The complex Wishart distribution and the symmetric group

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Abstract

Let V be the space of (r,r) Hermitian matrices and let Ω be the cone of the positive definite ones. We say that the random variable S, taking its values in $\overline{\Omega}$, has the complex Wishart distribution $\gamma_{p,\sigma}$ if $\mathbb{E}(\exp \operatorname{trace}(\theta S)) = (\det(I_r - \sigma\theta))^{-p}$, where σ and $\sigma^{-1} - \theta$ are in Ω, and where $p = 1, 2, \ldots, r-1$ or $p > r-1$. In this paper, we compute all moments of S. More specifically, if $h = (h_1, \ldots, h_n)$ is any word of (r,r) complex matrices and if we introduce the complex random variables $S(h) = \operatorname{trace}(Sh_1 \ldots Sh_n)$ and $S^{(-1)}(h) = \operatorname{trace}(S^{-1}h_1 \cdots S^{-1}h_n)$, we are able to compute in a simple way the expressions $\mathbb{E}(S(h^{(1)}) \cdots S(h^{(m)}))$ for any set of words $\{h^{(1)}, \cdots, h^{(m)}\}$. Similarly we compute $\mathbb{E}(S^{(-1)}(h^{(1)}) \cdots S^{(-1)}(h^{(m)}))$ whenever it exists. This provides a general answer to the questions raised by D. Maiwald and D. Kraus (2000) for moments of order ≤ 4.

Our technique is to use the multilinear forms $r_\pi(\sigma)(h_1, \ldots, h_k)$, where π belongs to the group $\mathcal{S}_k$ of permutations of $\{1, \ldots, k\}$. For instance, if $\pi = (4)(2,5)(6,3,1)$, the form $r_\pi(\sigma)$ is defined by

$$r_\pi(\sigma)(h_1, \ldots, h_6) = \operatorname{trace}(\sigma h_4)\operatorname{trace}(\sigma h_2 \sigma h_5)\operatorname{trace}(\sigma h_6 \sigma h_3 \sigma h_1).$$

Denote by $m(\pi)$ the number of cycles of π, and write q for $p - r$. Our theorems 2 and 3 can be presented in the following compact way by using the convolution product $*$ associated to the group algebra $\mathcal{A}(\mathcal{S}_k)$:

$$\mathbb{E}(r(S)) = p^m * r(\sigma), \qquad r(\sigma^{-1}) = (-1)^k(-q)^m * \mathbb{E}(r(S^{-1})).$$

Our proofs of these formulas are elementary. The remainder of the paper is devoted to the inversion of the second formula, i.e. to the computation of $\mathbb{E}(r_\pi(S^{-1}))$, for any π. To this end, we use the irreducible characters of $\mathcal{S}_k$.

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1 Introduction: the complex Wishart distributions.

Let V be the set of (r,r) Hermitian matrices, i.e. of the (r,r) matrices s with complex entries such that $s = s^*$, where * indicates the transposed conjugate of the matrix. Note that V is a real linear space, not a complex one. Its dimension is r^2. Let Ω be the open convex cone of the elements s of V which are positive definite, i.e. such that $z^*sz > 0$ for any z in $\mathbf{C^r} \setminus \{\mathbf{0}\}$. Its closure $\overline{\Omega}$ is the closed convex cone of the elements s of V which are positive, i.e. such that $z^*sz \geq 0$ for any z in $\mathbf{C^r}$. Let σ be in Ω and let p be a positive integer. The complex Wishart distribution $\gamma_{p,\sigma}$ on $\overline{\Omega}$ has been considered in the literature (for instance Goodman (1963)) as the distribution of

$$S = \frac{1}{2}(Z_1 Z_1^* + \cdots + Z_p Z_p^*), \tag{1.1}$$

where $Z_1, \ldots, Z_p$ are i.i.d. Gaussian centered random variables taking their values in $\mathbf{C^r}$, with the covariance such that, if we write $Z_j = X_j + iY_j$ and $\sigma_{jk} = a_{jk} + ib_{jk}$, where $X_j,\ Y_j,\ a_{jk},\ b_{jk}$ are real, then we have

$$\begin{bmatrix} \mathbb{E}(X_j X_k) & \mathbb{E}(X_j Y_k) \\ \mathbb{E}(Y_j X_k) & \mathbb{E}(Y_j Y_k) \end{bmatrix} = \begin{bmatrix} a_{jk} & b_{jk} \\ -b_{jk} & a_{jk} \end{bmatrix}.$$

From (1.1) it is easy to verify that the Laplace transform of S is

$$\mathbb{E}(\exp \operatorname{trace}(\theta S)) = (\det(I_r - \sigma\theta))^{-p}, \tag{1.2}$$

for all θ in V such that $\sigma^{-1} - \theta$ is in Ω. To compute this expected value, we use Gaussian integrals and the spectral theorem for Hermitian matrices. From its definition in (1.1) and for $p = 1, 2, \ldots, r-1$, it is clear that S is concentrated on the boundary $\overline{\Omega} \setminus \Omega$, i.e. on the set on singular positive Hermitian matrices. In order to give a description of the distribution of S for the other values of p, write s in V as $s = (x_{jk} + iy_{jk})_{1 \leq j,k \leq r}$, where x_{jk} and y_{jk} are real and where $y_{jk} = -y_{kj}$, and introduce the Lebesgue measure on V defined by

$$\lambda(ds) = \prod_{j=1}^{r} dx_{jj} \prod_{1 \leq j < k \leq r} dx_{jk} dy_{jk}.$$

Suppose now that $p > r - 1$, but is not necessarily an integer anymore. We prove in the appendix that for an appropriate constant $C(p,r)$

$$C(p,r) \det(\sigma)^{-p} (\det s)^{p-r} \exp(-\operatorname{trace} \sigma^{-1} s) \mathbf{1}_\Omega(s) \lambda(ds) \tag{1.3}$$

is a probability (our proof of the above statement differs from the proof given in Goodman (1963)). Replacing in (1.3) σ by $(\sigma^{-1} - \theta)^{-1}$, we obtain that for $\sigma^{-1} - \theta$ in Ω

$$\begin{aligned} C(p,r) \int_\Omega \exp(\operatorname{trace} \theta s) \det(\sigma)^{-p} (\det s)^{p-r} \exp(-\operatorname{trace} \sigma^{-1} s) \quad & \mathbf{1}_\Omega(s) \lambda(ds) \\ & = (\det(I_r - \sigma\theta))^{-p}. \end{aligned}$$

