Free singularities and logarithmic residues

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Background

Divisors

Let $D \subseteq (\mathbb{C}^m, 0)$ be a divisor and h be a reduced equation of D.

Definition (K. Saito): The logarithmic forms along D are:

$$\Omega^q(\log D) = \left\{ \omega \in \frac{1}{h}\Omega^q \; ; \; h\mathrm{d}\omega \in \Omega^{q+1} \right\}$$

Definition: D is called free if $\Omega^1(\log D)$ is free.

Examples: plane curves, normal crossings, discriminants of isolated complete intersection singularities...

Theorem (Terao, Aleksandrov): Assume D singular. *Equivalent statements:*

1. D is free.
2. The Jacobian ideal $\mathscr{J}_D$ is Cohen-Macaulay.
3. The singular locus $\mathscr{O}_D / \mathscr{J}_D$ is Cohen-Macaulay of dimension $m-2$.

Complete intersections

Let $C \subseteq (\mathbb{C}^m, 0)$ be a reduced complete intersection defined by $\mathscr{I}_C = \sum_{i=1}^k h_i \mathscr{O}_{\mathbb{C}^m}$. We set $h = h_1 \cdots h_k$.

Definition (Aleksandrov, Tsikh): The logarithmic forms along C are :

$$\Omega^q(\log C) = \left\{ \omega \in \frac{1}{h}\Omega^q \; ; \; \forall i, \mathrm{d}h_i \wedge \omega \in \frac{1}{h}\mathscr{I}_C \Omega^{q+1} \right\}$$

Logarithmic residues

Theorem (Saito (divisors), A-T (CI)): Let $\omega \in \frac{1}{h}\Omega^q$. Then $\omega \in \Omega^q(\log C)$ iff $\exists g \in \mathscr{O}_{\mathbb{C}^m}$ such that g is a non zero divisor in $\mathscr{O}_C$, $\exists \xi \in \Omega^{q-k}$ and $\eta \in \frac{1}{h}\mathscr{I}_C \Omega^q$ such that :

$$g\omega = \frac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k}{h_1 \cdots h_k} \wedge \xi + \eta$$

Definition: The residue of ω is $\mathrm{res}_C(\omega) = \frac{\xi}{g}\Big|_C$.
We set $\mathscr{R}_C = \mathrm{res}_C(\Omega^k(\log C))$.

Theorem: Let D be a reduced divisor. *Equivalent statements:*

1. the local fundamental group of the complement of D is abelian.
2. D is a normal crossing divisor in codimension 1.
3. $\mathscr{R}_D = \mathscr{O}_{\widetilde{D}}$, with $\widetilde{D}$ the normalization of D.

$1 \Rightarrow 2 \Rightarrow 3$ is proved by Saito, $2 \Rightarrow 1$ by Lê and Saito and $3 \Rightarrow 2$ by Granger and Schulze.

Questions

- **An analogue of freeness for complete intersections ?**
- **Describe $\mathscr{R}_D$ for non normal crossings divisors ?**

Results

Free singularities

Definition (Granger, Schulze): A reduced complete intersection C is called free if the Jacobian ideal $\mathscr{J}_C$ is Cohen-Macaulay.

Theorem (P.): *Equivalent statements:*

1. C is free
2. the projective dimension of $\Omega^k(\log C)$ is $k-1$.
3. $\mathscr{R}_C$ is Cohen-Macaulay and $\mathscr{R}_C^\vee := \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{R}_C, \mathscr{O}_C) = \mathscr{J}_C$

The following is used in the proof :

Proposition (Granger, Schulze, P.): $\mathscr{J}_C^\vee = \mathscr{R}_C$

Generalization (P.) Analogous statements for CM spaces

Symmetry theorem for curves

Let $C = C_1 \cup \cdots \cup C_p$ be a reduced curve with p irreducible components with conductor γ.

Definition: The value map is $\mathrm{val} : \mathrm{Frac}(\mathscr{O}_C) \to (\mathbb{Z} \cup \{\infty\})^p$ defined by $\mathrm{val}(g) = (\mathrm{val}_1(g), \ldots, \mathrm{val}_p(g))$ where val_i is the valuation along C_i.

