Generic one-dimensional perturbation of parabolic points with several petals

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From 1 to >1 variables,
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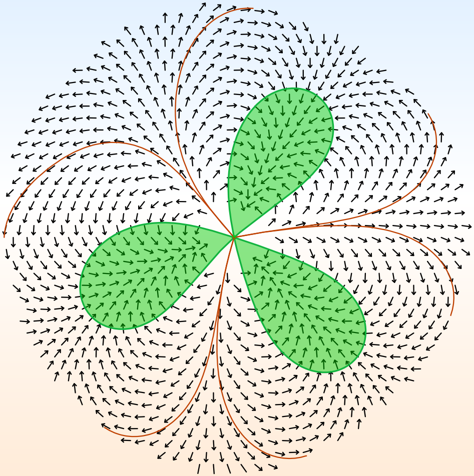
Parabolic dynamics

in 1D continuous/discrete time holomorphic dyn.



0 fixed

$\chi(z) = z^{k+1} + \mathcal{O}(z^{k+2})$ or
$f(z) = z + z^{k+1} + \mathcal{O}(z^{k+2})$.

The flow/dynamics is semi-conjugated to

$$Z \mapsto Z + t$$

by a non-holomorphic change of variable $\approx -1/kz^k$.

k attracting directions, k repelling directions

Local classification

By a change of variable $w = \phi(z)$ near the fixed point:

- $\exists\phi$ holomorphic such that

$$dw/dt = w^{k+1} + aw^{2k+1}$$

  $a \in \mathbb{C}$ is an invariant, related to the residue of the 1-form $dz/\chi(z)$,

- $\exists\phi$ formal power series such that

$$w_{n+1} = w_n + w_n^{k+1} + aw_n^{2k+1}$$

  but most of the time this series is divergent.
  a : unique formal invariant.
  Countable set of analytic invariants.

χ *vs* f

$$\chi \longrightarrow f$$

Via the time-1 map. (Singularity of χ) $\longrightarrow$ (fixed point of f). Parabolic for χ iff parabolic for f.

$$f \longrightarrow \chi \ ?$$

Not all parabolic points can be obtained this way: it is possible iff all the non-formal invariants are $= 0$.

Yet, comparison of a dynamical system f to a close enough vector field is useful, especially when studying the dynamics of maps close to f.

Straightening coordinates

For a vector field,

$$w = \int \frac{dz}{\chi(z)}$$

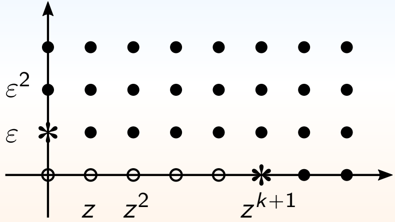
defines *straightening coordinates* in which $dw/dt = 1$. Atlas of a translation surface. The flow is $w \mapsto w + t$.

Fatou coordinates: For a dynamical system, with a parabolic point, f can be holomorphically conjugated to $w \mapsto w + 1$ on some domains called petals. *Not* a translation surface.

Generic perturbations

χ_0 parabolic at $z = 0$, $\varepsilon \in \mathbb{C}$ close to 0, χ_ε perturbation of χ_0, $(\varepsilon, z) \mapsto \chi_\varepsilon(z)$ analytic

Coefficients of the power series expansion $\chi_\varepsilon = \sum a_{i,j} z^i \varepsilon^j$:



Generic condition considered here:

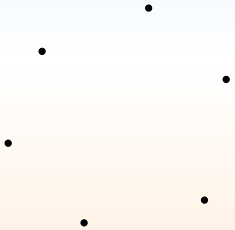
$$a_{0,1} \neq 0$$

Discrete time case: same condition for $f_\varepsilon - f_0$ in place of $\chi_\varepsilon - \chi_0$.