This shows that for $p = r, r+1, \ldots$, the probability density function of the $\gamma_{p,\sigma}$ random variable S defined in (1.1) is given explicitly in (1.3). Summarizing the results given above, we say that for $p \in \Lambda = \{1, 2, \dots, r-1\} \cup (r-1, \infty)$, and $\sigma \in \Omega$, the complex Wishart distribution $\gamma_{p,\sigma}$ with shape parameter p and scale parameter σ is defined by its Laplace transform $(\det(I_r - \sigma\theta))^{-p}$. For $p > r-1$ its density is (1.3). For p in the singular part $\{1, 2, \dots, r-1\}$ of Λ, it has the distribution of S as defined in (1.1). The fact that the set of values of p such that $(\det(I_r - \sigma\theta))^{-p}$ is the Laplace transform of a positive measure is Λ is known as Gyndikin's theorem (see Casalis and Letac (1994) for an accessible proof). Our definition of $\gamma_{p,\sigma}$ as the complex Wishart distribution, slightly differs from the traditional definition given in Goodman (1963) because of the factor 1/2 in (1.1). The correspondence between the two definitions is $W_r(\frac{n}{2}, 2\sigma) = \gamma_{p,\sigma}$.

2 The moments and the inverse moments.

This paper has been written partly to give a complete answer to the question considered in Maiwald and Kraus (2000), where the authors provide some exact and some approximate formulas for lower moments of the complex Wishart distribution. Our paper extends their results by providing exact formulas for moments of any order, and by trying to give insight into the mechanisms that give us the desired moments. In order to state our results, we need to introduce some notation for the symmetric group $\mathcal{S}_k$ of permutations π of $\{1, 2, \dots, k\}$.

We recall that a permutation π of a set $\{a_1, \dots, a_l\}$ of l objects is called a cycle if there exists a bijection φ from $\{1, 2, \dots, l\}$ to $\{1, \dots, l\}$ such that for all j one has $\pi(a_{\varphi(j)}) = a_{\varphi(j+1)}$, with the convention $\varphi(l+1) = \varphi(1)$. The permutation π can also be denoted by π_φ. Note that φ is not unique. More specifically, we say that the two bijections φ and φ' are equivalent if there exists an integer q such that $\varphi'(j) = \varphi(j+q)$ for all j, where $j+q$ is taken modulo l. Clearly $\pi_\varphi = \pi_{\varphi'}$ if and only if φ and φ' are equivalent. Note also that if the objects $\{a_1, \dots, a_l\}$ are (r, r) complex matrices, we use the well known property of commutativity for traces to see that

$$\operatorname{trace}\left(a_{\varphi(1)}a_{\varphi(2)}\dots a_{\varphi(l)}\right) = \operatorname{trace}\left(a_{\varphi'(1)}a_{\varphi'(2)}\dots a_{\varphi'(l)}\right) \tag{2.4}$$

when φ and φ' are equivalent. When $\pi = \pi_\varphi$ is a cycle, we denote the common value of (2.4) in a symbolic way by

$$\operatorname{trace}\left(\prod_{j\in\pi} a_j\right). \tag{2.5}$$

Now, it is a classical result of group theory that any element π of $\mathcal{S}_k$ can be written in a unique way as the product of cycles built on a set partition of $\{1, 2, \dots, k\}$. Denote by $C(\pi)$ the set of these cycles, and denote by $m(\pi)$ the size of $C(\pi)$, i.e. the number of cycles. For $\pi \in \mathcal{S}_k$, for $\sigma \in \Omega$ and for a sequence $(h_1, \dots, h_k)$ of (r, r) complex matrices, we now define the quantity

$$r_\pi(\sigma)(h_1, \dots, h_k) = \prod_{c\in C(\pi)} \operatorname{trace}\left(\prod_{j\in c} \sigma h_j\right). \tag{2.6}$$

Let us illustrate this with an example. Suppose that $k = 6$ and that π is given by $\pi(1) = 6$, $\pi(2) = 5$, $\pi(3) = 1$, $\pi(4) = 4$, $\pi(5) = 2$ and $\pi(6) = 3$. Then $m(\pi) = 3$ and with obvious notation, we write these three cycles as $(1,6,3)$, $(2,5)$ and (4). More commonly, we write $\pi = (1,6,3)(2,5)(4)$. The order on the set $C(\pi)$ and the circular order inside the cycles are irrelevant and we can also write $\pi = (4)(2,5)(6,3,1)$ for instance. For the particular example above, $r_\pi(\sigma)(h_1,\dots,h_6) = \mathrm{trace}\,(\sigma h_4)\mathrm{trace}\,(\sigma h_2\sigma h_5)\mathrm{trace}\,(\sigma h_1\sigma h_6\sigma h_3)$.

The two main results of this paper are given in Theorem 2 and Theorem 4 below, which link arbitrary moments and inverse moments of the complex Wishart distribution, respectively, with the symmetric group $\mathcal{S}_k$. Theorem 1 is a particular case of Theorem 2, but is actually a crucial ingredient in its proof and deserves a separate treatment.

Theorem 1. Let V be the space of (r,r) Hermitian matrices, let σ be in the cone Ω of positive definite elements of V and let p be in the Gyndikin set $\Lambda = \{1,2,\dots,r-1\} \cup (r-1,\infty)$. Consider a random variable S in V with the complex Wishart distribution $\gamma_{p,\sigma}$ and let $(h_1,\dots,h_k)$ be arbitrary (r,r) complex matrices. Then

$$\mathbb{E}(\mathrm{trace}\,(Sh_1)\mathrm{trace}\,(Sh_2)\dots\mathrm{trace}\,(Sh_k)) = \sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(h_1,\dots,h_k). \tag{2.7}$$

