

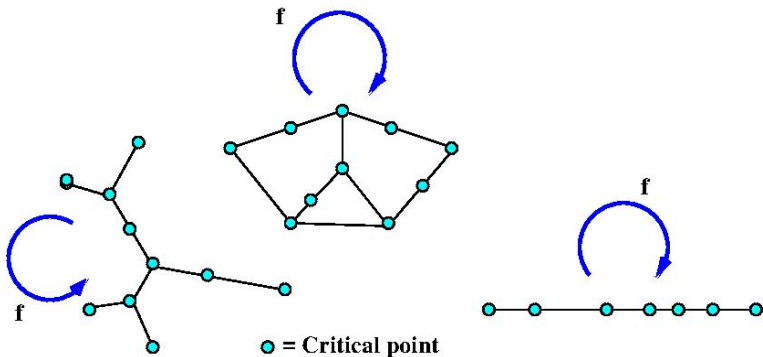
# The Milnor-Thurston determinant and the Ruelle transfer operator

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## Motivation and set-up

We consider  $(I, f)$ , a piecewise continuous and strictly monotone map of a 1 dimensional space. We may take  $I$  to be an Interval:



## Analytic structures related to $h_{\text{top}}(f)$

The dynamical system  $(I, f)$  has a topological entropy  $h_{\text{top}}(f)$ . We are interested in related analytic structures. Here is the zoo:

- **$L(t)$ : Lap number generating function.**
- $D(t)$ : Milnor-Thurston kneading determinant.
- $\zeta_{AM}(t)$ : Artin-Mazur topological zeta-function.
  
- $\mathcal{L}$ : Ruelle transfer operator for  $(I, f)$ .
- $\zeta_R(t)$ : Ruelle dynamical zeta-function.

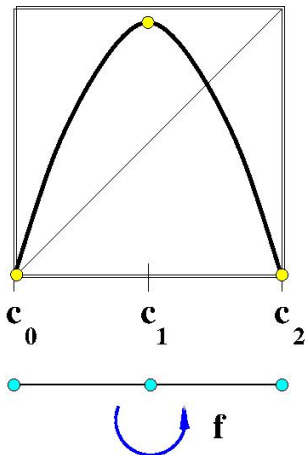
# Backward iterates of critical points

$$\text{Crit}(f) = \{c_0, c_1, c_2\}$$

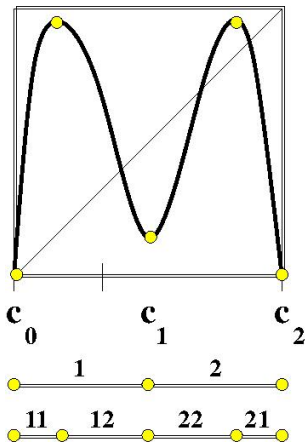
"Partition" into open intervals:

$$I_1 = (c_0, c_1), \quad I_2 = (c_1, c_2)$$

$$Z_1 = \{I_1, I_2\}$$



# Backward iterates of critical points



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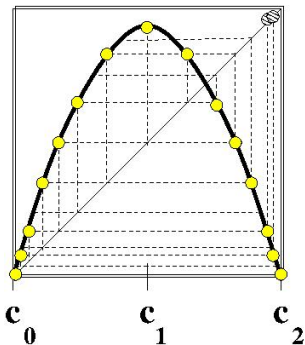
$$Z_1 = \{I_1, I_2\}$$

Refinement by backward iteration of critical points:

$$Z_2 = f^{-1}Z_1 \vee Z_1$$

$$Z_2 = \{I_{11}, I_{12}, I_{22}, I_{21}\}$$

# Backward iterates of critical points



$$Z_n = f^{-(n-1)}Z_1 \vee \dots \vee Z_1$$

Misiurewicz and Szlenk:

$$h_{\text{top}} = \lim_{n \rightarrow \infty} \frac{1}{n} \log \#Z_n$$

Lap number generating function:

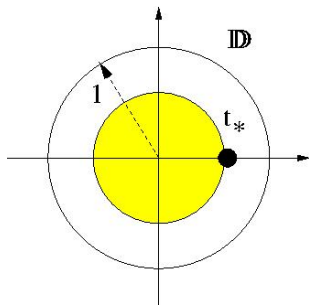
$$L(t) = \sum_{n \geq 0} t^n \#Z_n \in \mathbb{Z}_+[[t]]$$

An a priori simple analytic structure of the Lap generating function, example:

$$L(t) = 1 + 2t + 4t^2 + 8t^3 + 14t^4 + \dots$$

Coefficients are non-negative integers.