Position of the singuarities

$$\chi_\varepsilon(z) = bz^{k+1} + c\varepsilon + \dots \text{ or } f_\varepsilon(z) = z + bz^{k+1} + c\varepsilon + \dots$$

Singularities/fixed points: $z^{k+1} \approx -\frac{c}{b}\varepsilon$. (Left pic)



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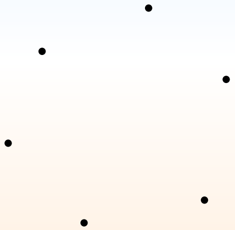
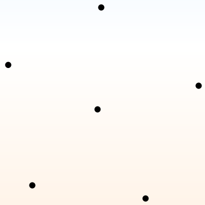
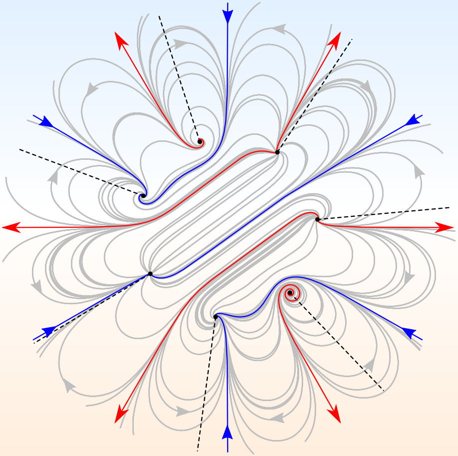


$\neq$



If $f_\varepsilon = g_\varepsilon^k$ with $g_0(0) = 0$, $g_0'(0) = \exp(i2\pi p/k)$ and g_ε generic in some sense, the position of the fixed points is different. (Right pic)

Prototype for vector fields



$$\chi_\varepsilon = z^{k+1} - \varepsilon$$

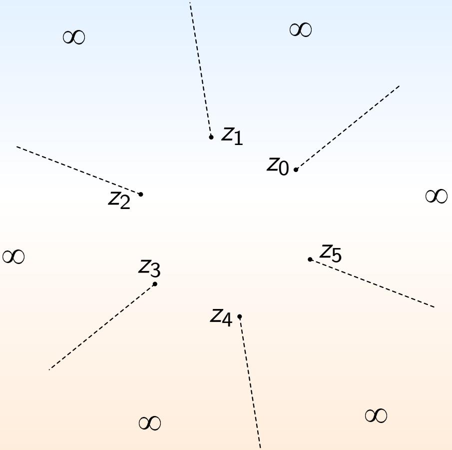
ε complex

k attracting directions
k repelling directions
$k+1$ singularities near 0

$\longleftarrow$ $k+1 = 7$

Douady-Sentenac invariant

Analysis of the prototype



Straightening coordinates $w = \int dz/(z^{k+1} - \varepsilon)$, here $k + 1 = 6$

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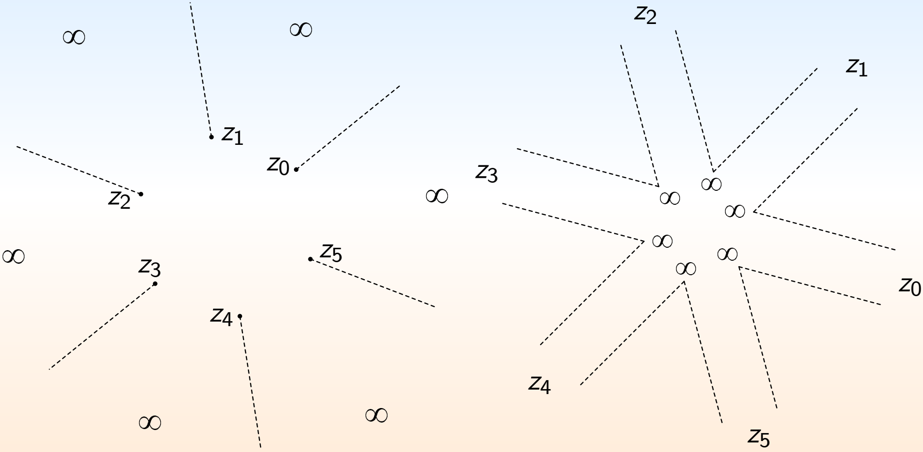

