Real singular Del Pezzo surfaces and threefolds fibred by rational curves

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1. Introduction

Let $f\colon W \to X$ be a real smooth projective threefold fibred by rational curves. Suppose that $W(\mathbb{R})$ is orientable. Then, by [4, Theorem 1.1], a connected component $M \subset W(\mathbb{R})$ is obtained from a Seifert fibred manifold or a connected sum of lens spaces by taking connected sums with a finite number of copies of $\mathbb{P}^3(\mathbb{R})$ and a finite number of copies of $S^1 \times S^2$.

Let $\nu := \nu(M)$ be the integer defined as follows:

(1) If $g\colon M \to F$ is a Seifert fibration, ν denotes the number of multiple fibres of g

(2) If M is a connected sum of lens spaces, ν denotes the number of lens spaces

Theorem 1. *If X is a geometrically rational surface, then $\nu \leq 4$*

This result answers in affirmative a question of Kollár who proved in 1999 that $\nu \leq 6$ and suggested that 4 would be the sharp bound.

We derive Theorem 1 from a careful study of real singular Del Pezzo surfaces with only rational double points as singularities (see Theorem 2).

Thanks to the Minimal Model Program over $\mathbb{R}$, the original setting $f\colon W \to X$ is replaced by the following: W is a real projective 3-fold with terminal singularities such that K_W is Cartier along $W(\mathbb{R})$ and $f\colon W \to X$ is a rational curve fibration over $\mathbb{R}$ such that $-K_W$ is f-ample.

Let M be a connected component of the topological normalization $\overline{W(\mathbb{R})}$ (see next section) and assume that M is a Seifert fibred 3-dimensional manifold or a connected sum of lens spaces. Then, by [3, Thm. 2.6], there exists a Werther fibration $g\colon M \to F$ over a 2-manifold with boundary. Werther fibrations are defined in [3], but for our purpose it is sufficient to recall that $g|_{g^{-1}(F\setminus\partial F)}$ is a Seifert fibration. Now there is an injection from the set of multiple fibres of $g|_{g^{-1}(F\setminus\partial F)}$ to the set of singular points of X contained in $f(M)$ which are of type A^+ and globally separating (see next section). Under this injection, the multiplicity of the Seifert fibre equals $\mu + 1$ if the singular point is of type A^+_μ.

Theorem 1 is now a corollary of the following:

Theorem 2. *Let X be a projective surface defined over $\mathbb{R}$. Suppose that X is geometrically rational with Du Val singularities. Then a connected component M of $\overline{X(\mathbb{R})}$ contains at most 4 Du Val singular points which are not of type A^- and globally nonseparating.*

2. Rational surfaces with Du Val singularities

On a surface, a rational double point is called a Du Val singularity. Over $\mathbb{C}$, these singularities are classified by their Dynkin diagrams, namely A_μ, $\mu \geq 1$, D_μ, $\mu \geq 4$, E_6, E_7, E_8.

Over $\mathbb{R}$, there are more possibilities. In particular, a surface singularity will be said to be *of type A_μ^+* if it is real analytically equivalent to

$$x^2 + y^2 - z^{\mu+1} = 0,\ \mu \geq 1\ ;$$

and *of type A_μ^-* if it is real analytically equivalent to

$$x^2 - y^2 - z^{\mu+1} = 0,\ \mu \geq 1\ .$$

The type A_1^+ is real analytically isomorphic to A_1^-; otherwise, singularities with different names are not isomorphic.

Definition 1. *Let V be a simplicial complex with only a finite number of points $x \in V$ where V is not a manifold. Define the* topological normalization

$$\overline{n}\colon \overline{V} \to V$$

as the unique proper continuous map such that $\overline{n}$ is a homeomorphism over the set of points where V is a manifold and $\overline{n}^{-1}(x)$ is in one-to-one correspondence with the connected components of a good punctured neighborhood of x in V otherwise.

Definition 2. *Let X be a real algebraic surface with isolated singularities, and let $x \in X(\mathbb{R})$ be a singular point of type $A_\mu^\pm$ with μ odd. The topological normalization $\overline{X(\mathbb{R})}$ has two connected components locally near x. We will say that x is* globally separating *if these two local components are on different connected components of $\overline{X(\mathbb{R})}$ and* globally nonseparating *otherwise.*

Definition 3. *Let X be a projective surface with Du Val singularities, let*

$$\mathcal{P} := \operatorname{Sing} X \setminus \big\{x \textit{ is of type } A_\mu^-,\ \textit{and } x \textit{ is globally nonseparating}\big\}\ .$$

Lemma 1 (Kollár). *Let $\overline{n}\colon \overline{X(\mathbb{R})} \to X(\mathbb{R})$ be the topological normalization, and define $M_1, M_2, \ldots, M_r$ be the connected components of $\overline{X(\mathbb{R})}$. The unordered sequence of numbers $\#(\overline{n}^{-1}(\mathcal{P}) \cap M_i) := m_i$, $i = 1, 2, \ldots, r$ is an invariant of extremal birational contractions of projective surfaces with Du Val singularities.*

Applying the Minimal Model Program over $\mathbb{R}$ to a geometrically rational surface with Du Val singularities, we reduce the proof of Theorem 2 to the study of singular Del Pezzo surfaces of degree one.

Recall that a Del Pezzo surface X is by definition a surface whose anticanonical divisor is ample. We add the adjective *Du Val* to emphasize that we allow X to have Du Val singularities. Let X be a real Du Val Del Pezzo surface and let $S \to X$ be the minimal resolution of singularities. The smooth surface S has nef anticanonical divisor and is called a *weak Del Pezzo surface* by many authors.

We obtain a Del Pezzo surface X of degree 1 by blowing up a finite number of pairs of conjugate imaginary smooth points and some real smooth point (there are several choices to do this).

The anticanonical model of a Del Pezzo surface X of degree 1 is a ramified double covering $q\colon X \to Q$ of a quadric cone $Q \subset \mathbb{P}^3$ whose branch locus is the union of the vertex of the cone and a cubic section not passing through the vertex (see e.g. [2, Exposé V]).

Let X' be the singular elliptic surface obtained from X by blowing up the pull-back by q of the vertex of the cone.

The surface X' is a ramified double covering of the Hirzebruch surface $\mathbb{F}_2$ whose branch curve is the union of the unique section of negative selfintersection, the section at infinity Σ_∞, and a trisection B disjoint from Σ_∞.

If the trisection is irreducible, then it has at most 4 singular points because it has genus 4. The heart of the proof is the study of normal forms for the reducible B. After studying the normal forms for the reducible branch curves, it appears that in almost all cases, the number of singular points is less than or equal to 4. There will remain only two cases to examine separately. In one case, the fifth point turns out to be of type $A_1^+ \cong A_1^-$ globally nonseparating and in the second case, the fifth point turns out to be of type A_2^-.

References

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