Theorem 1 is of interest in its own right because it yields immediately the simplest moments of $\gamma_{p,\sigma}$. Let $S = (S_{jl})_{1\le j,l\le r} = (X_{jl} + iY_{jl})_{1\le j,l\le r}$, be an element of Ω and suppose that we want to calculate the real number

$$\mathbb{E}(\prod_{1\le j,l\le r} X_{jl}^{a_{jl}} Y_{jl}^{b_{jl}}) \tag{2.8}$$

for some integers a_{jl} and b_{jl}. One can verify that if $E_{k,N}$ is the complex space of homogeneous polynomials of degree N with respect to the $2k$ variables $x_1,\dots,x_k,y_1,\dots,y_k$, and if $I_{k,N}$ is the set of sequences of $2k$ nonegative integers $a_1,\dots,a_k,b_1,\dots,b_k$, then the two sets of polynomials

$$\{x_1^{a_1}\cdots x_k^{a_k} y_1^{b_1}\cdots y_k^{b_k};\ (a_1,\dots,a_k,b_1,\dots,b_k) \in I_{k,N}\},$$

and

$$\{(x_1+iy_1)^{a_1}\cdots(x_k+iy_k)^{a_k}(x_1-iy_1)^{b_1}\cdots(x_k-iy_k)^{b_k};\ (a_1,\dots,a_k,b_1,\dots,b_k) \in I_{k,N}\},$$

both form a basis of $E_{k,N}$. Thus, the elements of the first basis can be expressed as linear combinations of the elements of the second one. As a consequence, since S is Hermitian, in order to calculate (2.8), it is enough to compute complex moments of the form

$$\mathbb{E}(\prod_{1\le j,l\le r} S_{jl}^{a_{jl}} \overline{S}_{jl}^{b_{jl}}). \tag{2.9}$$

This can be done by applying Theorem 1 to suitable matrices $h_1,\dots,h_k$ (see (4.27) below).

The statement of Theorem 2 includes in particular all the formulas of Section 4 of Maiwald and Kraus (2000). Indeed, for $k=1$ and $k=2$, (2.10) below gives the same results as their formulas (15)-(17). For $k=3$, their formula (18) gives the coefficient of

p^3 and p^2 in (2.10) for $\pi = (1,2)(3)$. For $k = 4$, their formula (19) gives the coefficient of p^4 and p^3 in (2.10) for $\pi = (1,2)(3,4)$.

Theorem 2. With the notations of Theorem 1, for all $\pi \in \mathcal{S}_k$ we have

$$\mathbb{E}(r_\pi(S)(h_1, \dots, h_k)) = \sum_{\pi' \in \mathcal{S}_k} p^{m(\pi'^{-1} \circ \pi)} r_{\pi'}(\sigma)(h_1, \dots, h_k). \tag{2.10}$$

Our next result, Theorem 3, links $\mathbb{E}(r_\pi(S^{-1}))$ and $r_\pi(\sigma^{-1})$ and is a first step towards the proof of Theorem 4 which is the main result for the moments of the inverse complex Wishart distribution.

Theorem 3. With the notations of Theorem 1, writing $q = p - r$ and assuming that $q > k$, we have, for all $\pi \in \mathcal{S}_k$,

$$r_\pi(\sigma^{-1})(h_1, \dots, h_k) = (-1)^k \sum_{\pi' \in \mathcal{S}_k} (-q)^{m(\pi'^{-1} \circ \pi)} \mathbb{E}(r_{\pi'}(S^{-1})(h_1, \dots, h_k)). \tag{2.11}$$

In the statement of the last theorem, Theorem 4, we use the characters of the symmetric group $\mathcal{S}_k$. For the convenience of the reader, this concept will be recalled in Section 6 below. Our reference is Simon(1995), hereafter denoted $[S]$. To state this theorem, let us introduce some notations.

We denote by I_k the set of sequences $\mathbf{i} = (i_1, \dots, i_k)$ of k non negative integers such that $i_1 + 2i_2 + \cdots + ki_k = k$. To each π in $\mathcal{S}_k$ we associate the element $\mathbf{i} = \mathbf{i}(\pi) = (i_1, \dots, i_k)$ of I_k where i_j is the number of cycles of length j in π. Such an element $\mathbf{i}$ of I_k is called the *portrait* of π. For instance $(k, 0, \dots, 0)$ is the portrait of the identity and is abbreviated as (k). It is easy to see ([S], Th VI 1.2) that the size $\sharp\mathbf{i}$ of the set of π's with portrait $\mathbf{i}$ is

$$\sharp\mathbf{i} = \frac{k!}{\prod_{j=1}^k i_j! j^{i_j}}. \tag{2.12}$$

If a function on $\mathcal{S}_k$, say $\pi \mapsto a(\pi)$, depends only on the portrait $\mathbf{i}$ of π, we write $a(\mathbf{i}) = a(\pi)$, that is a is considered as a function on I_k. In particular, since for $h_1 = \cdots = h_k = h$, the right hand side of (2.6) depends only on the portrait of π, we adopt the simplified notation

$$r_{\mathbf{i}}(\sigma)(h) = r_\pi(\sigma)(h, h, \dots, h) = (\mathrm{trace}\,(\sigma h))^{i_1} (\mathrm{trace}\,(\sigma h)^2)^{i_2} \dots (\mathrm{trace}\,(\sigma h)^k)^{i_k} \tag{2.13}$$

where $\mathbf{i}$ is the portrait of π.

We introduce also the set M_k of sequences $\mathbf{m} = (m_1, \dots, m_k)$ of integers such that $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and such that $m_1 + \cdots + m_k = k$.

Finally, we consider the matrix

$$C_k = (\chi^{\mathbf{m}}(\mathbf{i}))_{\mathbf{m} \in M_k,\ \mathbf{i} \in I_k}$$

of characters $\chi^{\mathbf{m}}(\mathbf{i})$ of $\mathcal{S}_k$, and we denote by $d_{\mathbf{m}}$ the dimension of the irreducible representation associated with $\mathbf{m}$ in M_k, i.e. (see [S] Th. VI.2.3, page 98)

$$d_{\mathbf{m}} = \frac{k!}{\prod_{i=1}^k (m_i - i + k)} \prod_{1 \leq i < j \leq k} (m_i - m_j - i + j). \tag{2.14}$$

Theorem 4. We keep the same asumptions and notations as in Theorem 3.
1) For $\mathbf{m} \in M_k$ we denote

$$f_{\mathbf{m}} = \frac{d_{\mathbf{m}}}{k!} \prod_{j=1}^{k} \prod_{i=1}^{m_j} (q + j - i). \tag{2.15}$$

Then for all π in $\mathcal{S}_k$ we have

$$(-1)^k (-q)^{m(\pi)} = \sum_{\mathbf{m} \in M_k} f_{\mathbf{m}} \chi^{\mathbf{m}}(\mathbf{i}). \tag{2.16}$$

2) For π in $\mathcal{S}_k$ we denote

$$f^{(-1)}(\pi) = \sum_{\mathbf{m} \in M_k} \frac{d_{\mathbf{m}}^2}{k!^2 f_{\mathbf{m}}} \chi^{\mathbf{m}}(\pi). \tag{2.17}$$

