Free singularities

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Divisors

Let $D \subseteq (\mathbb{C}^m, 0)$ be a divisor with reduced equation h, Ω^q the module of holomorphic q-forms and Θ the module of holomorphic vector fields of $(\mathbb{C}^m, 0)$.

Definition (K. Saito): The modules of logarithmic q-forms and of logarithmic vector fields along D are respectively:

$$\Omega^q(\log D) = \left\{ \omega \in \frac{1}{h}\Omega^q \ ;\ h\mathrm{d}\omega \in \Omega^{q+1} \right\}$$

$$\mathrm{Der}(-\log D) = \{ \delta \in \Theta \ ;\ \delta(h) \in (h) \}$$

Proposition (K.Saito): $\Omega^1(\log D)$ and $\mathrm{Der}(-\log D)$ are dual to each other as $\mathbb{C}\{x_1, \ldots, x_m\}$-modules.

Definition: D is called free if $\Omega^1(\log D)$ is free ($\Leftrightarrow \mathrm{Der}(-\log D)$ is free).

Examples: plane curves, normal crossing divisors, discriminants of isolated complete intersection singularities...

Theorem (Terao, Aleksandrov): Assume D singular.
Equivalent statements:
1. D is free.
2. The Jacobian ideal $\mathscr{J}_D \subseteq \mathscr{O}_D = \mathbb{C}\{\underline{x}\}/(h)$ is Cohen-Macaulay.
3. The singular locus $\mathscr{O}_D/\mathscr{J}_D$ is Cohen-Macaulay of dimension $m-2$.

Complete intersections

Logarithmic forms

Let $C \subseteq (\mathbb{C}^m, 0)$ be a reduced complete intersection of codimension k defined by $\mathscr{I}_C = \sum_{i=1}^k h_i \mathscr{O}_{\mathbb{C}^m}$. We set $h = h_1 \cdots h_k$.

Definition (Aleksandrov, Tsikh) The logarithmic q-forms along C are :

$$\Omega^q(\log C) = \left\{ \omega \in \frac{1}{h}\Omega^q \ ;\ \forall i, \mathrm{d}h_i \wedge \omega \in \frac{1}{h}\mathscr{I}_C \Omega^{q+1} \right\}$$

Definition (Granger, Schulze) The logarithmic k-vector fields along C are :

$$\mathrm{Der}^k(-\log C) = \left\{ \delta \in \bigwedge^k \Theta \ ;\ \mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k(\delta) \in \mathscr{I}_C \right\}$$

We have the following exact sequence:

$$0 \to \mathrm{Der}^k(-\log C) \to \bigwedge^k \Theta \to \mathscr{J}_C \to 0 \qquad (1)$$

where the Jacobian ideal $\mathscr{J}_C \subseteq \mathscr{O}_C$ is generated by the $k \times k$ minors of the Jacobian matrix of $(h_1, \ldots, h_k)$.

We generalize the duality between logarithmic forms and vector fields as follows:

Proposition (P.): There is a perfect pairing:

$$\Omega^k(\log C) \times \mathrm{Der}^k(-\log C) \to \frac{1}{h}\mathscr{J}_C$$

Logarithmic residues

Theorem (Saito (divisors), A-T (CI)): Let $\omega \in \frac{1}{h}\Omega^q$. Then $\omega \in \Omega^q(\log C)$ iff $\exists g \in \mathscr{O}_{\mathbb{C}^m}$ such that g is a non zero divisor in $\mathscr{O}_C$, $\exists \xi \in \Omega^{q-k}$ and $\eta \in \frac{1}{h}\mathscr{I}_C\Omega^q$ such that :

$$g\omega = \frac{\mathrm{d}h_1 \wedge \cdots \wedge \mathrm{d}h_k}{h_1 \cdots h_k} \wedge \xi + \eta$$

Definition: The residue of ω is $\mathrm{res}_C(\omega) = \frac{\xi}{g}\big|_C$. We set $\mathscr{R}_C = \mathrm{res}_C(\Omega^k(\log C))$.

