# CREMONA TRANSFORMATIONS AND DIFFEOMORPHISMS OF SURFACES

JÁNOS KOLLÁR AND FRÉDÉRIC MANGOLTE

The simplest *Cremona transformation* of projective 3-space is the involution

$$\sigma\colon (x_0 : x_1 : x_2 : x_3) \mapsto \Bigl(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}\Bigr),$$

which is a diffeomorphism outside the tetrahedron $(x_0x_1x_2x_3 = 0)$. More generally, if $L_i := \sum_j a_{ji}x_j$ are linear forms defining the faces of a tetrahedron, we get the Cremona transformation

$$\sigma_{\mathbf{L}}\colon (x_0 : x_1 : x_2 : x_3) \mapsto \Bigl(\frac{1}{L_0} : \frac{1}{L_1} : \frac{1}{L_2} : \frac{1}{L_3}\Bigr) \cdot \bigl(a_{ij}\bigr)^{-1},$$

which is a diffeomorphism outside the tetrahedron $(L_0L_1L_2L_3 = 0)$. The vertices of the tetrahedron are called the *base points*. If $Q$ is a quadric surface in $\mathbb{P}^3$, its image under a Cremona transformation is, in general, a sextic surface. However, if $Q$ passes through the 4 base points, then its image $\sigma_{\mathbf{L}}(Q)$ is again a quadric surface in $\mathbb{P}^3$ passing through the 4 base points. In many cases, we can view $\sigma_{\mathbf{L}}$ as a map of $Q$ to itself.

The aim of this paper is to show that these Cremona transformations generate both the group of automorphisms and the group of diffeomorphisms of the sphere, the torus and of all non-orientable surfaces.

Let us start with the sphere $S^2 := (x^2 + y^2 + z^2 = 1) \subset \mathbb{R}^3$ and view this as the set of real points of the quadric $Q := (x^2 + y^2 + z^2 = t^2) \subset \mathbb{P}^3$ in projective 3-space. Pick 2 conjugate point pairs $p, \bar{p}, q, \bar{q}$ on the complex quadric $Q(\mathbb{C})$ and let $\sigma_{p,q}$ denote the Cremona transformation with base points $p, \bar{p}, q, \bar{q}$. As noted above, $\sigma_{p,q}(Q)$ is another quadric surface. The faces of the tetrahedron determined by these 4 points are disjoint from $S^2$, hence $\sigma_{p,q}$ is a diffeomorphism from $S^2$ to the real part of $\sigma_{p,q}(Q)$. Thus $Q$ and $\sigma_{p,q}(Q)$ are projectively equivalent and the corresponding Cremona transformation $\sigma_{p,q}$ can be viewed as a diffeomorphism of $S^2$ to itself, well defined up to left and right multiplication by $O(3,1)$. It is also convenient to allow the points $p, q$ to coincide; see (9) for a precise definition. Let us call these the Cremona transformations with *imaginary base points*. Our first result is that, algebraically, these generate the automorphism group.

**Theorem 1.** *The Cremona transformations with imaginary base points $\sigma_{p,q}$ and $O(3,1)$ generate* $\operatorname{Aut}(S^2)$.

Most diffeomorphisms of $S^2$ are not algebraic, so the best one can hope for is that these Cremona transformations generate $\operatorname{Diff}(S^2)$ as topological group. Equivalently, that $\operatorname{Aut}(S^2)$ is a dense subgroup of $\operatorname{Diff}(S^2)$. The results of [Lukackiĭ77, Lukackiĭ79], pointed out to us by M. Zaidenberg, imply that the group of algebraic automorphisms is dense in the group of diffeomorphisms for the sphere and the torus. His methods, reviewed in (15), use the $SO(3,\mathbb{R})$ action on the sphere and the torus action on itself.

In order to go futher, first we need to deal with diffeomorphisms with fixed points. Building on [Biswas-Huisman07], it is proved in [Huisman-Mangolte09a] that $\mathrm{Aut}(S^2)$ is $n$-transitive for any $n \geq 1$. Using this, it is easy to see (19) that the density property also holds with assigned fixed points.

**Corollary 2.** $\mathrm{Aut}(S^2, p_1, \dots, p_n)$ *is dense in* $\mathrm{Diff}(S^2, p_1, \dots, p_n)$ *for any finite set of distinct points* $p_1, \dots, p_n \in S^2$*, where* $\mathrm{Aut}(\ )$ *denotes the group of algebraic automorphisms of* $S^2$ *fixing* $p_1, \dots, p_n$ *and* $\mathrm{Diff}(\ )$ *the group of diffeomorphisms fixing* $p_1, \dots, p_n$.

Note that, for a real algebraic variety $X$, the semigroup of algebraic diffeomorphisms is usually much bigger than the group of algebraic automorphisms $\mathrm{Aut}(X)$. For instance, $x \mapsto x + \frac{1}{x^2+1}$ is an algebraic diffeomorphism of $\mathbb{R}$ (and also of $\mathbb{RP}^1 \sim S^1$), but its inverse involves square and cube roots. The difference is best seen in the case of the circle $S^1 = (x^2 + y^2 = 1)$.

Essentially by the Weierstrass approximation theorem, any differentiable map $\phi\colon S^1 \to S^1$ can be approximated by polynomial maps $\Phi\colon S^1 \to S^1$. By contrast, the group of algebraic automorphisms of $S^1$ is the real orthogonal group $O(2,1) \cong PGL(2,\mathbb{R})$, which has real dimension 3. Thus $\mathrm{Aut}(S^1)$ is a very small closed subgroup in the infinite dimensional group $\mathrm{Diff}(S^1)$.

The Cremona transformations with real base points do not give diffeomorphisms of $S^2$; they are not even defined at the real base points. Instead, they give generators of the mapping class groups of non-orientable surfaces.

Let $R_g$ be a non-orientable, compact surface of genus $g$ without boundary. Coming from algebraic geometry, we prefer to think of it as $S^2$ blown up at $g$ points $p_1, \dots, p_g \in S^2$. Topologically, $R_g$ is obtained from $S^2$ by replacing $g$ discs centered at the $p_i$ by $g$ Möbius bands. Up to isotopy, a blow-up form of $R_g$ is equivalent to giving $g$ disjoint embedded Möbius bands $M_1, \dots, M_g \subset R_g$.

There are two ways to think of a Cremona transformation with real base points as giving elements of the mapping class group of $R_g$.

Let us start with the case when there are four real base points $p_1, \dots, p_4$. We can factor the Cremona transformation $\sigma_{p_1,\dots,p_4}$ as

$$\sigma_{p_1,\dots,p_4}\colon Q \overset{\pi_1}{\leftarrow} B_{p_1,\dots,p_4}Q \overset{\pi_2}{\rightarrow} Q$$

where on the left $\pi_1\colon B_{p_1,\dots,p_4}Q \to Q$ is the blow up of $Q$ at the 4 points $p_1, \dots, p_4$ and on the right $\pi_2\colon B_{p_1,\dots,p_4}Q \to Q$ contracts the birational transforms of the circles $Q \cap L_i$ where the $\{L_i\}$ are the faces of the tetrahedron with vertices $\{p_i\}$. In Figure 1, the $\bullet$ represent the 4 base points. On the left hand side, the 4 exceptional curves $E_i$ lie over the four points marked $\bullet$. On the right hand side, the images of the $E_i$ are 4 circles, each passing through 3 of the 4 base points. Since $\sigma_{p_1,\dots,p_4}$ is an involution, dually, the four points marked $\bullet$ on the right hand side map to the 4 circles on the left hand side.