Then the inverse formula of (2.11) is

$$\mathbb{E}(r_\pi(S^{-1})(h_1, \dots, h_k)) = \sum_{\pi' \in \mathcal{S}_k} f^{(-1)}(\pi'^{-1} \circ \pi) r_{\pi'}(\sigma^{-1})(h_1, \dots, h_k). \tag{2.18}$$

In particular for π=identity we obtain

$$\mathbb{E}(\mathrm{trace}\,(S^{-1}h_1)\mathrm{trace}\,(S^{-1}h_2)\dots\mathrm{trace}\,(S^{-1}h_k)) = \sum_{\pi' \in \mathcal{S}_k} f^{(-1)}(\pi') r_{\pi'}(\sigma^{-1})(h_1, \dots, h_k). \tag{2.19}$$

Note that we can use (2.19) in Theorem 4 to compute moments of S^{-1} of the form $\mathbb{E}(\prod_{1 \le j,l \le r} S_{jk}^{-a_{jl}} \overline{S}_{jl}^{-b_{jl}})$, just as we used Theorem 1 to compute similar moments of S. For $k \le 4$, Theorem 4 gives as a particular case formulas 50-54 of Maiwald and Kraus (2000).

To conclude this section, we observe that, for $k = 1$ and $k = 2$, the previous theorems give results which have already appeared in the litterature in the more general context of Wishart distributions on Jordan algebras. Indeed the present space V of (r, r) Hermitian matrices can be considered as a Jordan algebra with Peirce constant $d = 2$. In particular, the following formulas

$$\begin{aligned} \mathbb{E}(S) &= p\sigma \\ \mathbb{E}(S^{-1}) &= \sigma^{-1}/q \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}(\mathrm{trace}\,(Sh_1)\mathrm{trace}\,(Sh_2)) &= p^2 \mathrm{trace}\,(\sigma h_1)\mathrm{trace}\,(\sigma h_2) + p\ \mathrm{trace}\,(\sigma h_1 \sigma h_2), \\ \mathbb{E}(\mathrm{trace}\,(Sh_1 S h_2)) &= p^2 \mathrm{trace}\,(\sigma h_1 \sigma h_2) + p\ \mathrm{trace}\,(\sigma h_1)\mathrm{trace}\,(\sigma h_2) \\ \mathbb{E}(\mathrm{trace}\,(S^{-1}h_1)\mathrm{trace}\,(S^{-1}h_2)) &= \frac{1}{q^2 - 1} \mathrm{trace}\,(\sigma^{-1}h_1)\mathrm{trace}\,(\sigma^{-1}h_2) + \frac{1}{q^3 - q} \mathrm{trace}\,(\sigma^{-1}h_1\sigma^{-1}h_2), \\ \mathbb{E}(\mathrm{trace}\,(S^{-1}h_1 S^{-1}h_2)) &= \frac{1}{q^3 - q} \mathrm{trace}\,(\sigma^{-1}h_1\sigma^{-1}h_2) + \frac{1}{q^2 - 1} \mathrm{trace}\,(\sigma^{-1}h_1)\mathrm{trace}\,(\sigma^{-1}h_2) \end{aligned}$$

have already appeared in Letac and Massam (1998), formula (6.1), and in Letac and Massam (2000), formulas (5.1), (6.1) and (6.2). In Section 8, we will comment on some Jordan algebra aspects of the problems considered here.

3 Proof of Theorem 1.

We observe first that for $p \in \Lambda$, there exists an unbounded measure μ_p on $\overline{\Omega}$ such that for all θ in $-\Omega$ the Laplace transform of μ_p is

$$L_{\mu_p}(\theta) = \int_{\overline{\Omega}} e^{\mathrm{trace}\,(\theta s)} \mu_p(ds) = (\det(-\theta))^{-p} = (\det(-\theta)^{-1})^p. \tag{3.20}$$

The definition of $\gamma_{p,\sigma}$ through its Laplace transform as we have given it in Section 1 shows that $\gamma_{p,\sigma}(ds) = (\det\sigma)^{-p} \exp(-\mathrm{trace}\,(\sigma^{-1}s))\mu_p(ds)$. In other words, for a fixed p, $\{\gamma_{p,\sigma};\ \sigma \in \Omega\}$ is the natural exponential family generated by μ_p. For this reason, it is useful to use the natural parametrization of this family given by $\theta \in -\Omega$. The more traditional parameter $\sigma \in \Omega$ is such that

$$\sigma = \sigma(\theta) = (-\theta)^{-1}. \tag{3.21}$$

Let κ be the function defined on $-\Omega$ by $\kappa(\theta) = \log\det(-\theta)^{-1} = \log\det\sigma$, so that $L_{\mu_p} = \exp(p\kappa)$. We will use in the sequel the following two rules of differentiation of κ and σ with respect to θ in the direction $h \in V$:

$$\sigma'(\theta)(h) = \sigma h \sigma, \quad \kappa'(\theta)(h) = \mathrm{trace}\,(\sigma h). \tag{3.22}$$

These formulas follow immediately from the rules of differentiation of a determinant and of the inverse of a matrix. Formula (3.22) provides a remarkable way to differentiate the function $\theta \mapsto \mathrm{trace}\,(\sigma h_1 \sigma h_2 \ldots \sigma h_k)$ in the direction h_{k+1}. The differential of that function in the direction h_{k+1} is equal to

$$\mathrm{trace}\,(\sigma h_{k+1}\sigma h_1 \ldots \sigma h_k) + \mathrm{trace}\,(\sigma h_1 \sigma h_{k+1} \sigma h_2 \ldots \sigma h_k) + \cdots + \mathrm{trace}\,(\sigma h_1 \sigma h_2 \ldots \sigma h_{k+1} \sigma h_k).$$

It has been obtained by substituting $\sigma h_{k+1}\sigma$ to σ, k times in $\theta \mapsto \mathrm{trace}\,(\sigma h_1 \sigma h_2 \ldots \sigma h_k)$.