We have the following exact sequence:

$$0 \to \frac{1}{h}\mathscr{I}_C \Omega^k \to \Omega^k(\log C) \to \mathscr{R}_C \to 0 \qquad (2)$$

Characterizations of freeness

Definition (Granger, Schulze): A reduced complete intersection C is called free if the Jacobian ideal $\mathscr{J}_C$ is Cohen-Macaulay.

Theorem (P.): *Equivalent statements:*
1. C is free,
2. the projective dimension of $\mathrm{Der}^k(-\log C)$ is $k-1$,
3. the projective dimension of $\Omega^k(\log C)$ is $k-1$.

Sketch of proof: The equivalence between 1. and 2. comes from the exact sequence (1). Indeed, $\mathrm{depth}\left(\bigwedge^k \Theta\right) = m$, $\mathrm{depth}(\mathscr{J}_C) \leqslant m-k$, so that by the depth lemma, $\mathrm{depth}(\mathrm{Der}^k(-\log C)) = \mathrm{depth}(\mathscr{J}_C) + 1$. The Aulander-Buchsbaum formula then gives the result on the projective dimension.

The other equivalence needs more work, and is based on the study of the long exact sequence obtained by appliying $\mathrm{Hom}_{\mathscr{O}_{\mathbb{C}^m}}(-, \mathscr{O}_{\mathbb{C}^m})$ to the short exact sequence (2).

The proof also uses the following duality, which can be proved using the perfect pairing and the exact sequences (1) and (2):

Proposition (Granger, Schulze, P.): $\mathscr{J}_C^\vee := \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{J}_C, \mathscr{O}_C) = \mathscr{R}_C$

Example: Reduced curves are free singularities

Theorem (P.): Let C be a quasi-homogeneous complete intersection **curve**. Then a free resolution of $\Omega^{m-1}(\log C)$ is known. For example, if $m = 3$, we have:

$$0 \to \mathbb{C}\{\underline{x}\}^5 \to \mathbb{C}\{\underline{x}\}^8 \to \Omega^2(\log C) \to 0.$$

Generalization (P.) Analogous results on **freeness for reduced Cohen-Macaulay subspaces** are satisfied, which are based on the following definition of logarithmic forms:

Definition (Aleksandrov) The logarithmic q-forms along a reduced CM subspace contained in a reduced complete intersection C defined by $\mathscr{I}_C = \sum h_i \mathbb{C}\{\underline{x}\}$ are :

$$\Omega^q(\log X/C) = \left\{ \omega \in \frac{1}{h}\Omega^q \ ;\ \mathscr{I}_X \omega \subseteq \frac{1}{h}\mathscr{I}_C \Omega^q \text{ and } \mathrm{d}(\mathscr{I}_X) \wedge \omega \in \frac{1}{h}\mathscr{I}_C \Omega^{q+1} \right\}$$

Set of values of ideals along curves

Computing the logarithmic forms is in general difficult.With the following result we can compute the values of the logarithmic residues directly from the Jacobian ideal for curves.

Symmetry theorem

Let $C = C_1 \cup \cdots \cup C_p$ be a reduced curve with p irreducible components and conductor γ.

Definition: The value map is $\mathrm{val} : \mathrm{Frac}(\mathscr{O}_C) \to (\mathbb{Z} \cup \{\infty\})^p$ defined by $\mathrm{val}(g) = (\mathrm{val}_1(g), \ldots, \mathrm{val}_p(g))$ where val_i is the valuation along C_i.

Notation: For a fractional ideal I, $\mathrm{val}(I) = \{\mathrm{val}(g) \ ;\ g \in I\} \cap \mathbb{Z}^p$.
For $v \in \mathbb{Z}^p$ we set $\Delta_i(v, I) = \left\{ \alpha \in \mathrm{val}(I) \ ;\ \alpha_i = v_i \text{ and } \forall j \neq i, \alpha_j > v_j \right\}$

Generalizing Delgado's symmetry theorem, we prove :

Theorem (P.): C is Gorenstein iff for all fractional ideal I and for all $v \in \mathbb{Z}^p$:

$$v \in \mathrm{val}(I^\vee) \iff \bigcup_{i=1}^p \Delta_i(\gamma - v - \underline{1}, I) = \emptyset$$

