Topology of uniruled real algebraic threefolds

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Constraints on the real locus

X projective variety defined over $\mathbb{R}$, $\dim X = n$
$X(\mathbb{R}) :=$ *real locus* of X

If X is non singular and if $X(\mathbb{R}) \neq \emptyset$

$$\Rightarrow \begin{cases} X \text{ and } X(\mathbb{R}) \text{ are compact } \mathcal{C}^\infty\text{-manifolds} \\ \dim_{\mathbb{R}} X(\mathbb{R}) = \dim_{\mathbb{C}} X = n \end{cases}$$

Example of constraint for $n = 2$:

Theorem (Comessatti, 1914)

X geometrically rational non singular surface
($= \mathbb{C}$-birational to $\mathbb{P}^2_{\mathbb{C}}$)
$L \subset X(\mathbb{R})$ an orientable connected component
$\Rightarrow L$ is homeomorphic to the sphere S^2 or to the torus $S^1 \times S^1$

Question :
What topology for the real locus when X "close" to $\mathbb{P}^n$?

Uniruled and rationally connected varieties

X projective variety defined over $\mathbb{R}$, $\dim X = n$

Definition

X geometrically rational $\Leftrightarrow$ $\mathbb{C}$-birational to $\mathbb{P}^n_{\mathbb{C}}$

X rationally connected (r. c.)
$\Leftrightarrow \forall x, y \in X$, $\exists$ rational curve $C \subset X$ such that $x, y \in C$

X uniruled $\Leftrightarrow \forall\, x \in X$, $\exists$ rational curve $C \subset X$ such that $x \in C$

Remarks

- geometrically rational $\Rightarrow$ r. c. $\Rightarrow$ uniruled.
- $X \to B$ with uniruled general fiber $\Rightarrow$ X uniruled,
- If $n < 4$, X uniruled $\Leftrightarrow \mathrm{kod}(X) = -\infty$.

Examples

$\mathbb{P}^n$, hypersurfaces in $\mathbb{P}^{n+1}$ of degree $\leq n+1$,
Fano, conic fibrations, rational surface fibrations...

$n = 2$, uniruled surfaces

Theorem (Comessatti, 1914)

X uniruled non singular projective surface
$L \subset X(\mathbb{R})$ an orientable connected component $\Rightarrow g(L) < 2$

i.e., if $L = {}_{\Lambda}\backslash\mathbb{H}^2$ orientable and X uniruled, then $L \not\subset X(\mathbb{R})$

where $\Lambda < \mathrm{Isom}(\mathbb{H}^2)$ discrete subgroup acting without fixed point

Conversely, if $L = S^2, S^1 \times S^1$ or a non orientable surface, there exists a rational surface X such that $X(\mathbb{R}) \sim L$.
Consider $S^2, S^1 \times S^1 = \{(x, y, z, t) \in \mathbb{P}^3(\mathbb{R}), \quad x^2 + y^2 \pm z^2 = t^2\}$,
$B_P S^2 = \mathbb{R}\mathbb{P}^2$, $B_Q \mathbb{R}\mathbb{P}^2 =$ Klein bottle, then iterate...

$n = 3$, history

1. Kollár ($\sim$ 1999) theorems and conjectures
   - MMP over $\mathbb{R} \longrightarrow$ Mori fibrations
   - conic fibrations
   - del Pezzo fibrations
2. Viterbo, Eliashberg (1999)
   $_\Lambda\backslash\mathbb{H}^{n\geq 3} \not\subset X(\mathbb{R})$ if X uniruled non singular projective manifold
3. Huisman, M– (2005, 2005)
   uniruled models of Seifert manifolds and of $\#$lens spaces
4. Catanese, M– (2008, 2009)
   constraints if X r. c. + Comessatti's thm. for singular surfaces
5. M–, Welschinger (2011)
   $_\Lambda\backslash\mathrm{Sol} \not\subset X(\mathbb{R})$ if X del Pezzo fibration

$n = 3$, uniruled threefolds

L compact topological manifold without boundary of dimension 3

- $L :=$ Seifert manifold $\Leftrightarrow \exists g\colon L \to F$, locally trivial S^1-fibration up to a finite number k of multiple fibers (multiplicities k_j)
- $L :=$ lens space $\Leftrightarrow L$ cyclic quotient of S^3 by some $\mathbb{Z}_{k_j}$

Theorem (Kollár, 1999)

X non singular projective threefold such that $X(\mathbb{R})$ orientable
$L \subset X(\mathbb{R})$ connected component