More generally, consider some cycle c of $\pi \in \mathcal{S}_k$, say $c = (\varphi_1, \ldots, \varphi_l)$. If $\nu = \varphi_i$, we write c_ν for the cycle $(\varphi_1, \ldots, \varphi_{i-1}, k+1, \varphi_i, \ldots, \varphi_l)$ of $\mathcal{S}_{k+1}$. Then the differential of $\theta \mapsto \mathrm{trace} \prod_{j\in c}(\sigma h_j)$ taken in the direction h_{k+1} is

$$\sum_{\nu \in c} \mathrm{trace} \prod_{j \in c_\nu} (\sigma h_j). \tag{3.23}$$

The heart of the proof is the following algebraic lemma:

Lemma 5. Let $\sigma(\theta) = (-\theta)^{-1}$, with $\det\theta \neq 0$. Then the kth differential form of $\theta \mapsto (\det\sigma)^p$ is the multilinear form

$$(h_1, \ldots, h_k) \mapsto (\det\sigma)^p \Big(\sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(h_1, \ldots, h_k) \Big). \tag{3.24}$$

Proof of Lemma 5. Using (3.22), the result is clear for $k = 1$. Let us now assume that it is true for k. Let us take the differential of (3.24) with respect to θ in the direction h_{k+1}. Using (3.22) and (3.23), we obtain

$$\begin{aligned} & p(\det\sigma)^p \operatorname{trace}(\sigma h_{k+1})(\sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(h_1,\ldots,h_k)) \\ + \ & (\det\sigma)^p(\sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} \sum_{c\in C(\pi)} (\sum_{\nu\in c} \operatorname{trace} \prod_{j\in c_\nu}(\sigma h_j))(\prod_{c_1\in C(\pi)\setminus\{c\}} \operatorname{trace}\prod_{l\in c_1} h_l). \end{aligned}$$

We now split $\mathcal{S}_{k+1}$ in two parts, the first part containing the π's in $\mathcal{S}_{k+1}$ such that $\pi(k+1) = k+1$, and the second part containing the remaining permutations. Clearly the two preceeding lines correspond to each one of the two parts and it follows that the $(k+1)$th differential of $\theta \mapsto (\det\sigma)^p$ is

$$(h_1,\ldots,h_{k+1}) \mapsto (\det\sigma)^p(\sum_{\pi\in\mathcal{S}_{k+1}} p^{m(\pi)} r_\pi(\sigma)(h_1,\ldots,h_{k+1})).$$

The induction argument is therefore completed and the lemma is proved.

Let us now prove Theorem 1. We will do so first for $(h_1,\ldots,h_k)$ Hermitian. If S is $\gamma_{p,\sigma}$ distributed, taking the k-th differential of (3.20) in the directions $(h_1,\ldots,h_k)$ yields immediately

$$\mathbb{E}(\operatorname{trace}(Sh_1)\operatorname{trace}(Sh_2)\ldots\operatorname{trace}(Sh_k)) = \frac{1}{L_{\mu_p}(\theta)} L^{(k)}_{\mu_p}(\theta)(h_1,\ldots,h_k),$$

which can be rewritten

$$L^{(k)}_{\mu_p}(\theta)(h_1,\ldots,h_k) = (\det\sigma)^p \mathbb{E}(\operatorname{trace}(Sh_1)\operatorname{trace}(Sh_2)\ldots\operatorname{trace}(Sh_k)). \tag{3.25}$$

Putting together (3.20), Lemma 5 and (3.25) gives the proof of (2.7) for Hermitian matrices h_j.

Let us now prove (2.7) for matrices which are not necessarily Hermitian. Consider the Hermitian matrices $a_j(0) = \frac{1}{2}(h_j + h_j^*)$ and $a_j(1) = \frac{1}{2i}(h_j - h_j^*)$, so that $h_j = a_j(0) + ia_j(1)$. Consider also all sequences $\epsilon = (\epsilon_1,\ldots,\epsilon_k) \in \{0,1\}^k$. We write $|\epsilon| = \epsilon_1 + \cdots + \epsilon_k$.

Using (2.7) for Hermitian matrices and the linearity of traces we have

$$\begin{aligned} & \mathbb{E}(\prod_{j=1}^k \operatorname{trace}(S(a_j(0) + ia_j(1)))) \\ = & \sum_{\epsilon\in\{0,1\}^k} i^{|\epsilon|}\mathbb{E}(\prod_{j=1}^k \operatorname{trace}(Sa_j(\epsilon_j))) \\ = & \sum_{\epsilon\in\{0,1\}^k} i^{|\epsilon|}(\sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(a_1(\epsilon_1),\ldots,a_k(\epsilon_k)) \\ = & \sum_{\epsilon\in\{0,1\}^k} (\sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(i^{\epsilon_1}a_1(\epsilon_1),\ldots,i^{\epsilon_k}a_k(\epsilon_k)) \\ = & \sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(a_1(0)+ia_1(1),\ldots,a_k(0)+ia_k(1)), \end{aligned}$$

which proves (2.7) for arbitrary complex matrices.

4 Proof of Theorem 2.

We begin with a corollary to Theorem 1. We first need to introduce some notation. The set $\{1, 2, \ldots, r\}^{2k}$ of double sequences $B = (a_1, b_1, a_2, b_2, \ldots, a_k, b_k)$ of integers in $\{1, 2, \ldots, r\}$ is denoted by D_k. For π in the symmetric group $\mathcal{S}_k$, we define the transformation l_π on D_k by

$$l_\pi(B) = (a_1, b_{\pi(1)}, a_2, b_{\pi(2)}, \ldots, a_k, b_{\pi(k)})$$

Thus $(\pi, B) \mapsto l_\pi(B)$ is a group action of $\mathcal{S}_k$ on D_k, and $\pi \mapsto l_\pi$ is an injective antihomomorphism (that is to say $l_{\pi' \circ \pi} = l_\pi \circ l_{\pi'}$) between $\mathcal{S}_k$ and the group of permutations of D_k.

Let us also adopt the following notation: if $F = (F_{ab})_{1 \leq a,b \leq r}$ is an (r, r) matrix and if $B = (a_1, b_1, a_2, b_2, \ldots, a_k, b_k)$ is in D_k, we write

$$F(B) = F_{a_1 b_1} F_{a_2 b_2} \ldots F_{a_k b_k}.$$

Since $l_\pi(B)$ is in D_k, $F(l_\pi(B))$ is also well defined. Suppose for instance that $k = 6$ and that $\pi = (1, 6, 3)(2, 5)(4)$. Then

$$F(l_\pi(B)) = F_{a_1 b_6} F_{a_6 b_3} F_{a_3 b_1} F_{a_2 b_5} F_{a_5 b_2} F_{a_4 b_4}.$$

We are now in position to state the corollary of Theorem 1. It extends the formulas given between the formulas (14) and (15) in Maiwald and Kraus (2000) for $k = 2, 3, 4$.