- $L$  analytic for:  $|t| < t_* = e^{-h_{\text{top}}}$
- $L$  diverges for  $t > t_* = e^{-h_{\text{top}}}$



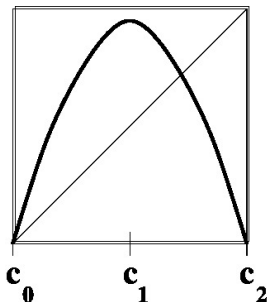
## Analytic structures related to $h_{\text{top}}(f)$

Returning to the zoo:

- $L(t)$ : Lap number generating function.
- $D(t)$ : **Milnor-Thurston kneading determinant.**
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## Dual picture: Forward iterates of critical points



$$c_0 < c_1 < \cdots < c_d < c_{d+1}.$$

Intervals of monot.:  $I_k = (c_k, c_{k+1})$ ,

$$f_k : I_k \rightarrow I = (c_0, c_{d+1})$$

is strictly monotone and continuous.  
Need not be defined at  $c_k$  and  $c_{k+1}$ .

But  $f(c_k^+)$  and  $f(c_{k+1}^-)$  are well-defined.

Introduce "directed" points to keep track of directed limits:

$$x^+ = (x, +1), \quad x^- = (x, -1)$$

## Lifting the map to a directed point map

A directed point  $\widehat{x}$  denotes either  $(x, +1)$  or  $(x, -1)$  (limit from the right/left).

On each directed interval  $[c_k^+, c_{k+1}^-]$  the map either preserves or reverses orientation, also at endpoints. Set:

- $s(f, \widehat{x}) = +1$  if  $f$  preserves the orientation at  $\widehat{x}$ ,
- $s(f, \widehat{x}) = -1$  if  $f$  reverses the orientation at  $\widehat{x}$ .

We "lift"  $f$  to a map on the space of directed points:

$$\widehat{f}(\widehat{x}) = \widehat{f}((x, \epsilon)) = \left( \lim_{t \rightarrow 0^+} f(x + \epsilon t), s(f, \widehat{x}) \epsilon \right).$$

## Kneading invariant

When  $x < y$ , declare  $x < x^+ < y^- < y$  and for  $\hat{x} \in \hat{I}$  and  $u \in \mathbb{R}$ :

$$\sigma(\hat{x}, u) = \begin{cases} +1/2 & \text{if } \hat{x} < u \\ -1/2 & \text{if } \hat{x} > u \end{cases}$$

and for  $c \in \text{Crit}(f)$  the kneading invariant (coefficients  $= \pm \frac{1}{2}$ ):

$$\theta_c(\hat{x}, t) = \sum_{n \geq 0} t^n s(f^n, \hat{x}) \sigma(\hat{f}^n(\hat{x}), c).$$

The kneading "determinant" (in our unimodal case):

$$D(t) = \theta_{c_1}(c_1^+, t) - \theta_{c_1}(c_1^-, t).$$

Recall: The lap generating function

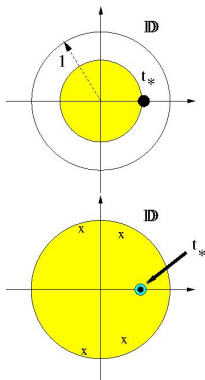
$$L(t) = 1 + 2t + 4t^2 + 8t^3 + 14t^4 + \dots$$

diverges for  $t > t_* = e^{-h_{\text{top}}}$

while  $D(t)$  is analytic in  $\mathbb{D} = \{|t| < 1\}$   
since coefficients are in  $\{-1, 0, 1\}$ .

Cancellations of backward and forward  
orbit contributions:  $D(t) \times L(t)$  is  
analytic in  $\mathbb{D}$  with no roots!

- $e^{-h_{\text{top}}}$  is a pole of  $L(t)$ .
- $e^{-h_{\text{top}}}$  is the smallest root of  $D(t)$ .