1. X *uniruled*
   $\Rightarrow$ *up to connected sums with* $\mathbb{RP}^3$ *and* $S^1 \times S^2$, *up to finitely many exceptions, and up to infinitely many torus bundles and* $\mathbb{Z}/2$*-quotients of them,*
   L is a Seifert manifold or a connected sum of lens spaces
2. *Let* $k := \#\{\textit{multiple fibers}\}$ *or* $\#\{\textit{lens spaces}\}$
   X *rationally connected* $\Rightarrow k \leq 6$

$n = 3$, uniruled threefolds, converse result

Theorem (Huisman, M–, 2005)

L any connected sum of $\mathbb{RP}^3$ and $S^1 \times S^2$ with a Seifert manifold or with any connected sum of lens spaces

$$\Rightarrow \exists \text{ uniruled real projective threefold } X \text{ such that } L \subset X(\mathbb{R})$$

$n = 3$, rationally connected threefolds

$X \longrightarrow S$
$\mathbb{P}^1$-fibred projective threefold, $X(\mathbb{R})$ orientable
$L \subset X(\mathbb{R})$ connected component $k := k(L)$, k_j, $j = 1 \dots k$ multiplicities

Theorem (Catanese, M–, 2007, 2008)

X r. c. ($\Leftrightarrow S$ geometrically rational) $\Rightarrow$

- $k(L) \leq 4$,
- $\sum(1 - \frac{1}{k_j}) \leq 2$,
- $L \to S^1 \times S^1$ *Seifert* $\Rightarrow k(L) = 0$.

Suspension of a diffeomorphism of the torus $S^1 \times S^1$

$S^1 := \{|z| = 1\} \subset \mathbb{C}$, $\quad S^1 \times S^1 := \{|u| = 1, |v| = 1\} \subset \mathbb{C} \times \mathbb{C}$

$\mathrm{Gl}_2(\mathbb{Z})$ acts on $S^1 \times S^1$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto [(u, v) \mapsto (u^a v^b, u^c v^d)]$

For $M \in \mathrm{Gl}_2(\mathbb{Z})$, let

$L := \left(S^1 \times S^1\right) \times [0, 1]/((u, v), 0) \sim (M \cdot (u, v), 1)$

$p \colon L \to S^1 = [0, 1]/(0 \sim 1)$ is then a torus bundle.

Let λ be an eigenvalue of M

- $|\lambda| = 1$, M periodic $\Rightarrow L = {}_{\Lambda}\backslash\mathbb{E}^3$ and is also Seifert fibred
- $|\lambda| = 1$, M non periodic $\Rightarrow L = {}_{\Lambda}\backslash\mathrm{Nil}$ and is also Seifert fibred
- $|\lambda| \neq 1$, i.e. M hyperbolic $\Rightarrow L = {}_{\Lambda}\backslash\mathrm{Sol}$ is NOT Seifert fibred

Sol-manifolds

The Lie group Sol is the set $\mathbb{R}^3$ endowed with the semi-direct product induced by the action :
$\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ (z,(x,y)) \mapsto (e^z x, e^{-z} y)$
The group law is :

$$((\alpha,\beta,\lambda),(x,y,z)) \mapsto \left(e^{\lambda}x+\alpha, e^{-\lambda}y+\beta, z+\lambda\right)$$

Definition

L is a Sol-manifold
$\Leftrightarrow \exists \Lambda \subset \mathrm{Isom}(\mathrm{Sol})$ discrete subgroup of isometries acting without fixed point such that

$$L = {}_{\Lambda}\backslash \mathrm{Sol}$$

Classification of closed Sol-manifolds

1. Suspensions of hyperbolic diffeomorphisms e.g. $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
2. Sapphires $L \to [0,1]$, $\mathbb{Z}/2$-quotients of case 1.

n=3, Homogeneous differentiable manifolds

Recall:
Let G be a Lie group corresponding to one of the eight Thurston's geometries, $\Lambda \subset \mathsf{Isom}(G)$ discrete subgroup of isometries acting without fixed point, $L = {}_{\Lambda}\backslash G$
$\Rightarrow$ L is Seifert fibred, or $G = \mathrm{Sol}$, or $G = \mathbb{H}^3$.