Example: $h = (x^3 - y^2)(x^3 - y^4)$, $I = \mathscr{J}_C$ and $I^\vee = \mathscr{R}_C$

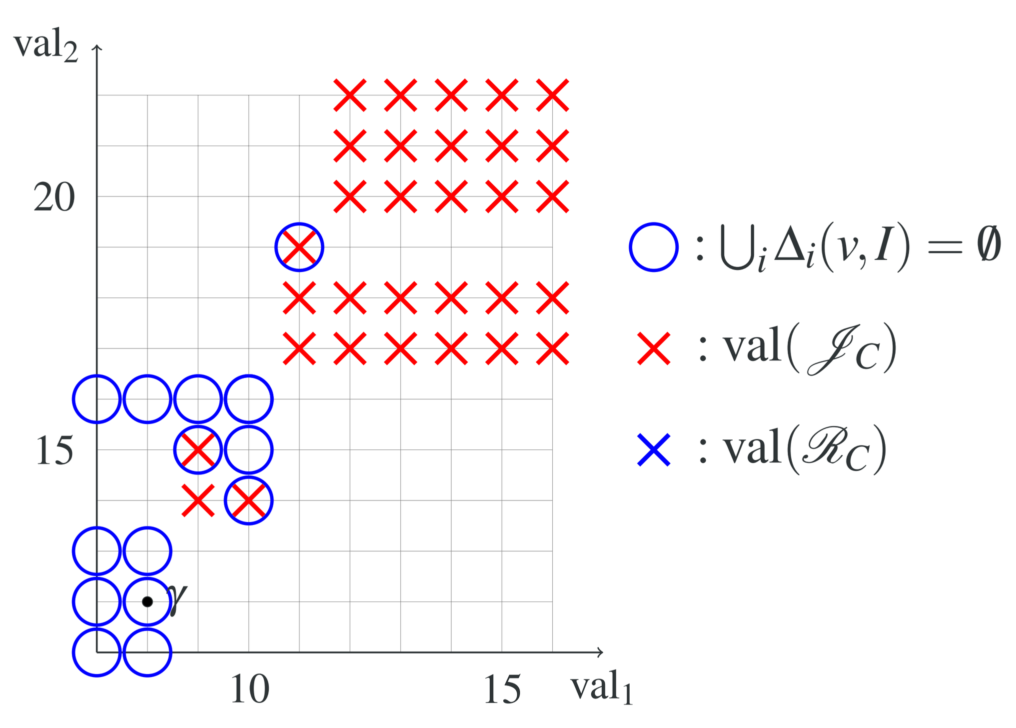

Figure : $\mathrm{val}(\mathscr{J}_C)$

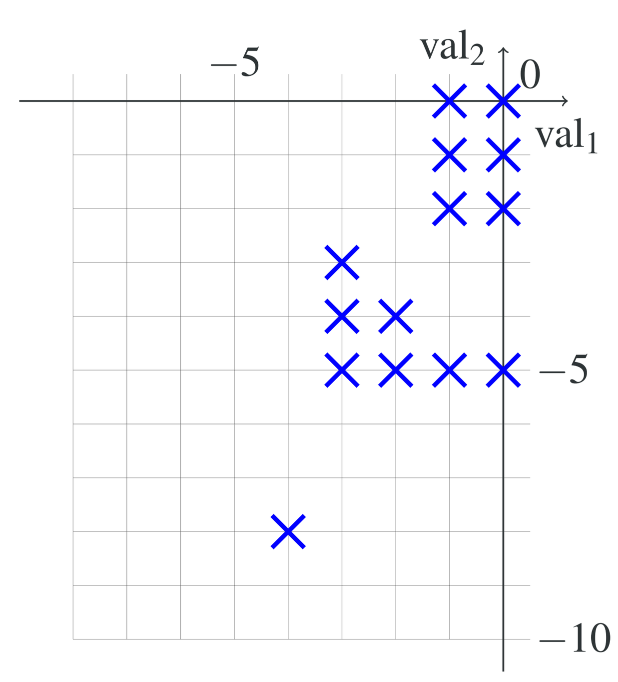

Figure : $\mathrm{val}(\mathscr{R}_C)$

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