A Cremona transformation $\sigma_{p_1,p_2,q,\bar{q}}$ with 2 real and a conjugate complex pair of base points act similarly. Here only two Möbius bands are altered.

In general, we can think of the above real Cremona transformation $\sigma_{p_1,\dots,p_4}$ as a topological operation that replaces the set of $g$ Möbius bands $\big(M_1, \dots, M_g\big)$ by a new set $\big(M_1', \dots, M_4', M_5, \dots, M_g\big)$. In this version, $\sigma_{p_1,\dots,p_4}$ is the identity on the surfaces but acts nontrivially on the set of isotopy classes of $g$ disjoint Möbius bands.

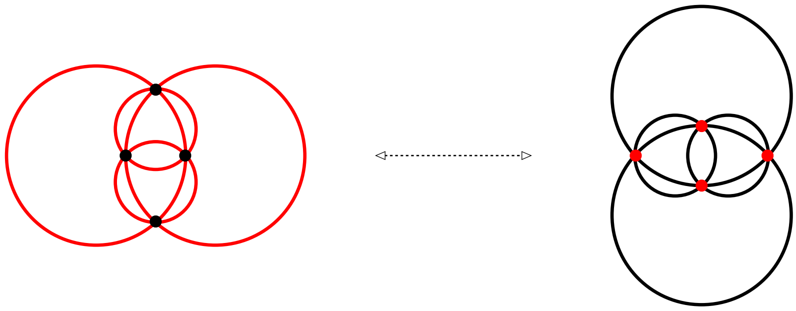

FIGURE 1. Cremona transformation with four real base points.

One version of our result says that the transformations $\sigma_{p_1,\dots,p_4}$ and $\sigma_{p_1,p_2,q,\bar{q}}$ act transitively on the set of isotopy classes of $g$ disjoint Möbius bands.

The other way to view $\sigma_{p_1,\dots,p_4}$ is as follows. First, we obtain an isomorphism

$$\sigma'_{p_1,\dots,p_4} : B_{p_1,\dots,p_g} S^2 \cong B_{q_1,\dots,q_g} S^2$$

for some $q_1, \dots, q_g \in S^2$. Under this isomorphism, the exceptional curve $E(p_i) \subset B_{p_1,\dots,p_g} S^2$ is mapped to the exceptional curve $E(q_i) \subset B_{q_1,\dots,q_g} S^2$ for $i \geq 5$ and to the circle passing through the points $\{q_j : 1 \leq j \leq 4, j \neq i\}$ for $i \leq 4$. As we noted above, there is an automorphism $\Phi \in \mathrm{Aut}(S^2)$ such that $\Phi(q_i) = p_i$ for $1 \leq i \leq n$. Thus

$$\Phi \circ \sigma'_{p_1,\dots,p_4} : B_{p_1,\dots,p_g} S^2 \xrightarrow{\cong} B_{p_1,\dots,p_g} S^2$$

is an automorphism of $B_{p_1,\dots,p_g} S^2$ which maps $E(p_i)$ to $E(q_i)$ for $i \geq 5$ and to a simple closed curve passing through the points $\{p_j : 1 \leq j \leq 4, j \neq i\}$ for $i \leq 4$.

**Theorem 3.** *For any $g$, the Cremona transformations with $4, 2$ or $0$ real base points generate the (non-orientable) mapping class group $\mathcal{M}_g$.*

Finally, we can put these results together to obtain a general approximation theorem for diffeomorphisms of such real algebraic surfaces.

**Theorem 4.** *Let $R$ be a compact, smooth, real algebraic surface birational to $\mathbb{P}^2$ and $q_1, \dots, q_n \in R$ distinct marked points. Then the group of algebraic automorphisms $\mathrm{Aut}(R, q_1, \dots, q_n)$ is dense in $\mathrm{Diff}(R, q_1, \dots, q_n)$.*

*As a topological manifold, here $R$ can be the sphere, the torus or any non-orientable surface $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$.*

**5** (Other algebraic varieties)**.** Similar assertions definitely fail for most other algebraic varieties. Real algebraic varieties of general type have only finitely many birational automorphisms. (See [Ueno75] for an introduction to these questions.) For varieties whose Kodaira dimension is between 0 and the dimension, every birational automorphism preserves the Iitaka fibration. If the Kodaira dimension is 0 (e.g., Calabi-Yau varieties, Abelian varieties), then every birational automorphism preserves the canonical class, that is, a volume form, up to sign. The automorphism group is finite dimensional but may have infinitely many connected components. In particular, using [Comessatti14], for surfaces we obtain the following.

**Proposition 6.** *Let $S$ be a smooth real algebraic surface. If $S(\mathbb{R})$ is an orientable surface of genus $\geq 2$ then $\mathrm{Aut}(S)$ is not dense in $\mathrm{Diff}\big(S(\mathbb{R})\big)$.* □

If $X$ has Kodaira dimension $-\infty$, then every birational automorphism preserves the MRC fibration [Kollár96, Sec.IV.5]. Thus the main case when density could hold is when the variety is rationally connected [Kollár96, Sec.IV.3]. It is clear that the analog of (1) fails even for most geometrically rational real algebraic surfaces. Consider, for instance, the case when $R \to \mathbb{P}^1$ is a minimal conic bundle with at least 8 singular fibers. Then $\mathrm{Aut}(R)$ is infinite dimensional, but every automorphism of $R$ preserves the conic bundle structure [Iskovskikh96, Thm. 1.6(iii)]. Conic bundles with 4 singular fibers are probably the only other case where the analog of (4) holds.

The results of [Lukackiĭ77] imply that $\mathrm{Aut}(S^n)$ is dense in $\mathrm{Diff}(S^n)$ for every $n \geq 2$ and, similarly, $\mathrm{Aut}(T^n)$ is dense in $\mathrm{Diff}(T^n)$ for every $n \geq 2$, where $T^n$ is the $n$-dimensional torus. It is not clear to us what happens with other varieties birational to $\mathbb{P}^n$.

**7** (History of related questions)**.** There are many results in real algebraic geometry that endow certain topological spaces with a real algebraic structure or approximate smooth maps by real algebraic morphisms. In particular real rational models of surfaces were studied in [Bochnak-Coste-Roy87], [Mangolte06] and approximations of smooth maps to spheres by real algebraic morphisms were investigated in [Bochnak-Kucharz87a, Bochnak-Kucharz87b], [Bochnak-Kucharz-Silhol97], [Kucharz99], [Joglar-Kollár03], [Joglar-Mangolte04], [Mangolte06].

An indication that $\mathrm{Aut}(S^2)$ is surprisingly large comes from [Biswas-Huisman07], with a more precise version developed in [Huisman-Mangolte09a].

**8** (Plan of the proofs)**.** In Section 1 we prove that the Cremona transformations with imaginary base points generate $\mathrm{Aut}(S^2)$. Next, in Section 2, we prove (4) for the identity components. If $\phi\colon R \to R$ is homotopic to the identity, then $\phi$ can be written as the composite of diffeomorphisms $\phi_i\colon R \to R$ such that each $\phi_i$ is the identity outside a small open set $W_i \subset R$. Moreover, we can choose the $W_i$ in such a way that for every $i$ there is a morphism $\pi_i\colon R \to S^2$ that is an isomorphism on $W_i$. The map $\phi_i$ then pushes down to a diffeomorphism of $S^2$. We take an approximation there and lift it to $R$.