Theorem (M–, Welschinger, 2011)

An orientable closed Sol*-manifold does not embed in the real locus of a projective threefold fibered over a curve with rational fibers.*

$n = 3$ collect results

Recall:
L is Seifert fibred
$\Rightarrow L = {}_{\Lambda}\backslash G$ such that $G = S^3, S^2 \times \mathbb{E}^1, \mathbb{E}^3, \mathbb{H}^2 \times \mathbb{E}^1, \mathrm{SL}(2, \mathbb{R})$

Theorem

1. *X non singular projective threefold, $X(\mathbb{R})$ orientable $L \subset X(\mathbb{R})$ connected component*
   - 1.1 *X uniruled $\Rightarrow$ up to finitely many exceptions, L is a connected sums of $\mathbb{RP}^3$'s and $S^1 \times S^2$'s with a Seifert manifold or with a connected sum of lens spaces*
   - 1.2 *X rationally connected and $L \to B$ Seifert with orientable orbit space $\Rightarrow L$ is not ${}_{\Lambda}\backslash \mathbb{H}^2 \times \mathbb{E}^1$ nor ${}_{\Lambda}\backslash \mathrm{SL}(2, \mathbb{R})$*
2. *L any connected sum of $\mathbb{RP}^3$ and $S^1 \times S^2$ with any Seifert manifold or with any connected sum of lens spaces $\Rightarrow \exists$ uniruled real projective threefold X such that $L \subset X(\mathbb{R})$*

$n = 2$, Comessatti's thm. for singular surfaces

manifold = charts are diffeomorphisms
orbifold = charts are finite coverings
Du Val singularities = canonical singularities for surfaces
= quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$
X geometrically rational surface
$M \subset \overline{X(\mathbb{R})}$ a connected component of the topological normalization

Theorem (Comessatti, 1914)

X nonsingular and M orientable
$\Rightarrow M$ is a sphere or a torus

Theorem (Catanese, M–, 2008, 2009)

X with Du Val singularities and M orientable orbifold
$\Rightarrow M$ is spherical or euclidean

Uniruled varieties II

X non singular algebraic variety $\dim_{\mathbb{C}} X = n$,
W underlying differential manifold $\dim_{\mathbb{R}} W = 2n$

X projective variety $\Rightarrow \exists m, W \subset \mathbb{P}^m(\mathbb{R})$,
$\omega =$ restriction of the standard kähler form of $\mathbb{P}^m(\mathbb{R})$
$\Rightarrow (W, \omega)$ symplectic variety

$L \subset X(\mathbb{R}) \Rightarrow L$ lagrangian $\subset W$ ($\Leftrightarrow \dim_{\mathbb{R}} L = n$ et $\omega|_L \equiv 0$)

Definition

Let W be a closed symplectic manifold
W is uniruled iff it has a non vanishing genus 0 mixed Gromov-Witten invariant $\langle [pt]_k ; [pt], \omega^k \rangle_E^W$, where $E \in H_2(W, \mathbb{Z})$,
$[pt]_k$ Poincaré dual of the point class in $\overline{\mathcal{M}}_{0,k+1}$

Theorem (Kollár 1998)

X projective
X uniruled $\Leftrightarrow \exists E \in H_2(W, \mathbb{Z}), \exists k$ such that $\langle [pt]_k ; [pt], \omega^k \rangle_E^W \neq 0$.

Theorem (M–, Welschinger, 2011)

If (W^6, ω) *is uniruled*
and
if the suspension of a hyperbolic diffeomorphism of the two-torus L
Lagrangian embeds in (W, ω)*,*

then (W, ω) *contains a symplectic disc* D *with* $\partial D \subset L$
such that $[\partial D] \neq 0$ *in* $H_1(L; \mathbb{Q})$*.*

Corollary (Rational surface fibrations)

$X \to C$ rational surface fibration, $\dim_{\mathbb{C}} X = 3$
Assume that $L \subset X(\mathbb{R})$ Sol-manifold

Lemma

If $L \to C(\mathbb{R})$ restriction of $X \to C$ then $\exists$

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C \end{array} \qquad L' \subset X'(\mathbb{R}), L' \text{ Sol-}\textit{torus bundle}$$

such that $g(C') > 0$ and $H_1(L', \mathbb{Q}) \hookrightarrow H_1(C', \mathbb{Q}) \hookrightarrow H_1(X', \mathbb{Q})$

We deduce :
If D disc in X' with boundary in L'
$\Rightarrow \partial D$ vanishes in $H_1(X', \mathbb{Q})$
$\Rightarrow \partial D$ vanishes in $H_1(L', \mathbb{Q})$
$\Rightarrow$ contradiction by the main theorem