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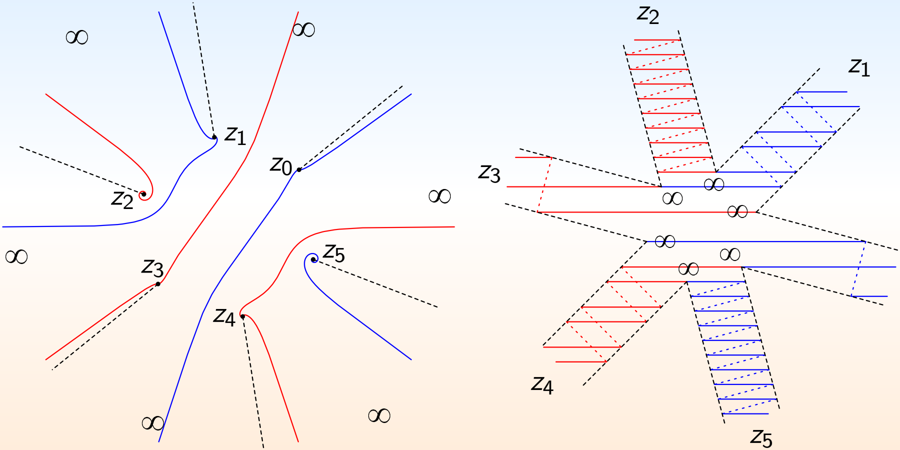

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Bifurcations of the prototype

$$dz/dt = \chi(z) = z^{k+1} - \varepsilon$$

Rescaling space and time one may assume that $|\varepsilon| = 1$.

$$\varepsilon = e^{i\theta}.$$

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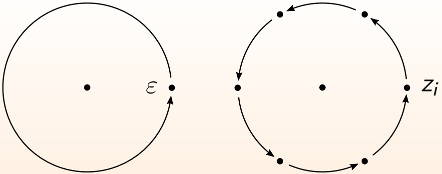
When ε makes one turn then each singularity z_i makes $1/(k+1)$ turns and thus exchange place with the next one.



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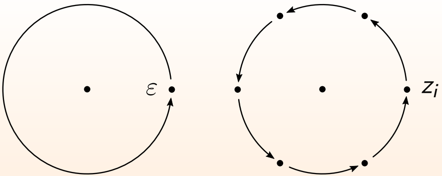
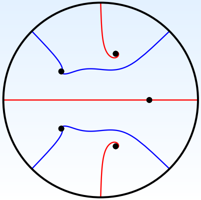


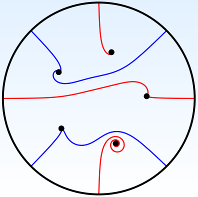
However the vector field is *not* invariant by the z-rotation by $1/(k+1)$ turn; the correct symmetry is obtained by comparing ε to the position of the $k+1$ singularities: $\theta \in [0, 2\pi/k]$.

Bifurcations of the prototype

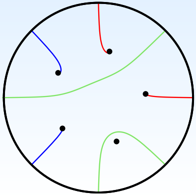
$k+1$ odd



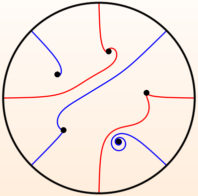
(g) $\theta = 0$



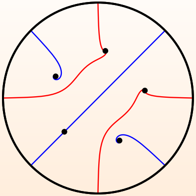
(h) $\theta \in (0, \frac{\pi}{8})$



(i) $\theta = \frac{\pi}{8}$



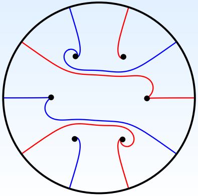
(j) $\theta \in (\frac{\pi}{8}, \frac{\pi}{4})$



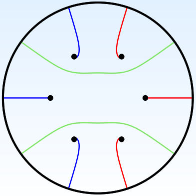
(k) $\theta = \frac{\pi}{4}$

Bifurcations of the prototype

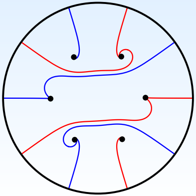
$k+1$ even



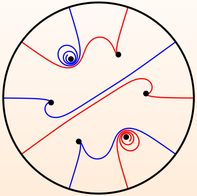
(l) $\theta = -\delta$ for small $\delta > 0$



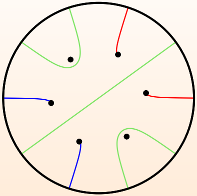
(m) $\theta = 0$



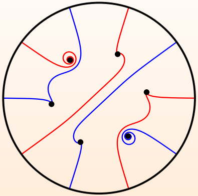
(n) $\theta = \delta$ for small $\delta > 0$