The case $R = S^1 \times S^1$ follows from [Lukackiĭ79].

Generators of the mapping class group of non-orientable surfaces have been written down by [Chillingworth69] and [Korkmaz02]. In Section 3 we describe a somewhat different set of generators. We thank M. Korkmaz for his help in proving these results.

Theorem 3 is proved in Section 4. We show by explicit constructions that our generators are given by Cremona transformations.

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## 1. Generators of $\mathrm{Aut}(S^2)$

Max Noether proved that the involution $(x, y, z) \mapsto \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ and $PGL_3$ generate the group of birational self-maps $\mathrm{Bir}(\mathbb{P}^2)$ over $\mathbb{C}$. Using similar ideas, [Ronga-Vust05] proved that $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{R}})$ is generated by linear automorphisms and certain real algebraic automorphisms of degree 5. In this section, we prove that $\mathrm{Aut}(S^2)$ is generated by linear automorphisms and by the $\sigma_{p.q}$. The latter are real algebraic automorphisms of degree 3.

**Example 9** (Cubic involutions of $\mathbb{P}^3$)**.** On $\mathbb{P}^3$ take coordinates $(x, y, z, t)$. We need two types of cubic involutions of $\mathbb{P}^3$. Let us start with the Cremona transformation

$$(x, y, z, t) \mapsto \left(\tfrac{1}{x}, \tfrac{1}{y}, \tfrac{1}{z}, \tfrac{1}{t}\right) = \tfrac{1}{xyzt}(yzt, ztx, txy, xyz)$$

whose base points are the 4 "coordinate vertices". We will need to put the base points at complex conjugate point pairs, say $(1, \pm i, 0, 0), (0, 0, 1, \pm i)$. Then the above involution becomes

$$\tau\colon (x, y, z, t) \mapsto \big((x^2 + y^2)z, (x^2 + y^2)t, (z^2 + t^2)x, (z^2 + t^2)y\big).$$

Check that

$$\tau^2(x, y, z, t) = (x^2 + y^2)^2(z^2 + t^2)^2 \cdot (x, y, z, t),$$

thus $\tau$ is indeed a rational involution on $\mathbb{P}^3$.

Consider a general quadric passing through the points $(1, \pm i, 0, 0), (0, 0, 1, \pm i)$. It is of the form

$$Q = Q_{abcdef}(x, y, z, t) := a(x^2 + y^2) + b(z^2 + t^2) + cxz + dyt + ext + fyz.$$

By direct computation,

$$Q_{abcdef}\big(\tau(x, y, z, t)\big) = (x^2 + y^2)(z^2 + t^2) \cdot Q_{abcdfe}(x, y, z, t).$$

(Note that $ef$ changes to $fe$. Thus, if $e = f$, then $\tau$ restricts to an involution of the quadric $(Q = 0)$ but not in general.)

Assume now that we are over $\mathbb{R}$. We claim that $\tau$ is regular on the real points if $a, b \neq 0$. The only possible problem is with points where $(x^2 + y^2)(z^2 + t^2) = 0$. Assume that $(x^2 + y^2) = 0$. Then $x = y = 0$ and so $Q(x, y, z, t) = 0$ gives that $b(z^2 + t^2) = 0$ hence $z = t = 0$, a contradiction.

Whenever $Q$ has signature $(3, 1)$, we can view $(Q = 0)$ as a sphere and then $\tau$ gives a real algebraic automorphism of the sphere $S^2$, which is well defined up to left and right multiplication by $O(3, 1)$. A priori the automorphisms depend on $a, b, c, d, e, f$, so let us denote them by $\tau_{abcdef}$.

Given $S^2$, the above $\tau_{abcdef}$ depends on the choice of the base points, that is, 2 conjugate pairs of points on the complex quadric $S^2(\mathbb{C})$. The group $O(3, 1)$ has real dimension 6. Picking 2 complex points has real dimension 8. So the $\tau_{abcdef}$ should give a real 2-dimensional family of automorphisms modulo $O(3, 1)$.

We also need a degenerate version of the Cremona transformation when the 4 base points come together to a pair of points. With base points $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$, we get

$$(x, y, z, t) \mapsto (xz^2, yt^2, zt^2, z^2t).$$

If we put the base points at $(1, \pm i, 0, 0)$ then we get the transformation

$$\sigma'\colon (x, y, z, t) \mapsto \big(y(z^2 - t^2) + 2xzt, x(t^2 - z^2) + 2yzt, t(z^2 + t^2), z(z^2 + t^2)\big).$$

Take any quadric of the form

$$Q = Q'_{abcdef}(x,y,z,t) := a(x^2+y^2) + bz^2 + czt + dt^2 + e(xt+yz) + f(xz-yt).$$

By direct computation,

$$Q'_{abcdef}\big(\sigma'(x,y,z,t)\big) = (z^2+t^2)^2 \cdot Q'_{adcbef}(x,y,z,t).$$

As before, if $Q'$ has signature $(3,1)$, we can view $(Q'=0)$ as a sphere and then $\sigma'$ gives a real algebraic automorphism of the sphere $S^2$, which is well defined up to left and right multiplication by $O(3,1)$. Let us denote them by $\sigma_{abcdef}$. Despite the dimension count, the group $O(3,1)$ does not act with a dense orbit on the set of complex conjugate point pairs and complex conjugate directions. Indeed, after complexification, the quadric becomes $\mathbb{P}^1 \times \mathbb{P}^1$ and we can chose the two points to be $p_1 := (0,0)$ and $p_2 := (\infty,\infty)$. The subgroup fixing these two points is $\mathbb{C}^* \times \mathbb{C}^*$ and the diagonal acts trivially on the tangent directions at both of the points $p_i$. Thus the $\sigma_{abcdef}$ form a 1-dimensional family.

**Theorem 10.** *The group of algebraic automorphisms of $S^2$ is generated by $O(3,1)$, the $\tau_{abcdef}$ and $\sigma_{abcdef}$.*

**Remark 11.** It is possible that the $\tau_{abcdef}$ alone generate $\mathrm{Aut}(S^2)$. In any case, as the 4 base points come together to form 2 pairs, the $\tau_{abcdef}$ converge to the corresponding $\sigma_{abcdef}$. Thus the $\tau_{abcdef}$ generate a dense subgroup of $\mathrm{Aut}(S^2)$ (in the $C^\infty$-topology.)

One reason to use the $\sigma_{abcdef}$ is that, as the proof shows, the $\tau_{abcdef}$ and $\sigma_{abcdef}$ together generate $\mathrm{Aut}(S^2)$ in an "effective manner." By this we mean the following.

Any rational map $\Phi\colon S^2 \dashrightarrow S^2$ can be given by 4 polynomials

$$\Phi(x,y,z,t) = \big(\Phi_1, \Phi_2, \Phi_3, \Phi_4\big).$$

Note that $\Phi$ does not determine the $\Phi_i$ uniquely, but there is a "minimal" choice. We can add any multiple of $x^2+y^2+z^2-t^2$ to the $\Phi_i$ and we can cancel common factors. We choose $\max_i\{\deg \Phi_i\}$ to be minimal and call it the *degree* of $\Phi$. It is denoted by $\deg \Phi$. (It is easy to see that these minimal $\Phi_i$ are unique up to a multiplicative constant.) Note that $\deg \Phi = 1$ iff $\Phi \in O(3,1)$.