Corollary 6. For all $B \in D_k$ we have

$$\mathbb{E}(S(B)) = \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \sigma(l_\pi(B)). \tag{4.26}$$

Proof. We introduce here the following notation which will also be useful later: for $(a, b) \in \{1, 2, \ldots, r\}^2$, let

$$h^{ab} = (h^{ab}_{ts})_{1 \leq t,s \leq r} \tag{4.27}$$

be the (r, r) matrix with $h^{ab}_{ba} = 1$ and $h^{ab}_{ts} = 0$ if $(t, s) \neq (b, a)$. The matrix h^{ab} is designed so that $\mathrm{trace}\, S h^{ab} = S_{ab}$.

We now apply Theorem 1 to $h_j = h^{a_j b_j}$, where $B = (a_1, b_1, a_2, b_2, \ldots, a_k, b_k)$. Since $\mathrm{trace}\, S h^{ab} = S_{ab}$, it is clear that the first member of (2.7) is equal to $\mathbb{E}(S(B))$. To compute the second member of (2.7), we observe that if $(A^{(1)}, \ldots, A^{(n)})$ is a sequence of (r, r) matrices, then it can easily be proved by induction on n that

$$\mathrm{trace}\,(A^{(1)} A^{(2)} \ldots A^{(n)}) = \sum_{1 \leq j_1, \ldots, j_n \leq r} A^{(1)}_{j_1 j_2} A^{(2)}_{j_2 j_3} \cdots A^{(n-1)}_{j_{n-1} j_n} A^{(n)}_{j_n j_1}.$$

Applying this formula to $A^{(1)} = F$, $A^{(2)} = h^{a_1 b_1}$, $A^{(3)} = F$ etc, we get that for any (r, r) matrix F

$$\mathrm{trace}\,(F h^{a_1 b_1} F h^{a_2 b_2} \cdots F h^{a_k b_k}) = F_{a_1 b_2} \cdots F_{a_{k-1} b_k} F_{a_k b_1}. \tag{4.28}$$

We notice that the second member of (4.28) is $F(l_\pi(B))$ when π is the cycle $(1,2,\dots,k)$. Applying this to each cycle of $\pi \in \mathcal{S}_k$ and using the definition of r_π given in (2.6) we obtain

$$r_\pi(F)(h^{a_1b_1},\dots,h^{a_kb_k}) = F(l_\pi(B)). \tag{4.29}$$

Letting $F=\sigma$ and summing both sides of (4.29) over all $\pi\in\mathcal{S}_k$, yields (4.26) and the corollary is proved.

We are now in position to prove Theorem 2. For $B=(a_1,b_1,a_2,b_2,\dots,a_k,b_k)$ in D_k given, we will prove Theorem 2 first for $h_j = h^{a_jb_j}$. For these h_j, it follows immediately from (4.29) applied to $F=S$ that the left hand side of (2.10) is $\mathbb{E}(S(l_\pi(B)))$ while the right hand side of (2.10) is

$$\sum_{\pi'\in\mathcal{S}_k} p^{m(\pi^{-1}\circ\pi')}\sigma(l_{\pi'}(B))$$

(Recall that $m(\pi)=m(\pi^{-1})$). Making the change of variable $\pi''=\pi^{-1}\circ\pi'$ in this last sum we obtain

$$\sum_{\pi''\in\mathcal{S}_k} p^{m(\pi'')}\sigma(l_{\pi\circ\pi''}(B)).$$

Now, we use the fact that $\pi\mapsto l_\pi$ is an antihomomorphism. Applying (4.26) to $B'=l_\pi(B)$ yields (2.10) for $h_j=h^{a_jb_j}$.

In order to obtain (2.10) for any sequence $(h_1,\dots,h_k)$ of (r,r) complex matrices, we use linearity in a standard way and the proof of Theorem 2 is complete.

5 Proof of Theorem 3.

Similarly to what we did for Theorem 2, we first prove Theorem 3 for $\pi=$ identity and for Hermitian matrices h_j. The idea of the proof is to use Stokes formula, roughly in the following way: if a function f defined on $\overline{\Omega}$ is zero on the boundary $\partial\Omega=\overline{\Omega}\setminus\Omega$ and is smooth enough, then $\int_\Omega f'(s)(h)\lambda(ds)=0$. Actually, we use this principle for the k th differential of f, obtaining

$$\int_\Omega f^{(k)}(s)(h_1,\dots,h_k)\lambda(ds)=0. \tag{5.30}$$

The formula is applied to $s\mapsto f(s)=C(p,r)(\det s)^q\exp(\mathrm{trace}\,(\theta s))$, where $C(p,r)$ is the constant in (1.3) and $\theta=-\sigma^{-1}$ is in $-\Omega$. Since f is the product of the two functions $u(s)=(\det s)^q$ and $v(s)=C(p,r)\exp(\mathrm{trace}\,(\theta s))$, we first apply Leibnitz formula:

$$(uv)^{(k)}(s)(h_1,\dots,h_k)=\sum_{T\subset\{1,\dots,k\}} u^{(T)}(s)(h_T)v^{(T')}(s)(h_{T'}), \tag{5.31}$$

with the following notations: $T'=\{1,\dots,k\}\setminus T$ is the complementary set of T, h_T is the set of variables $(h_j)_{j\in T}$ and $u^{(T)}(s)$ means the differential whose order is the size of T. For the particular choices that we have made for u and v, these differentials are easy to compute. Indeed, the differentials of v are such that

$$\begin{aligned} v^{(k)}(s)(h_1,\dots,h_k) &= v(s)\times\mathrm{trace}\,(\theta h_1)\dots\mathrm{trace}\,(\theta h_k)\\ &= v(s)\times(-1)^k\mathrm{trace}\,(\sigma^{-1}h_1)\dots\mathrm{trace}\,(\sigma^{-1}h_k). \end{aligned}$$

To compute the differentials of u we use Lemma 5 in Section 3, with θ replaced by $-s$ so that $(\det s)^q = (\det(-\theta)^{-1})^{-q}$, and with p replaced by $-q$. We obtain

$$u^{(k)}(s)(h_1,\ldots,h_k) = u(s) \times (-1)^k \sum_{\pi\in\mathcal{S}_k} (-q)^{m(\pi)} r_\pi(s^{-1})(h_1,\ldots,h_k). \tag{5.32}$$

For simplicity, denote $v_j = \operatorname{trace}(\sigma^{-1}h_j)$, and for $T \subset \{1,\ldots,k\}$

$$u(T) = \int_\Omega u^{(T)}(s)(h_T)v(s)\lambda(ds).$$

Formula (5.32) implies in particular that

$$u(\{1,\ldots,k\}) = (-1)^k \sum_{\pi\in\mathcal{S}_k} (-q)^{m(\pi)} \mathbb{E}(r_\pi(S^{-1})(h_1,\ldots,h_k)). \tag{5.33}$$