Notation: For I a fractional ideal, $\mathrm{val}(I) = \{\mathrm{val}(g) \; ; \; g \in I\} \cap \mathbb{Z}^p$.
For $v \in \mathbb{Z}^p$ we set $\Delta_i(v, I) = \{\alpha \in \mathrm{val}(I) \; ; \; \alpha_i = v_i \text{ and } \forall j \neq i, \alpha_j > v_j\}$

Generalizing Delgado's symmetry theorem, we prove :

Theorem (P.): C is Gorenstein iff for all fractional ideal I,

$$\forall v \in \mathbb{Z}^p, \; v \in \mathrm{val}(I^\vee) \iff \bigcup_{i=1}^p \Delta_i(\gamma - v - \underline{1}, I) = \emptyset$$

Example: $h = (x^3 - y^2)(x^3 - y^4)$, $I = \mathscr{J}_C$ and $I^\vee = \mathscr{R}_C$

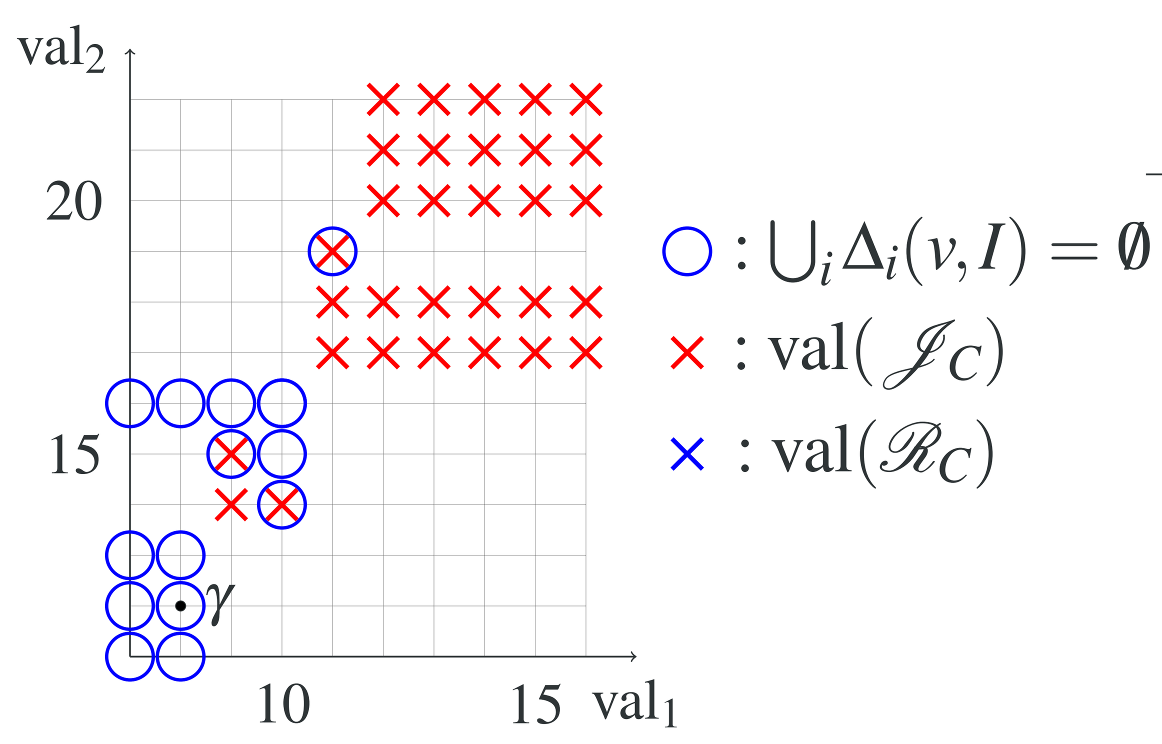

Figure : $\mathrm{val}(\mathscr{J}_C)$

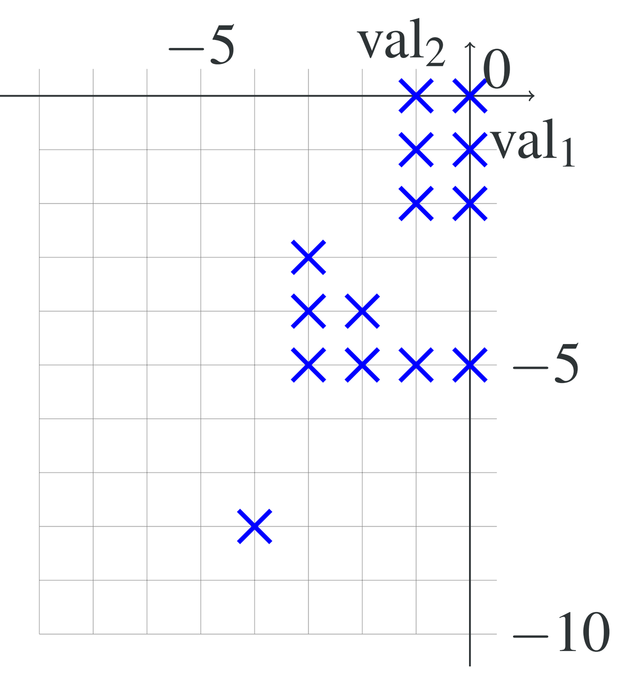

Figure : $\mathrm{val}(\mathscr{R}_C)$

Perspectives

- Families of free singularities ? Case of arrangements
- Projective dimension and Betti numbers of $\Omega^{q+k}(\log C)$
- Develop a theory of free projective complete intersection curves

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