By "effective" generation we mean that given any $\Phi \in \mathrm{Aut}(S^2)$ with $\deg \Phi > 1$, there is a $\rho$ which is either of the form $\tau_{abcdef}$ or $\sigma_{abcdef}$ such that

$$\deg\big(\Phi \circ \rho\big) < \deg \Phi.$$

**12** (Proof of (10))**.** The proof is an application of the Noether-Fano method. See [Kollár-Smith-Corti04, Secs. 2.2–3] for details.

Let $k$ be a field and $Q \subset \mathbb{P}^3$ a quadric defined over $k$. Assume that $\mathrm{Pic}(Q) = \mathbb{Z}[H]$ where $H$ is the hyperplane class. Let $Q'$ be any other quadric and $\Phi\colon Q \dashrightarrow Q'$ a birational map. Then $\Gamma := \Phi^*|H_{Q'}|$ is a 3-dimensional linear system on $Q$ and $\Gamma \subset |dH_Q|$ for some $d$. Let $p_i$ be the (possibly infinitely near) base points of $\Gamma$ (over $\bar{k}$) and $m_i$ their multiplicities. As in [Kollár-Smith-Corti04, 2.8], we have the equalities

$$\Gamma^2 - \sum m_i^2 = \deg Q' \quad \text{and} \quad \Gamma \cdot K_Q + \sum m_i = \deg K_{Q'}.$$

In our case, these become

$$\sum m_i^2 = 2d^2 - 2 \quad \text{and} \quad \sum m_i = 4d - 4.$$

Next we see how the transformations $\tau_{abcdef}$ and $\sigma_{abcdef}$ change the degree of a linear system $\Gamma$.

**Example 13** (Cremona transformation on a quadric)**.** For the $\tau_{abcdef}$ series, pick 4 distinct points $p_1, \dots, p_4 \in Q$ such that no two are on a line in $Q$, not all 4 on a conic and assume that $s := m_1 + \dots + m_4 > 2d$. Blow up the 4 points and contract the 4 conics that pass through any 3 of them. The $p_i$ are replaced by 4 base points of multiplicities $2d - s + m_i$. Their sum is $8d - 4s + s = 8d - 3s$. Thus $4d - 4 = \sum m_i$ is replaced by $\sum m_i - s + (8d - s)$, hence $d$ becomes $d - (s - 2d) < d$.

For $\sigma_{abcdef}$, pick 2 distinct points $p_1, p_2 \in Q$ and 2 infinitely near points $p_3 \to p_1$ and $p_4 \to p_2$ such that no two are on a line in $Q$, not all 4 on a conic and assume that $s := m_1 + \dots + m_4 > 2d$. Blow up the points $p_1, p_2$ and then the points $p_3, p_4$. After this, we can contract the two conics that pass through $p_1 + p_2 + p_3$ (resp. $p_1 + p_2 + p_4$) and we can also contract the birational transforms of the exceptional curves over $p_1$ and $p_2$. The rest of the computation is the same. The $p_i$ are replaced by 4 base points of multiplicities $2d - s + m_i$. Their sum is $8d - 4s + s = 8d - 3s$. Thus $4d - 4 = \sum m_i$ is replaced by $\sum m_i - s + (8d - s)$ hence $d$ becomes $d - (s - 2d) < d$.

Thus, as long as we can find $p_1, \dots, p_4 \in Q$ (or infinitely near) such that $m_1 + \dots + m_4 > 2d$, we can lower $\deg \Phi$ by a suitable degree 3 Cremona transformation.

In order to find such $p_i$, assume first to the contrary that $m_i \leq d/2$ for every $i$. Then

$$2d^2 - 2 = \sum m_i^2 \leq \tfrac{d}{2} \sum m_i = \tfrac{d}{2}(4d - 4) = 2d^2 - 2d.$$

This is a contradiction, unless $d = 1$ and $\Phi$ is a linear isomorphism.

If we work over $\mathbb{R}$ and we assume that there are no real base points, then we have at least one complex conjugate pair of base points with multiplicity $m_i > d/2$. We are done if we have found 2 such pairs.

In any case, up to renumbering the points, we have $m_1 = m_2 = \frac{d}{2} + a$ for some $d/2 \geq a > 0$. Assume next that all the other $m_j \leq \frac{d}{2} - a$. Then

$$\begin{aligned} 2d^2 - 2 = \sum m_i^2 &\leq 2\big(\tfrac{d}{2} - a\big)^2 + \big(\tfrac{d}{2} - a\big)\big(\sum m_i - d + 2a\big) \\ &= 2\big(\tfrac{d}{2} - a\big)^2 + \big(\tfrac{d}{2} - a\big)\big(4d - 4 - d + 2a\big). \end{aligned}$$

By expanding, this becomes

$$(a + 2)(d - 4) \leq -6.$$

Thus $d \in \{1, 2, 3\}$. If $d = 3$ then $a + 2 \geq 6$ so $d/2 \geq a \geq 4$ gives a contradiction. If $d = 2$ then we get $a = 1$. Thus $\Gamma$ consists of quadric sections with singular points at $p_1, p_2$. These are necessarily reducible (they have $p_a = 1$ with 2 singular points), again impossible.

We also need to show that no two of the points lie on a line and not all 4 are on a conic. For any line $L \subset Q(\mathbb{C})$, $(L \cdot \Gamma) = d$ gives that

$$\sum_{i : p_i \in L} m_i \leq d.$$

In particular, $m_i \leq d$ for every $i$ and if $p_i, p_j$ are on a line then $m_i + m_j \leq d$. Thus out of $p_1, \dots, p_4$ only $p_3, p_4$ could be on a line. But $p_3, p_4$ are conjugates, thus they would be on a real line. There is, however, no real line on $S^2$.

Similarly, for any conic $C \subset Q(\mathbb{C})$, $(C \cdot \Gamma) = 2d$ gives that $\sum_{i : p_i \in C} m_i \leq 2d$. Thus not all 4 points are on a conic.

**Remark 14.** Note that we started the proof over an arbitrary field, but at the end we had to assume that that we worked over $\mathbb{R}$. For a quadric surface $Q$ with Picard number one, the above method should give generators for the group $\mathrm{Bir}^*(Q)$ of those birational self-maps that are regular along $Q(k)$. However, for other fields $k$, other generators also appear if there are more than 2 conjugate base points.

## 2. The identity component

The purpose of this section is to prove (4) for the identity components. For the sphere and the torus these were done by Lukackiĭ. Next we prove (4) for the identity components in the case $R$ is the non-orientable surface $R_g$.

**15** (The results of Lukackiĭ)**.** The paper [Lukackiĭ77, Thm. 2] proves that $SO(n+1,1)$ is a maximal closed subgroup of $\mathrm{Diff}_0(S^n)$. In particular, $O(n+1,1)$ and anything else generate a dense subgroup of $\mathrm{Diff}(S^n)$.

Since this result seems not to have been well known, let us give a quick review of the steps of the proof.

We start with the Lie algebra of polynomial vector fields $H^0(S^n, T_{S^n})$. Its structure as an $\mathfrak{so}(n+1)$ representation was described by [Kirillov57], including the highest weight vectors.