Note that this implies that $u(\emptyset) = 1$. Using (5.30), (5.31) and the computations of the differentials of u and v we have

$$\sum_{T\subset\{1,\ldots,k\}} u(T) \prod_{j\in T'} (-v_j) = 0.$$

More generally, from the last formula, we get that, for all non empty $T_1 \subset \{1,\ldots,k\}$,

$$\sum_{T\subset T_1} u(T) \prod_{j\in T_1\setminus T} (-v_j) = 0.$$

It is easily seen by induction on the size of T_1 that the solution of this linear system is exactly $u(T_1) = \prod_{j\in T_1} v_j$, since $u(\emptyset) = 1$. In particular

$$u(\{1,2,\ldots,k\}) = \prod_{j=1}^{k} \operatorname{trace}(\sigma^{-1}h_j)$$

and therefore (5.33) yields the desired formula (2.11) of Theorem 3 for π = identity and Hermitian matrices h_j.

The remainder of the proof of Theorem 3 follows the same lines as the proofs of the two previous theorems: that is, as in Theorem 1, in a first step, we prove (2.11) for π = identity and for arbitrary h_j, by a linearity argument. In a second step, keeping π = identity and choosing h_j to be $h_j = h^{a_jb_j}$, as defined in (4.27), (2.11) from the first step becomes

$$\sigma^{-1}(B) = (-1)^k \sum_{\pi\in\mathcal{S}_k} (-q)^{m(\pi)} \mathbb{E}(S^{-1}(l_\pi(B)),$$

where $B = (a_1, b_1, \ldots, a_k, b_k)$. In the third step, we apply the formula above to $B' = l_\pi(B)$ in order to obtain (2.11) for the particular $h_j = h^{a_jb_j}$ but for arbitrary π. Finally a linearity argument leads us to the proof of (2.11) in Theorem 3.

6 The case $k = 3$ and the irreducible representations of $\mathcal{S}_k$.

The remainder of the paper is devoted to the computation of the $\mathbb{E}(r_\pi(S^{-1})(h_1, \ldots, h_k))$, i.e. to the inversion of the $(k!, k!)$ matrix $M(q) = ((-1)^k(-q)^{m(\pi'^{-1}\circ\pi)})$ appearing in (2.11) and to the proof of Theorem 4. In this section, we use the case $k = 3$ to show that methods from group theory arise naturally in this problem.

Let us denote the elements of $\mathcal{S}_3$ by

$$\pi_1 = (1)(2)(3),\ \pi_2 = (1,2,3),\ \pi_3 = (1,3,2),\ \pi_4 = (1)(2,3),\ \pi_5 = (2)(1,3),\ \pi_6 = (3)(1,2).$$

Following the order given by the π_i, and denoting by i_3 and j_3 the identity (3,3) matrix and the matrix containing only ones, respectively, we can write the matrix $M(q)$ by blocks as

$$M(q) = \begin{bmatrix} (q^3 - q)i_3 + qj_3 & -q^2 j_3 \\ -q^2 j_3 & (q^3 - q)i_3 + qj_3 \end{bmatrix},$$

and therefore the matrix $(M(q))^{-1}$ is easily calculated:

$$(M(q))^{-1} = \frac{1}{q(q^2-1)(q^2-4)} \begin{bmatrix} (q^2 - 4)i_3 + 2j_3 & qj_3 \\ qj_3 & (q^2 - 4)i_3 + 2j_3 \end{bmatrix}. \tag{6.34}$$

Denoting $P(q) = q(q^2 - 1)(q^2 - 4)$, this leads to the following explicit solution of the moment problem for S^{-1} and for $k = 3$:

$$\begin{bmatrix} \mathbb{E}(r_{\pi_1}(S^{-1})) \\ \mathbb{E}(r_{\pi_2}(S^{-1})) \\ \mathbb{E}(r_{\pi_3}(S^{-1})) \\ \mathbb{E}(r_{\pi_4}(S^{-1})) \\ \mathbb{E}(r_{\pi_5}(S^{-1})) \\ \mathbb{E}(r_{\pi_6}(S^{-1})) \end{bmatrix} = \frac{1}{P(q)} \begin{bmatrix} q^2-2 & 2 & 2 & q & q & q \\ 2 & q^2-2 & 2 & q & q & q \\ 2 & 2 & q^2-2 & q & q & q \\ q & q & q & q^2-2 & 2 & 2 \\ q & q & q & 2 & q^2-2 & 2 \\ q & q & q & 2 & 2 & q^2-2 \end{bmatrix} \begin{bmatrix} r_{\pi_1}(\sigma^{-1}) \\ r_{\pi_2}(\sigma^{-1}) \\ r_{\pi_3}(\sigma^{-1}) \\ r_{\pi_4}(\sigma^{-1}) \\ r_{\pi_5}(\sigma^{-1}) \\ r_{\pi_6}(\sigma^{-1}) \end{bmatrix}$$

It is also interesting to compute the eigenspaces and eigenvalues of $M(q)$. We obtain the eigenvalue $q^3 - 3q^2 + 2q$, associated with the eigenvector $[1, 1, 1, 1, 1, 1]$, the eigenvalue $q^3 + 3q^2 + 2q$ associated with the eigenvector $[1, 1, 1, -1, -1, -1]$, and finally $q^3 - q$, with multiplicity 4, associated with the 4-dimensional space orthogonal to the two preceeding eigenvectors. Note that these eigenspaces do not depend on q.

To compute $\mathbb{E}(r_\pi(S^{-1})(h_1, \ldots, h_k))$ for higher values of k,we will have to use methods which are less elementary than before, and we will have to rely on classical results for finite groups, which we will apply later to the group $G = \mathcal{S}_k$.

Given a finite group G, its algebra $\mathcal{A}(G)$ is the complex linear space of functions from G to $\mathbf{C}$. This space is endowed with a product operation called *convolution* and defined by

$$f * g(x) = \sum_{y \in G} f(y^{-1}x)g(y).$$

One can easily verify that this product on $\mathcal{A}(G)$ is associative. Its unit is the indicator δ_e of the neutral element e of G. We give $\mathcal{A}(G)$ a Hermitian structure by defining the scalar product

$$\langle f, g\rangle = \frac{1}{o(G)} \sum_{x\in G} \overline{f(x)} g(x).$$

On G, we say that x is equivalent to x' if and only if there exists y in G such that $x' = yxy^{-1}$. The equivalence classes of this relation are simply called *classes*. The subspace $\mathcal{Z}(G)$ of $\mathcal{A}(G)$ of the functions f such that f is constant on each class is called the *center* of $\mathcal{A}(G)$, and an element of $\mathcal{Z}(G)$ is called a *central* function. The reason for this name is that $f * g = g * f$ for all g in $\mathcal{A}(G)$ if and only if f is central (see [S] p. 39).