With  $d$  critical points,  $\text{Crit}(f) = \{c_0, \dots, c_{d+1}\}$  one may introduce a  $(d+1) \times (d+1)$  kneading matrix:

$$R_{jk}(t) = \begin{cases} \theta_{c_k}(c_0^+, t) + \theta_{c_k}(c_{d+1}^-, t) & , \quad j = 0 \\ \theta_{c_k}(c_j^+, t) - \theta_{c_k}(c_j^-, t) & , \quad 1 \leq j \leq d \end{cases}$$

and a Milnor-Thurston kneading determinant:

$$D(t) = \det R_{jk}(t).$$

Again a magic property (much harder to prove):  $D(t)L(t)$  is analytic in  $\mathbb{D}$  and has no roots for  $|t| < e^{-h_{\text{top}}}$ . Once again:

- $t_* = e^{-h_{\text{top}}}$  is the smallest root of  $D(t)$ .

## Analytic structures related to $h_{\text{top}}(f)$

Back at the zoo ...:

- $L(t)$ : Lap number generating function.
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The Artin-Mazur topological zeta-function (tacitly assuming finitely many fixed points):

$$\zeta_{AM}(t) = \exp \left( \sum_{n \geq 1} \frac{t^n}{n} \# \text{Fix}(f^n) \right).$$

Yet again magic cancellations:

$$D(t) \times \zeta_{AM}(t)$$

is analytic in  $\mathbb{D}$  with no roots!



## Analytic structures related to $h_{\text{top}}(f)$

Continuing the tour at the zoo ...:

- $L(t)$ : Lap number generating function.
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## The Ruelle transfer operator and zeta-function

For  $\phi \in BV(I)$ , a function of bounded variation on  $I$ , we set:

$$\mathcal{L}\phi(y) = \sum_{x:f(x)=y} \phi(x)$$

Acting on the constant function we simply count pre-images:

$$\text{Card}\{x : f^n(x) = y\} = \mathcal{L}^n \mathbf{1}(y).$$

One has:

- $r_{\text{sp}}(\mathcal{L}) = \lim_{n \rightarrow \infty} \|\mathcal{L}^n\|_{BV}^{1/n} = e^{h_{\text{top}}}$ .
- $\mathcal{L}$  is a positive operator  $\Rightarrow (r_{\text{sp}}(\mathcal{L}) - \mathcal{L})$  is non-invertible.

Baladi and Keller (1990) defines a zeta-function (when  $f$  is expanding):

$$\zeta_R(t) = \exp \left( \sum_{n \geq 1} \frac{t^n}{n} \# \text{Fix}(f^n) \right).$$

They show that on the space of BV-functions:

$$(\zeta_R(t))^{-1} = \det(1 - t\mathcal{L})$$

"det" is a "dynamical" determinant introduced by Ruelle. Both functions are analytic in  $\mathbb{D}$ . Zeros are in 1-1 correspondance with the reciprocal of the eigenvalues of  $\mathcal{L}$ , greater than 1 in absolute value.

- Now note that  $\zeta_R(t) = \zeta_{AM}(t)$ .

So ( $\sim$  = analytic, same roots in  $\mathbb{D}$ ):

$$D(t) \sim \zeta_{AM}(t)^{-1} = \zeta_R(t)^{-1} \sim \det(1 - t\mathcal{L}).$$

$D(t)$  not only determines  $h_{\text{top}}$  but also describes eigenvalues of  $\mathcal{L}$ !

- $D(t)$  is determined by forward orbits
- while  $\mathcal{L}$  uses backward iteration.

A thought: Since  $\mathcal{L}$  is based upon backward iterates of  $f$ , perhaps the dual operator  $\mathcal{L}'$  should use forward iterations by  $f$ ?

Quest starting in the 90s ...:

Express  $D(t)$  as a determinant of the dual Ruelle operator.

Several partial results: Baladi and Ruelle (1994), ..., Gouëzel (2001). Calculations use BV-functions, are indirect and difficult. They do not quite cover the original problem.

Reason:

- BV function space is too large  $\Rightarrow$  The dual space is too small.

Is it possible to tailor a Banach space better to our needs...?



Let  $\mathcal{S}$  be the space of piecewise constant functions on  $(c_0, c_{d+1})$  and  $X$  the closure of  $\mathcal{S}$  in  $BV$ .  $X'$  denotes the dual space.

### Theorem (R. 2015)

$D(t)$  equals a (regularized) determinant of  $\mathcal{L}'$  acting upon  $X'$ .  
This (regularized) determinant is analytic in  $\mathbb{D}$ .

### Corollary

$h_{\text{top}} > 0 \Rightarrow e^{-h_{\text{top}}}$  is the smallest zero of  $D(t)$ .