As we go from $\mathfrak{so}(n+1)$ to $\mathfrak{so}(n+1,1)$, we get extra unipotent elements and their action on the highest weight vectors can be computed explicitly. One obtains that $\mathfrak{so}(n+1,1)$ is a maximal Lie subalgebra of $H^0(S^n, T_{S^n})$. This implies that $SO(n+1,1)$ is a maximal connected closed subgroup of $\mathrm{Diff}_0(S^n)$. It is easy to check that $SO(n+1,1)$ is its own normalizer, which rules out all disconnected subgroups as well.

The paper [Lukackiĭ79] gives generators of the Lie algebra $H^0(T^n, T_{T^n})$ where $T^n$ denotes the $n$-dimensional torus

$$T^n := \bigl(x_1^2 + y_1^2 - 1 = \cdots = x_n^2 + y_n^2 - 1 = 0\bigr) \subset \mathbb{R}^{2n}.$$

This is again through explicit Lie theory. Up to coordinate changes by $GL(n,\mathbb{Z})$, the generators are the shears

$$g(x_1,\dots,x_{n-1}) \cdot \Bigl(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_n}\Bigr) \quad \text{and} \quad y_n \cdot \Bigl(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_n}\Bigr).$$

(Using polar angles $\phi_i$, the latter is the vector field $\sin\phi_n \cdot \bigl(\partial/\partial\phi_n\bigr)$.) Up to a factor of 2, this is exactly the tangent vector field corresponding to the unipotent group

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \subset PGL(2,\mathbb{R}) \cong O(2,1) \quad \text{acting on } S^1.$$

**Definition 16.** Let $X$ and $Y$ be real algebraic manifolds and let $I$ be any subset of $X$. A map $f$ from $I$ into $Y$ is *algebraic* if there is a Zariski open subset $U$ of $X$ containing $I$ such that $f$ is the restriction of an algebraic map from $U$ into $Y$.

Consider the standard sphere $S^2 \subset \mathbb{R}^3$ and let $L$ be a line through the origin. Choose coordinates such that $L$ is the $x$-axis and $S^2 := (x^2+y^2+z^2=1) \subset \mathbb{R}^3$. Let $M\colon [-1,1] \to O(2)$ be a real algebraic map. Then

$$\Phi_M\colon S^2 \to S^2 \quad \text{given by} \quad (x,y,z) \mapsto \bigl(x, (y,z)\cdot M(x)\bigr)$$

is an automorphism of $S^2$, called the *twisting map* with axis $L$ and associated to $M$. A conjugate of a twisting map by an element of $O(3,1)$ is also called a twisting map.

The following results are proved in [Huisman-Mangolte09a].

**Theorem 17.** *Notation as above.*

(1) *Any $C^\infty$ map $M_0 : [-1,1] \to O(2)$ can be approximated by real algebraic maps $M_s : [-1,1] \to O(2)$. Moreover, given finitely many points $t_i \in [-1,1]$, we can choose the $M_s$ such that $M_s(t_i) = M_0(t_i)$ for every $i$.*
(2) *Given distinct points $p_1, \dots, p_m$ and $q_1, \dots, q_m$ there are two twisting maps (with different axes) $\Phi_1$ and $\Phi_2$ such that $\Phi_1 \circ \Phi_2(q_i) = p_i$ for every $i$. Moreover,*
    (a) *if $p_j = q_j$ for some values of $j$ then we can assume that $\Phi_1(q_j) = \Phi_2(q_j) = q_j$ for these values of $j$, and*
    (b) *if $p_i$ is near $q_i$ for every $i$ then we can assume that the $\Phi_1, \Phi_2$ are near the identity.*
(3) *Let $R$ be any real algebraic surface that is obtained from $S^2$ by repeatedly blowing up $m$ real (possibly infinitely near) points and let $r_1, \dots, r_n$ be points in $R$. Then there are (nonunique) distinct points $p_1, \dots, p_m$ and $q_1, \dots, q_n$ and an isomorphism $\phi : R \to B_{p_1,\dots,p_m} S^2$ such that $\phi(r_i) = q_i$.*

By adding more points in (17.3) and compactness, we obtain the following stronger version:

**Corollary 18.** *Let $R$ be any real algebraic surface that is obtained from $S^2$ by repeatedly blowing up $m$ real (possibly infinitely near) points and let $r_1, \dots, r_n$ be points in $R$. There is a finite open cover $R = \cup_j W_j$ such that for every $j$ there are distinct points $p_{1j}, \dots, p_{mj}, q_{1j}, \dots, q_{nj} \in S^2$ and an isomorphism $\phi_j : R \to B_{p_{1j},\dots,p_{mj}} S^2$ such that $\phi_j(r_i) = q_{ij}$ and $\phi_j(W_j) \subset S^2 \setminus \{p_{1j}, \dots, p_{mj}\}$.* □

**19** (Proof of (2))**.** Let $p_1, \dots, p_n, q \in S^2$ be any finite set of distinct points, and let $\phi \in \mathrm{Diff}(S^2, p_1, \dots, p_n)$. By (15) there are automorphisms $\psi_s \in \mathrm{Aut}(S^2)$ such that $\psi_s$ converges to $\phi$.

For any $s$ and $i$, set $q_i^s := \psi_s(p_i)$. As $\psi_s$ converges to $\phi$, the $q_i^s$ converge to $p_i$ for every $i$. By (17.2.b) there are automorphisms $\Phi_s$ such that $\Phi_s(q_i^s) = p_i$ and $\Phi_s$ converges to the identity. Thus the composites $\Phi_s \circ \psi_s$ are in $\mathrm{Aut}(S^2, p_1, \dots, p_n)$ and they converge to $\phi$.

**Proposition 20.** *Let $R$ be any real algebraic surface that is obtained from $S^2$ by repeatedly blowing up $g$ real (possibly infinitely near) points and let $r_1, \dots, r_n$ be points in $R$. Then the group $\mathrm{Aut}_0(R, r_1, \dots, r_n)$ of algebraic automorphisms homotopic to identity is dense in $\mathrm{Diff}_0(R, r_1, \dots, r_n)$.*

*Proof.* Let $\phi\colon R \to R$ be a diffeomorphism fixing $r_1, \dots, r_n$, and homotopic to the identity. Choose $R = \cup_j W_j$ as in (18). By a partition of unity argument, $\phi$ can be written as the composite of diffeomorphisms $\phi_\ell\colon R \to R$ fixing $r_1, \dots, r_n$ such that each $\phi_\ell$ is the identity outside some $W_j \subset R$.

In particular, each $\phi_\ell$ descends to a diffeomorphism $\phi'_\ell$ of $S^2$ which fixes the points $p_{1j}, \dots, p_{gj}$ and $q_{1j}, \dots, q_{nj}$. By (2), we can approximate $\phi'_\ell$ by algebraic automorphisms $\Phi'_{\ell,s}$ fixing all the points $p_{1j}, \dots, p_{gj}$ and $q_{1j}, \dots, q_{nj}$. Since the $\Phi'_{\ell,s}$ fix $p_{1j}, \dots, p_{gj}$, they lift to algebraic automorphisms $\Phi_{\ell,s}$ of $R \cong B_{p_{1j},\dots,p_{gj}} S^2$ fixing the points $r_1, \dots, r_n$. The composite of the $\Phi_{\ell,s}$ then converges to $\phi$. □

## 3. Generators of the mapping class group

**Definition 21.** Let $R$ be a compact, closed surface and $p_1, \dots, p_n$ distinct points on $R$. The *mapping class group* is the group of connected components of those diffeomorphisms $\phi\colon R \to R$ such that $\phi(p_i) = p_i$ for $i = 1, \dots, n$.

$$\mathcal{M}(R, p_1, \dots, p_n) := \pi_0\bigl(\mathrm{Diff}(R, p_1, \dots, p_n)\bigr).$$

Up to isomorphism, this group depends only on the orientability and the genus of $R$. The orientable case has been intensely studied. Recent important results about the non-orientable case are in [Korkmaz02, Wahl08].