(o) $\theta = \frac{\pi}{5} - \delta$ for small $\delta > 0$



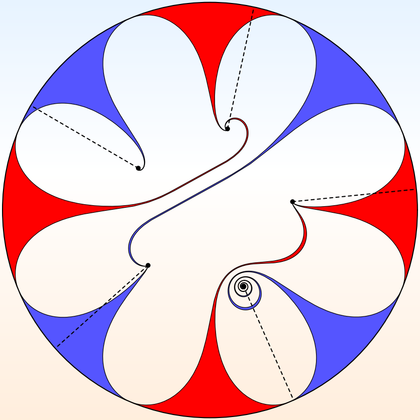
(p) $\theta = \frac{\pi}{5}$



(q) $\theta = \frac{\pi}{5} + \delta$ for small $\delta > 0$

Bifurcation of prototype

local version



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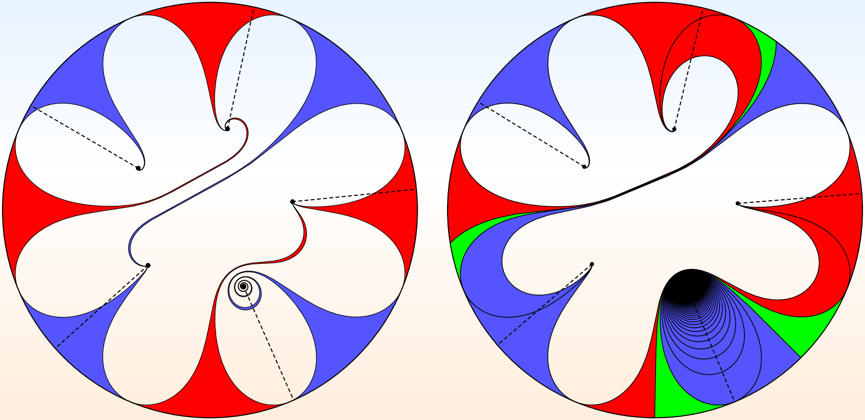

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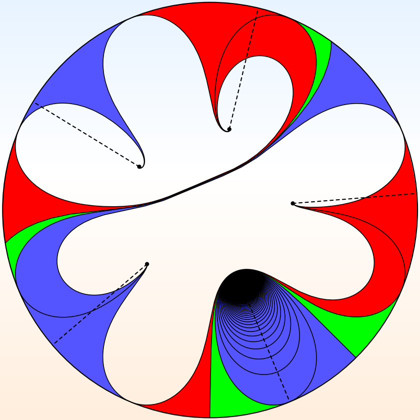
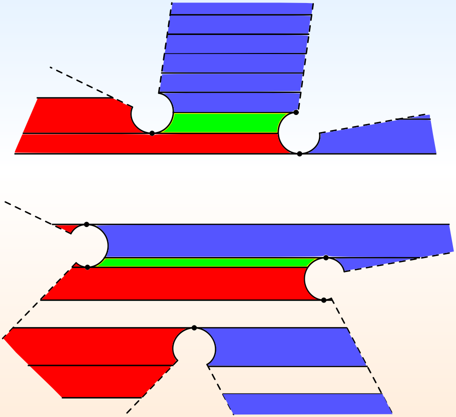
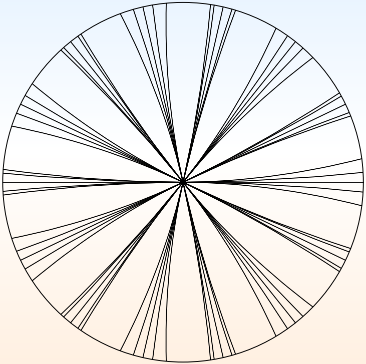




Bifurcation of prototype

Theorem

For $\rho > 0$ small the $\varepsilon \in B(0, \rho)^*$ for which the system is not structurally stable form a finite number of disjoint real-analytic curves from the origin to the boundary of the disk. They are organized in groups that tend to $\varepsilon = 0$ along the $2k$ directions. Each group contains at least three curves, one of which is a straight ray. The other ones come in pairs on each side of the ray, with a tangency at $\varepsilon = 0$ of order $2 - 1/(k+1)$.



Classification for vector fields

"Two generic one parameter families are equivalent iff their multipliers can be made to match."