Proof:

- $r_{\text{sp}}(\mathcal{L}') = r_{\text{sp}}(\mathcal{L}) = e^{-h_{\text{top}}}$ .
- $\mathcal{L}$  is a **positive** operator  $\Rightarrow e^{h_{\text{top}}} \in \text{spectrum of } \mathcal{L}$  (and of  $\mathcal{L}'$ ).
- For  $t \in \mathbb{D}$  we have:  $D(t) = 0 \Leftrightarrow 1/t$  is an eigenvalue of  $\mathcal{L}'$

## Functions of bounded variation

$\phi : I = (c_0, c_{d+1}) \rightarrow \mathbb{C}$  is said to be of bounded variation (BV) iff

$$\text{var } \phi = \sup \left\{ \sum_{i=1}^{N-1} |\phi(x_i) - \phi(x_{i+1})| : c_0 < x_1 < \dots < x_N < c_{d+1} \right\} < +\infty$$

When  $\text{var } \phi < +\infty$  we have existence of right and left limits:

- $\phi(x^+) = \lim_{t \rightarrow 0^+} \phi(x + t)$  for  $c_0 \leq x < c_{d+1}$ ,
- $\phi(x^-) = \lim_{t \rightarrow 0^+} \phi(x - t)$  for  $c_0 < x \leq c_{d+1}$ .

We introduce the "boundary" value of  $\phi$ :

$$\partial\phi = \phi(c_0^+) + \phi(c_{d+1}^-)$$

We define the BV norm:

$$\|\phi\| = \|\phi\|_{\text{BV}} = \text{var } \phi + |\partial\phi|.$$

Denote by  $\mathcal{S}$  the piecewise constant functions on  $I$  and let  $X$  be the completion of  $\mathcal{S}$  w.r.t  $\|\cdot\|$  (same as closure in BV).

Every  $\phi \in \mathcal{S}$  may be written as  $\sum_{\text{finite}} w_i \sigma_{\hat{u}_i}$  in the basis:

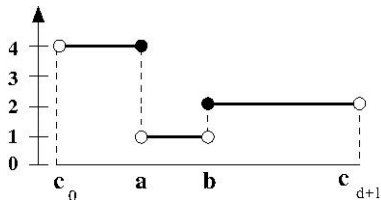
$$\sigma_{\hat{u}}(x) = \begin{cases} +1/2 & \text{if } \hat{u} < x \\ -1/2 & \text{if } \hat{u} > x \end{cases}, \quad \hat{u} \in \hat{I} = [c_0^+, c_{d+1}^-).$$

Example:

$$\phi = 3\sigma_{c_0^+} - 3\sigma_{a^+} + \sigma_{b^-}$$

The BV-norm of  $\phi$ :

$$\|\phi\| = 3 + 3 + 1 = 4$$



For  $\phi = \sum_{\text{finite}} w_i \sigma_{\hat{u}_i}$  we have:  $\|\phi\| = \sum_i |w_i|$ .



A representation of the dual space of  $X$  is obtained by acting upon the basis of  $X$ . For  $\ell \in X'$  define:

$$\widehat{\ell}(\widehat{u}) = \langle \ell, \sigma_{\widehat{u}} \rangle, \quad \widehat{u} \in \widehat{I}.$$

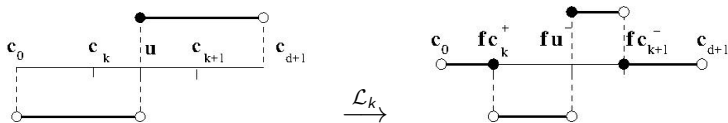
Then  $|\widehat{\ell}(\widehat{u})| \leq \|\ell\|_{X'}$  and for  $\phi = \sum_{\text{finite}} w_i \sigma_{\widehat{u}_i}$ :

$$|\langle \ell, \phi \rangle| = \left| \sum_i w_i \widehat{\ell}(\widehat{u}_i) \right| \leq \|\phi\|_{\text{BV}} \|\widehat{\ell}\|_{\infty}.$$

We have an isomorphism between  $X'$  and the bounded functions on the directed points of  $I$  with the uniform norm:

$$X' \cong B([c_0^+, c_{d+1}^-]).$$

Example: Acting with  $\mathcal{L}_k$ ,  $1 \leq k \leq d+1$  upon  $\sigma_u^-$ :