(In the literature, $\mathcal{M}_{g,n}$ is used to denote both the mapping class group of an orientable genus $g$ (hence with Euler characteristic $2-2g$) surface with $n$ marked points and the mapping class group of a non-orientable genus $g$ (hence with Euler characteristic $2-g$) surface with $n$ marked points.)

In preparation for the next section, we establish a somewhat new explicit set of generators in the non-orientable case.

Write $R$ as $B_{p_1,\dots,p_g}S^2$, the blow up of $S^2$ at $g$ points. We start by describing some elements of the mapping class group. For more details see [Lickorish65, Chillingworth69, Korkmaz02].

**Definition 22** (Dehn twist)**.** Let $R$ be any surface and $C \subset R$ a simple closed smooth curve such that $R$ is orientable along $C$. Cut $R$ along $C$, rotate one side around once completely and glue the pieces back together. This defines a diffeomorphism $t_C$ of $R$, see Figure 2.

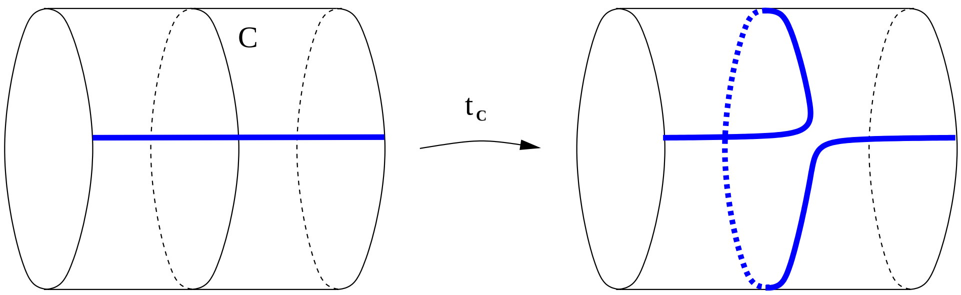


Figure 2. The effect of the Dehn twist around $C$ on a curve.

The inverse $t_C^{-1}$ corresponds to rotating one side the other way. Up to isotopy, the pair $\{t_C, t_C^{-1}\}$ does not depend on the choice of $C$ or the rotation. Either of $t_C$ and $t_C^{-1}$ is called a *Dehn twist* using $C$. On an oriented surface, with $C$ oriented, one can make a sensible distinction between $t_C$ and $t_C^{-1}$. This is less useful in the non-orientable case.

**Definition 23** (Crosscap slide)**.** Let $D$ be a closed disc and $p, q \in D$ two points. Take a simple closed curve $C$ in $D$ passing through $p, q$ and let $C'$ denote the corresponding curve in $B_qD$. Let $M_p$ be a small disc around $p$. Let $\{\phi_t : t \in [0,1]\}$ be a continuous family of diffeomorphisms of $B_qD$ such that $\phi_0$ is the identity, each $\phi_t$ is the identity near the boundary and as $t$ increases, the $\phi_t$ slide $M_p$ once around $C$. At $t = 1$, $M_p$ returns to itself with its orientation reversed, as in Figure 3. In

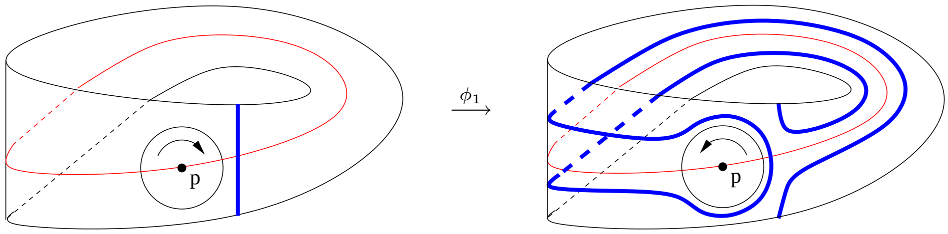


FIGURE 3. Cross-cap slide.

particular, $\phi_1(p) = p$. Thus $\phi_1$ can be lifted to a diffeomorphism of $B_{p,q}D$ which is not isotopic to the identity but is the identity near the boundary.

Let $R$ be any surface, $U \subset R$ a closed subset with $C^\infty$ boundary and $\tau : U \to B_{p,q}D$ a diffeomorphism. Then $\tau^{-1}\phi_1\tau \colon U \to U$ is the identity near the boundary of $U$, hence it can be extended by the identity on $R \setminus U$ to a diffeomorphism of $R$. Up to isotopy, this diffeomorphism does not depend on the choice of $C$, $\phi_t$ and $\tau$. It is called a *cross-cap slide* or a *Y-homeomorphism* using $U$. Note that for a cross-cap slide to exist, $R$ must be non-orientable and of genus at least 2.

**24** (Generators of the mapping class group)**.** Let $R_g$ be a non-orientable surface of genus $g \geq 1$. We write $R_g := B_{p_1,\dots,p_g}S^2$ with exceptional curves $E_i \subset R_g$ and let $\pi \colon R_g \to S^2$ be the blow down map.

The map $\pi$ gives a one-to-one correspondence between

- simple closed smooth curves $C_R \subset R_g$ whose intersection with any exceptional curve $E_i$ is transversal, and
- immersed curves $C = \pi(C_R) \subset S^2$ whose only self-intersections are at the $p_i$ and no two branches are tangent.

Generators of the mapping class group were first established by [Lickorish65] and simplified by [Chillingworth69]. The case with marked points was settled by [Korkmaz02].

The generators are the following

(1) Dehn twists along $C_R$ for certain smooth curves $C \subset S^2$ that pass through an even number of the $p_1, \dots, p_g$. (No self-intersections at the $p_i$.)
(2) Cross-cap slides using a disc $D \subset S^2$ that contains exactly 2 of the points $p_1, \dots, p_g$.

The results of [Chillingworth69] and of [Korkmaz02] are more precise in that only very few of these generators are used. In the unmarked case, the above formulation is established in the course of the proof and stated on [Chillingworth69, p.427].

We will need somewhat different generators. We thank M. Korkmaz for answering many questions and especially for pointing out that one should use the lantern relation (26) to establish the following.

**Proposition 25.** *The following elements generate the mapping class group of the marked surface* $(B_{p_1,\dots,p_g}S^2, q_1, \dots, q_n)$.

(1) *Dehn twists along* $C_R$ *for certain smooth curves* $C \subset S^2$ *that pass through* $0, 2$ *or* $4$ *of the points* $p_1, \dots, p_g$*. (No self-intersections at the* $p_i$*.)*
(2) *Cross-cap slides using a disc* $D \subset S^2$ *that contains exactly 2 of the points* $p_1, \dots, p_g$.