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An invariant: the eigenvalues $\chi'(z)$ of the singularities $\chi(z) = 0$

$$\varepsilon \mapsto \Lambda(\varepsilon) = \left\{\chi'(z_i)\,;\ z_i \text{ singularity near } 0\right\}.$$

By a change of var. and par.: $\Lambda = \hat{\Lambda} \circ \phi$. Conversely:

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Theorem (Ribón)

Given two families χ_ε and $\tilde{\chi}_\varepsilon$ as before with the same k, if $\exists$ a change of parameter $\tilde{\varepsilon} = \phi(\varepsilon)$ such that $\tilde{\Lambda}(\phi(\varepsilon)) = \Lambda(\varepsilon)$ holds near 0 then χ and $\tilde{\chi}$ are conjugate by a local change of variable $(\tilde{\varepsilon}, \tilde{z}) = (\phi(\varepsilon), \psi(\varepsilon, z))$.

A convenient change of variable

Lemma

There exists a change of variable $\hat{z} = \psi(\varepsilon, z)$ sending the singularities exactly on the roots of $\hat{z}^{k+1} - \varepsilon$.

Corollary

For any χ generic, its set of eigenvalues $\Lambda(\varepsilon)$ is of the form $\left\{\lambda(\eta)\,;\,\eta^{k+1} = \varepsilon\right\}$ where η is a function having a root of order exactly k at the origin. Conversely all such function λ can arise.

Proof: $\Lambda(\varepsilon) = \hat{\Lambda}(\varepsilon)$ and $\hat{\Lambda}(\varepsilon) = \left\{\hat{\chi}'_\varepsilon(\eta)\,;\,\eta^{k+1} = \varepsilon\right\}$.
$\hat{\chi}_\varepsilon(z) = (\hat{z}^{k+1} - \varepsilon)h(\varepsilon, \hat{z})$ so $\chi'_\varepsilon(\eta) = (k+1)\eta^k h(\eta^{k+1}, \eta)$.
For the converse let $\chi_\varepsilon(z) = (z^{k+1} - \varepsilon)\lambda(z)/z^k(k+1)$.

Classification for vector fields

Moduli space and normal forms

Reminder: the set of eigenvalues for χ_ε is

$$\Lambda(\varepsilon) = \left\{\lambda(\eta)\,;\,\eta^{k+1} = \varepsilon\right\}.$$

Model of the moduli space: identify λ and $\hat{\lambda}$ whenever there exists a change of variable $\varepsilon \mapsto \hat{\varepsilon}$ such that $\Lambda(\varepsilon) = \hat{\Lambda}(\hat{\varepsilon})$.

Example of set containing $\leqslant k$ representatives for each class:

$$\left\{\lambda \text{ germ at } 0\,;\, \lambda(z) = z^k + \sum_{n>k,\ (k+1)\nmid n+1} a_n z^n\right\}$$

to be quotiented by $\lambda \sim \lambda \circ \rho$ where ρ is the multiplication by a k-th root of unity.

Normal forms

For instance $d\tilde{z}/dt = (\tilde{z}^{k+1} - \varepsilon) \times Q_\varepsilon(\tilde{z})$ with $\deg Q_\varepsilon \leqslant k$ and $Q_0(0) = 1$.

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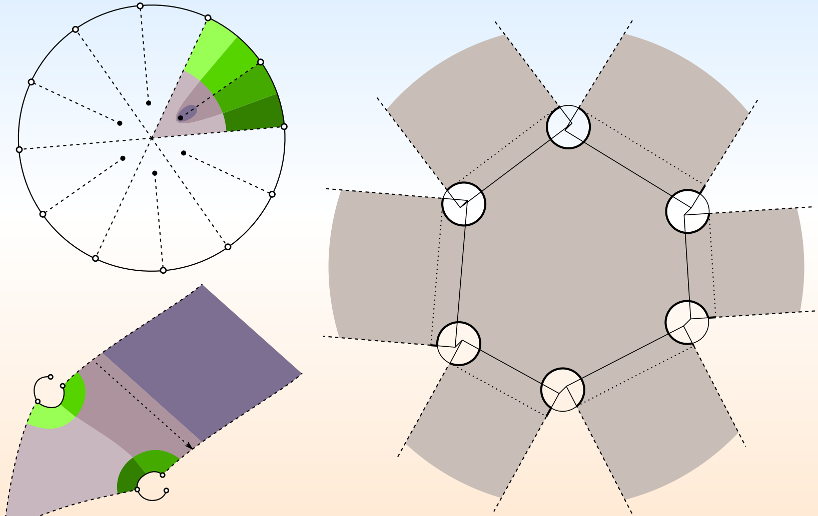
Its eigenvalues are $\left\{(k+1)\tilde{\eta}^{k+1} Q_{\eta^{k+1}}(\eta)\,;\, \eta^{k+1} = \varepsilon\right\}$ so to prove that χ can be put in this normal form by a change of variable (without changing the parameter) it is enough to prove that the set $\left\{\lambda(\eta)\,;\, \eta^{k+1} = \varepsilon\right\}$ coincides with the one above for an appropriate choice of Q_ε.

