho_2(s))$ of $\mathscr{R}_{C_s}$ for $s \in S'$.

Example: Equisingular deformation of $x^5 - y^6$:

$$F(x, y, s_1, s_2, s_3) = x^5 - y^6 + s_1 x^2 y^4 + s_2 x^3 y^3 + s_3 x^3 y^4$$

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$S_{\tau=20} = \{0\}$
$S_{\tau=19} = \{(0,0,s_3), s_3 \neq 0\}$
$S'_{\tau=18} = \{(s_1,s_2,s_3), s_1 \neq 0\}$ and $S''_{\tau=18} = \{(0,s_2,s_3), s_2 \neq 0\}$

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Stratum	$\tau - \delta$	negative values of $\mathscr{R}_{C_s}$
$S_{\tau=20}$	10	$-1, -2, -3, -4, \quad -7, -8, -9, \quad -13, -14, \quad -19$
$S_{\tau=19}$	9	$-1, -2, -3, -4, \quad -7, -8, -9, \quad -13, -14$
$S'_{\tau=18}$	8	$-1, -2, -3, -4, \quad -7, -8, -9, \quad -14$
$S''_{\tau=18}$	8	$-1, -2, -3, -4, \quad -7, -8, -9, \quad -13$

The relation with Kähler differentials

Let $C = C_1 \cup \ldots \cup C_p$ be a reduced **complete intersection curve** in $\mathbb{C}^m$ defined by a regular sequence $(f_1, \ldots, f_{m-1})$. The module of Kähler differentials on C is

$$\Omega^1_C := \frac{\Omega^1_{\mathbb{C}^m}}{(f_1, \ldots, f_{m-1})\Omega^1_{\mathbb{C}^m} + \sum \mathscr{O}_{\mathbb{C}^m} \mathrm{d} f_i}$$

Let $\varphi_i(t_i)$ be a parametrization of C_i, and $\omega = \sum_{i=1}^{p} a_i \mathrm{d} x_i \in \Omega^1_C$.

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$$\varphi_i^*(\omega) = \left(\sum_{i=1}^{p} a_i \circ \varphi_i(t_i) x_i'(t_i)\right) \mathrm{d}t_i$$

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$$\varphi_i^*(\omega) = \left(\sum_{i=1}^p a_i \circ \varphi_i(t_i) x_i'(t_i) \right) \mathrm{d}t_i$$

$$\mathrm{val}_i(\omega) = \mathrm{val}_i \left(\sum_{i=1}^p a_i \circ \varphi_i(t_i) x_i'(t_i) \right) + 1$$

The value of ω is then $\mathrm{val}(\omega) = (\mathrm{val}_1(\omega), \ldots, \mathrm{val}_p(\omega))$.

Proposition (M.Granger, P.)

$$\mathrm{val}(\mathscr{I}_C) = \gamma + \mathrm{val}(\Omega^1_C) - \underline{1}$$

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By the Symmetry Theorem, we obtain:

Proposition (P.)

Let C be a plane curve. Then:

$$\forall v \in \mathbb{Z}^p, v \in \mathrm{val}(\mathscr{R}_C) \iff \Delta(-v, \mathrm{val}(\Omega^1_C)) = \varnothing$$

In particular, the stratifications by logarithmic residues and Kähler differentials coincide.

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This stratification is used by A.Hefez and M.Hernandes for the analytic classification of plane curves with one or two branches: in a given stratum, analytic equivalence can be expressed in terms of "normal form" of the parametrization.

Definitions

Let $(C,0) \subset (\mathbb{C}^m,0)$ be a **reduced complete intersection** defined by a regular sequence $(f_1,\dots,f_k)$, and $(D,0)$ the divisor defined by $f = f_1 \cdots f_k$. We set $\widetilde{\Omega}^q = \sum_{j=1}^k \frac{1}{f_1 \cdots \widehat{f_j} \cdots f_k} \Omega^q_{\mathbb{C}^m}$

Definitions

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Definition (A.Aleksandrov, A.Tsikh)

$$\Omega^q(\log C) := \left\{ \omega \in \frac{1}{f}\Omega^q_{\mathbb{C}^m}; \forall i \in \{1,\ldots,k\}, \mathrm{d}f_i \wedge \omega \in \widetilde{\Omega}^{q+1} \right\}$$

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Proposition (A.Aleksandrov, A.Tsikh)

$\omega \in \Omega^q(\log C)$ iff there exist $g \in \mathbb{C}\{\underline{x}\}$, which induces a non zero divisor in $\mathscr{O}_C$, $\xi \in \Omega^{q-k}_{\mathbb{C}^m}$ and $\eta \in \widetilde{\Omega}^q$ such that:

$$g\omega = \frac{\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k}{f_1 \cdots f_k} \wedge \xi + \eta \quad (5)$$

Definition (A.Aleksandrov, A.Tsikh)

The multi-residue of $\omega \in \Omega^q(\log C)$ is

$$\mathrm{res}_C(\omega) = \frac{\xi}{g} \in \Omega_C^{q-k} \otimes_{\mathscr{O}_C} Q(\mathscr{O}_C)$$

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Definition (M.Granger, M.Schulze)

$$\mathrm{Der}^k(-\log C) := \left\{ \delta \in \bigwedge^k \Theta_{\mathbb{C}^m} ; \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k(\delta) \in (f_1, \ldots, f_k) \right\}$$

$$= \mathrm{Ker}\big(\bigwedge^k \Theta_{\mathbb{C}^m} \to \mathscr{J}_C \big)$$

Freeness for complete intersections

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A singular reduced complete intersection is free if its singular locus is Cohen Macaulay of codimension 1, which is equivalent to $\mathscr{I}_C$ being Cohen-Macaulay maximal.

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Theorem (P.)

A reduced complete intersection is free iff $\Omega^k(\log C)$ has projective dimension $k-1$.

Remark. For $k=1$, we recover the definition of Saito free divisor.

Complete intersection curves

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- The values of $\mathscr{R}_C$ and Ω^1_C determine each other
- $\Omega^k(\log D) \subset \Omega^k(\log C)$, and

$$\mathrm{res}_C(\Omega^k(\log D)) \subset \mathrm{res}_C(\Omega^k(\log C))$$

but this inclusion is strict in general (P.)

Thank you for your attention !