Proof. We have included all the cross-cap slides from (24). Thus we need to deal with Dehn twists along $C_R$ where $C \subset S^2$ is a simple closed curve passing through $m$ of the points $p_1, \dots, p_g$ with $m > 4$.

Using induction, it is enough to show that the Dehn twist along $C_R$ can be written as the product of Dehn twists along curves $C'_R$ where each $C' \subset S^2$ is a simple closed curve passing through fewer than $m$ of the points $p_1, \dots, p_g$.

Assume for simplicity that $C$ passes through $p_1, \dots, p_m$ with $m > 4$ (and even). For $I \subset \{1, \dots, m\}$ let $t_I$ be a Dehn twist using a simple closed curve $C_I$ passing through the $\{p_i : i \in I\}$ but none of the others. The precise choice of the curve will be made later. We show that, with a suitable choice of the curves, $t_{12345\dots m}$ is a product of the Dehn twists $t_{125\dots m}, t_{345\dots m}, t_{1234}, t_{5\dots m}, t_{12}, t_{34}$.

This is best shown by a picture for $m = 8$. In Figure 4, $t_{12345678}$ is a product of the Dehn twists $t_{125678}, t_{345678}, t_{1234}, t_{5678}, t_{12}, t_{34}$. The shaded region is a sphere with four holes, and corresponds to a neighborhood of the lift to $R_8$ of $C_{12345678}$. On each side of the picture are drawn the curves corresponding to the Dehn twists of the same side in (26.1):

a) $C_{12}$, $C_{34}$, $C_{5678}$, $C_{12345678}$, b) $C_{1234}$, $C_{125678}$, $C_{345678}$. □

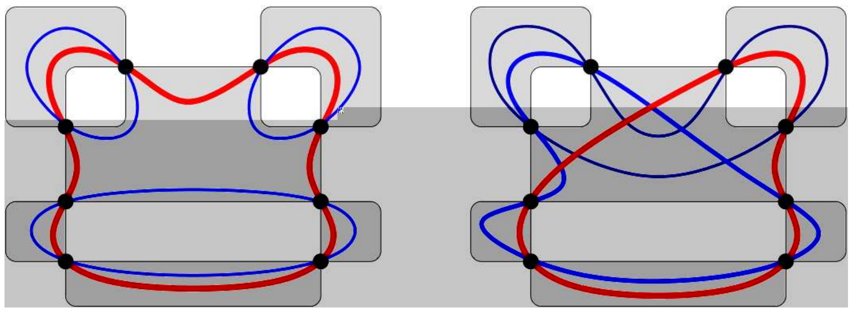

FIGURE 4. Lantern relation for $m = 8$.

**26** (Lantern relation of Dehn)**.** [Dehn38, Johnson79] Fix 4 points $q_0, \dots, q_3 \in S^2$. Let $t_i$ be the Dehn twist using a small circle around $q_i$ and for $i, j \in \{1, 2, 3\}$, let $t_{ij}$ be the Dehn twist using a simple closed curve that separates $q_i, q_j$ from the other 2 points. Then, with suitable orientations,

$$t_0 t_1 t_2 t_3 = t_{12} t_{13} t_{23} \ , \tag{26.1}$$

where the equality is understood to hold in $\mathcal{M}(S^2, q_0, \dots, q_3)$.

## 4. Automorphisms and the mapping class group

The main result of this section is the following.

**Theorem 27.** *Let $R$ be a real algebraic surface that is obtained from $S^2$ by blowing up points and $p_1, \dots, p_n \in R$ distinct marked points. Then the natural map*

$$\operatorname{Aut}(R, p_1, \dots, p_n) \twoheadrightarrow \mathcal{M}(R, p_1, \dots, p_n) \quad \textit{is surjective.}$$

Proof. We prove that all the generators of the mapping class group listed in (25) can be realized algebraically. There are 4 cases to consider:

(1) Dehn twists along $C_R \subset R$ for smooth curves $C \subset S^2$ that pass through either
   (a) none of the points $p_i$,
   (b) exactly 2 of the points $p_i$, or
   (c) exactly 4 of the points $p_i$.
(2) Cross-cap slides using a disc $D \subset S^2$ that contains exactly 2 of the points $p_i$.

We start with the easiest case (27.1.a).

**28** (Algebraic realization of Dehn twists)**.** Let $C \subset S^2$ be a smooth curve passing through none of the points $p_i$. After applying a suitable automorphism of $S^2$, we may assume that $C$ is the big circle $(x = 0)$.

Consider the map $g\colon [-1, 1] \to O(2)$ where $g(t) = \mathbf{1}$ for $t \in [-1, -\epsilon] \cup [\epsilon, 1]$ and $g(t)$ is the rotation by angle $\pi(1 + t/\epsilon)$ for $t \in [-\epsilon, \epsilon]$. Let $M\colon [-1, 1] \to O(2)$ be an algebraic approximation of $g$ such that the corresponding twisting (16) $\Phi_M$ is the identity at the points $p_i$. Then $\Phi_M$ is an algebraic realization of the Dehn twist around $C$.

On the torus, the same argument works for either of the $S^1$-factors. Up to isotopy and the natural $GL(2, \mathbb{Z})$-action, this takes care of all simple closed curves.

Next we deal with the hardest case (27.1.c).

**29** (4 pt case)**.** After applying a suitable automorphism of $S^2$, we may assume that $C$ is close to a circle in $S^2$ but the 4 points do not lie on a circle.

Let us take an annular neighborhood of $C$ and blow up the 4 points $p_1, \dots, p_4$. The resulting open surface is denoted by $W \subset B_{p_1,\dots,p_4}S^2$. It contains the curve $C_R$ and the 4 exceptional curves $E_1, \dots, E_4$.

If we cut the blown-up annulus $W$ along the 5 curves $A_1, \dots, A_4, D$ as indicated of the left hand side of Figure 5, we get the contractible surface $U$ indicated on the right hand side of Figure 5. The left and right hand sides of $U$ are identified to form a cylinder, giving a neighborhood of the curve $C_R \subset B_{p_1,\dots,p_4}S^2$. The big rectangle with lighter shading in $U$ on the right corresponds to the lighter shaded are in $W$ on the left. The 4 top and 4 bottom line segments of $U$ are identified to form 4 Möbius bands.

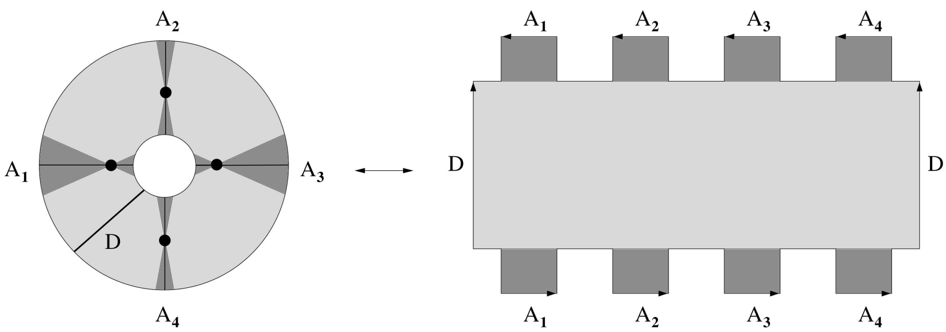


FIGURE 5. Two models of the annulus blown up in 4 points.