About the proof of the theorem

same eigenvalues $\Longrightarrow$ conjugate



Towards dynamical systems

Comparison to vector fields help analyse perturbation of parabolic points in holomorphic dynamics.

Two tasks we would like to address in the near future:

- Consequence for bifurcation loci in holomorphic dynamics.
- Classify generic one-parameter families of perturbation of an order k parabolic point.

Bifurcation loci

Given a family of rational maps R_ε of degree d, the bifurcation locus B is defined as the complement of the stability set S, the latter being the set of parameters on which, locally, the Julia follows an isotopy that is compatible with the dynamics.

A famous example of bifurcation locus is the boundary of the Mandelbrot set for the family $R_c(z) = z^2 + c$.

Bifurcation loci

The bifurcation locus been characterized by Mañé, Sad and Sullivan.

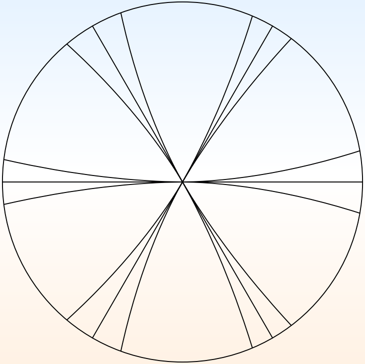
Its complement S is open and dense and is locally the intersection of the stability set S_i of the critical points $c_i(\varepsilon)$ of R_ε, where S_i is defined as the set of parameters on which the family of functions $R_\varepsilon^n(c_i(\varepsilon))$ indexed by n is equicontinuous (normal).

To finish, I will show on a program examples of what to expect using the family

$$f(z) = -\varepsilon + z + z^4 + Az^5$$

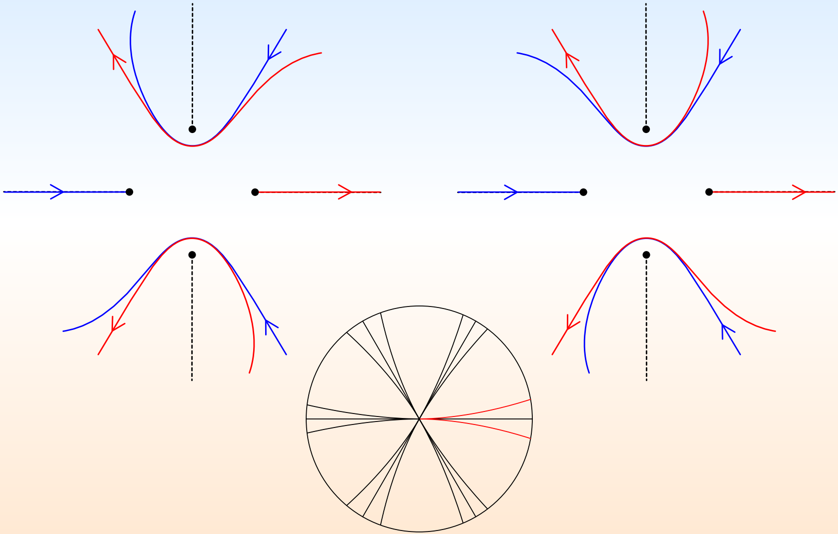
for some $A \in \mathbb{C}$ that was chosen arbitrarily.

Model bifurcation for $k+1=4$



Model bifurcation for $k+1=4$

First kind



Model bifurcation for $k + 1 = 4$

Second kind