The image is a linear combination of (at most) three basis functions. In the dual representation:

$$\begin{aligned}
 \widehat{\mathcal{L}}_k \widehat{\ell}(\widehat{u}) &= \langle \ell, \mathcal{L}_k \sigma_{\widehat{u}} \rangle \\
 &= \mathbf{1}_{\widehat{I}_k}(\widehat{u}) s(f, \widehat{u}) \widehat{\ell}(f\widehat{u}) \\
 &\quad - \sigma_{c_k}(\widehat{u}) s(f, c_k^+) \widehat{\ell}(fc_k^+) \\
 &\quad + \sigma_{c_{k+1}}(\widehat{u}) s(f, c_{k+1}^-) \widehat{\ell}(fc_{k+1}^-)
 \end{aligned}$$

Adding terms:  $\widehat{\mathcal{L}} = \sum_{k=1}^{d+1} \widehat{\mathcal{L}}_k$  we get:

$$\widehat{\mathcal{L}} = S - PS$$

in which we find a signed Koopman operator of norm  $\|S\| = 1$ :

$$S \widehat{\ell}(\widehat{u}) = s(f, \widehat{u}) \ell(\widehat{f} \widehat{u})$$

and a finite rank projection operator,  $P^2 = P$ :

$$\begin{aligned} P \widehat{\ell}(\widehat{u}) &= \sigma_{c_0}(\widehat{u}) \left( \widehat{\ell}(c_0^+) + \widehat{\ell}(c_{d+1}^-) \right) \\ &+ \sum_{k=1}^d \sigma_{c_k}(\widehat{u}) \left( \widehat{\ell}(c_k^+) - \widehat{\ell}(c_k^-) \right) \end{aligned}$$

Good news:  $\widehat{\mathcal{L}} = S - PS$  is a finite  $(d + 1)$  rank perturbation of the signed Koopman operator of norm 1. For  $t = 1/\lambda \in \mathbb{D}$ ,  $(1 - tS)$  is invertible:

$$(1 - tS)^{-1} = 1 + tS + t^2S^2 + \dots$$

To find eigenvalues with  $|\lambda| > 1$  of  $\widehat{\mathcal{L}}$  is equivalent to finding values of  $t = 1/\lambda \in \mathbb{D}$  for which:

$$\begin{aligned} & 1 - t\mathcal{L} && \text{is non-invertible} \\ = & (1 - tS) + tPS \\ = & (1 + tPS(1 - tS)^{-1})(1 - tS) \\ \Leftrightarrow & 1 + tPS(1 - tS)^{-1} && \text{is non-invertible} \end{aligned}$$

The projection  $P$  has finite rank so  $(1 + tPS(1 - tS)^{-1})$  is invertible iff it is invertible on

$$\text{im } P = \text{Span}\{\sigma_{c_0}, \dots, \sigma_{c_d}\}$$

Let us compute the matrix elements on  $\text{im } P$  of

$$G(t) = P + tPS(1 - tS)^{-1} = P(1 - tS)^{-1}$$

Note first that

$$\begin{aligned}(1 - tS)^{-1}\sigma_{c_k}(\hat{u}) &= (1 + tS + t^2S^2 + \dots)\sigma_{c_k}(\hat{u}) \\ &= \sum_{n \geq 0} t^n s(f^n, \hat{u})\sigma(\hat{f}^n(\hat{u}), c) = \theta_{c_k}(\hat{u}, t)\end{aligned}$$

is nothing but the kneading coordinate of  $\hat{u}$  w.r.t.  $c_k$ .

Applying  $P$  to the result we get:

$$\begin{aligned} G(t) \sigma_{c_k}(\hat{u}) &= P(1 - tS)^{-1} \sigma_{c_k}(\hat{u}) \\ &= \sigma_{c_0}(\hat{u}) (\theta_{c_k}(c_0^+, t) + \theta_{c_k}(c_{d+1}^-, t)) \\ &\quad + \sum_{j=1}^d \sigma_{c_j}(\hat{u}) (\theta_{c_k}(c_j^+, t) - \theta_{c_k}(c_j^-, t)) \\ &= \sum_{j=0}^d \sigma_{c_j}(\hat{u}) R_{jk}(t) \end{aligned}$$

where  $(R_{jk}(t))$  is the Milnor-Thurston kneading matrix. It is non-invertible precisely when its determinant vanishes which is what we wanted to show.

$h_{\text{top}}$  of interval maps  
Dual picture: Forward iterates  
Some kneading theory  
The Ruelle transfer operator. Backward iterates  
**The dual Ruelle operator**