Next, in Figure 6 we show the 4 exceptional curves.

Figure 7 shows the images of the curves $E_i$ after the Dehn twist around $C_R$.

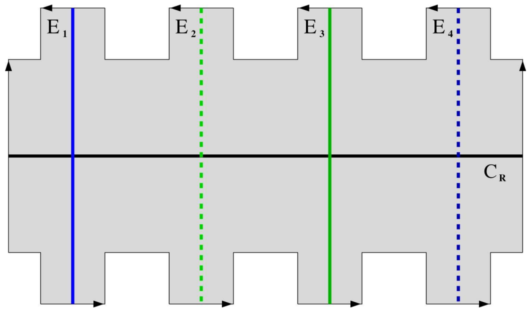


FIGURE 6. The 4 exceptional curves.

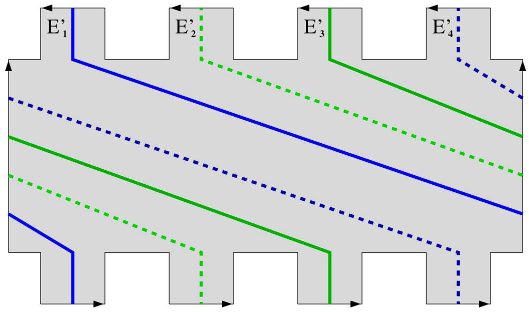


FIGURE 7. Effect of the Dehn twist around $C_R$.

These images can be deformed to obtain a configuration as in Figure 8. Note that now $E_i$ intersects $E'_j$ iff $i \neq j$.

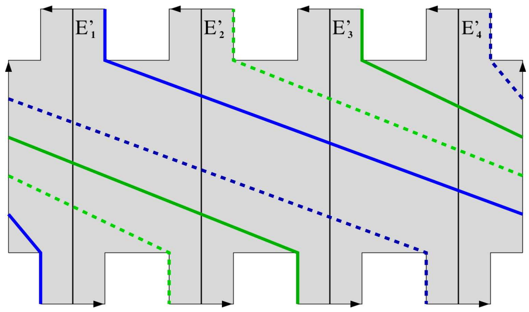


FIGURE 8. Deformation of Figure 7.

Next we convert this back to the annulus model $W$ on the left hand side of Figure 5.

We obtain Figure 9.

The images of the exceptional curves $E_1, \dots, E_4$ under the standard Cremona transformation with base points $p_1, \dots, p_4$ are shown in Figure 1.

We see by direct inspection that the two quartets of curves in Figures 1 and 9 are isotopic. Thus, if we first apply the Dehn twist and then the (inverse) Cremona transformation and a suitable isotopy, we get a diffeomorphism $\phi\colon R_n \to R_n$ such

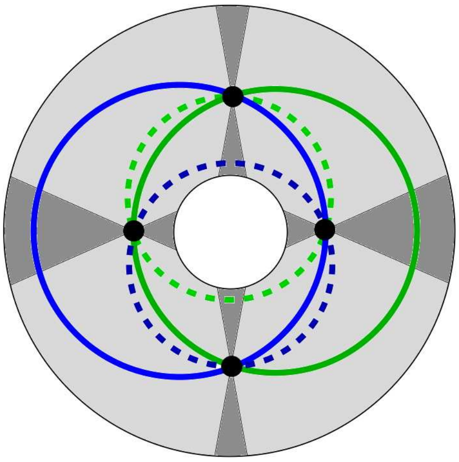

FIGURE 9. Images of the four exceptional curves.

that $\phi(E_i) = E_i$. That is, $\phi$ is lifted from a diffeomorphism of the g-pointed sphere $(S^2, p_1, \dots, p_n)$. By (2), any such diffeomorphism is isotopic to an algebraic automorphism. Hence the Dehn twist along $C_R$ is also algebraic.

**30** (2 pt case)**.** The proof is the same as in the 4 point case but the description is easier.

A neighborhood of $C$ gives an annulus with 2 blown-up points. After the Dehn twist we get two curves $E_1', E_2'$ as in Figure 10.

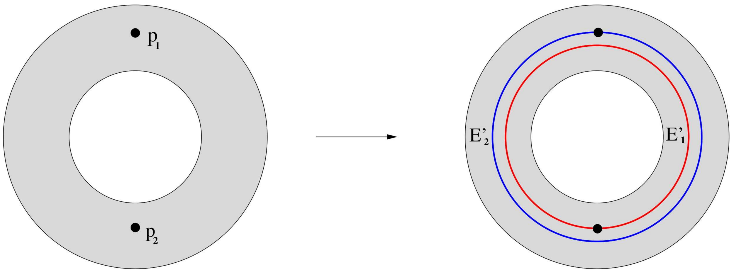


FIGURE 10. Cremona transformation with 2 real base points.

We can assume that the two curves $E_1', E_2'$ are close to being circles, that is, close to the intersections $S^2 \cap H_i$ for some planes for $i = 1, 2$. Let $q, \bar{q}$ be the 2 (complex conjugate) points where these 2 planes $H_i$ intersect the complexified sphere $Q$. Then the Cremona transformation with base points $p_1, p_2, q, \bar{q}$ is the inverse of the Dehn twist, again up to a diffeomorphism of $S^2$.

**31** (Crosscap slides)**.** Here the topological picture is given by Figure 11. Note that $E_1$ is mapped to itself and $E_2$ is mapped to the (almost) circle $E_2'$. Up to isotopy, we can replace $E_1$ with a small circle $E_1'$ passing through $p_1$.

As in (30), we obtain $q, \bar{q}$ such that the Cremona transformation with base points $p_1, p_2, q, \bar{q}$ is the inverse of the Dehn twist, up to a diffeomorphism of $S^2$.

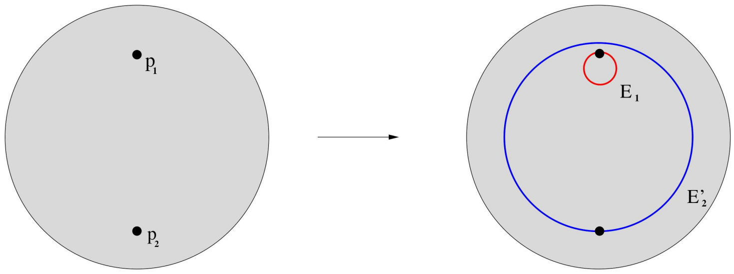


FIGURE 11. Cross-cap slides.

**32** (Proof of (4))**.** Let $\phi : (R, q_1, \dots, q_n) \to (R, q_1, \dots, q_n)$ be any diffeomorphism. By (20), there is an automorphism $\Phi_1 \in \operatorname{Aut}(R, q_1, \dots, q_n)$ such that $\Phi_1^{-1} \circ \phi$ is homotopic to the identity.

By (27), we can approximate $\Phi_1^{-1} \circ \phi$ by a sequence of automorphisms $\Psi_s \in \operatorname{Aut}(R, q_1, \dots, q_n)$. Thus $\Phi_1 \circ \Psi_s \in \operatorname{Aut}(R, q_1, \dots, q_n)$ converges to $\phi$. □

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Princeton University, Princeton NJ 08544-1000
*E-mail address*: `kollar@math.princeton.edu`

Université de Savoie, 73376 Le Bourget du Lac Cedex, France.
*E-mail address*: `mangolte@univ-savoie.fr`