For a given central function f, we now want to analyse the endomorphism $L(f)$ of $\mathcal{A}(G)$ defined by $g \mapsto f * g = L(f)(g)$, by finding in particular its eigenspaces and eigenvalues. To do so, we introduce the set $\hat{G}$ of all irreducible linear representations of G (up to equivalence). Consider also the regular linear representation defined as the family of unitary transformations $(R(y))_{y\in G}$ of the Hermitian space $\mathcal{A}(G)$ acting as follows: for all g in $\mathcal{A}(G)$, then $R(y)g(x) = g(xy)$. From the theory of finite groups, we know that R is the direct sum of all irreducible representations $D^{(\alpha)}$, i.e. each $\alpha \in \hat{G}$ is present in the canonical decomposition of R with multiplicity d_α, where d_α denotes the dimension of the representation $D^{(\alpha)}$ (See [S] section III.1, or more explicitely Fulton and Harris (1991) Corollary 2.18). Without loss of generality, let us choose a representative unitary matrix $D^{(\alpha)}(x) = (D^{(\alpha)}_{ij}(x))_{1\leq i,j\leq d_\alpha}$.

Then $\mathcal{A}(G)$ is the orthogonal direct sum of subspaces V_α of respective dimensions d_α^2 which are generated by the set of functions on G which are the coefficients of the representation, namely $\{D^{(\alpha)}_{ij}(x); 1 \leq i, j \leq d_\alpha\}$. Furthermore the $D^{(\alpha)}_{ij}$ are orthogonal, with $\langle D^{(\alpha)}_{ij}, D^{(\alpha)}_{ij}\rangle = 1/d_\alpha$ (see [S] page 36). Each V_α shares exactly a 1-dimensional linear space with $\mathcal{Z}(G)$, and this space is generated by the *character*

$$\chi^{(\alpha)}(x) = \text{trace}\, D^{(\alpha)}(x) = \sum_{i=1}^{d_\alpha} D^{(\alpha)}_{ii}(x)$$

of the corresponding irreducible representation. These characters satisfy $\langle \chi^{(\alpha)}, \chi^{(\alpha)}\rangle = 1$. Since the $(V_\alpha)_{\alpha\in\hat{G}}$ form an orthogonal direct sum, the characters are orthonormal. In fact, the characters form a basis of $\mathcal{Z}(G)$, (see [S], page 40). This shows in particular that the number of classes is equal to the number $o(\hat{G})$ of irreducible representations. Observe however that, for an arbitrary finite group, there is no natural correspondence between the set of equivalence classes of G and the set $\hat{G}$ of irreducible representations.

Denote by $\chi^\alpha(c)$ the common value of the character χ^α on the class c. An other consequence of the orthonormality of characters is that if $C_G = (\chi^\alpha(c))$ is the square matrix of characters, then

$$\frac{1}{o(G)} \text{Diag}(\sharp c) C_G^t = C_G^{-1}, \tag{6.35}$$

where $\sharp c$ is the number of elements of the class c.

To analyse the product $f*g$, recall that $f*g=0$ if $f\in V_\alpha$, $g\in V_\beta$ and $\alpha\neq\beta$, that $D^{(\alpha)}_{ij}*D^{(\alpha)}_{kl}=0$ if $j\neq k$ and that for all i,j,l in $1,2,\ldots,d_\alpha$

$$D^{(\alpha)}_{ij}*D^{(\alpha)}_{jl}=\frac{o(G)}{d_\alpha}D^{(\alpha)}_{il}.$$

This implies in particular that $\chi^{(\alpha)}*D^{(\alpha)}_{ij}=\frac{o(G)}{d_\alpha}D^{(\alpha)}_{ij}$ and that $\chi^{(\alpha)}*\chi^{(\alpha)}=\frac{o(G)}{d_\alpha}\chi^{(\alpha)}$.

Now, suppose that f is in $\mathcal{Z}(G)$. We write $f=\sum_{\alpha\in\hat{G}}f_\alpha\chi^{(\alpha)}$, with

$$f_\alpha=\langle f,\chi^{(\alpha)}\rangle=\frac{1}{o(G)}\sum_c\overline{f(c)}\chi^{(\alpha)}(c)\sharp c, \tag{6.36}$$

where the sum is taken on the set of classes and where $f(c)$ is the common value of f on the class c.

Similarly, for g in $\mathcal{A}(G)$ we write $g=\sum_{\alpha\in\hat{G}}\sum_{i,j=1}^{d_\alpha}g_{\alpha,i,j}D^{(\alpha)}_{ij}$, with

$$g_{\alpha,i,j}=d_\alpha\langle g,D^{(\alpha)}_{ij}\rangle.$$

Thus we get

$$f*g=\sum_{\alpha\in\hat{G}}f_\alpha\frac{o(G)}{d_\alpha}\sum_{i,j=1}^{d_\alpha}g_{\alpha,i,j}D^{(\alpha)}_{ij}.$$

This equation clearly shows that the eigenspaces of the endomorphism $g\mapsto f*g=L(f)(g)$ of $\mathcal{A}(G)$ are the V_α and the eigenvalues are

$$f_\alpha\frac{o(G)}{d_\alpha} \tag{6.37}$$

with multiplicity d_α^2. Note that $\mathcal{Z}(G)$ is a stable subspace of $L(f)$ and that $L(f)$ can be considered as an endomorphism of $\mathcal{Z}(G)$ as well. Indeed, suppose that g is in $\mathcal{Z}(G)$, and write $g=\sum_{\alpha\in\hat{G}}g_\alpha\chi^{(\alpha)}$. Then

$$f*g=\sum_{\alpha\in\hat{G}}f_\alpha\frac{o(G)}{d_\alpha}g_\alpha\chi^{(\alpha)}.$$

Thus for $L(g)$ restricted to $\mathcal{Z}(G)$, the eigenvectors are the $\chi^{(\alpha)}$ and the eigenvalues obviously are again given by (6.37). Finally, we note that $L(f)$ is invertible if and only if for any α in $\hat{G}$ we have $f_\alpha\neq 0$. Under these conditions, we have $(L(f))^{-1}=L(f^{(-1)})$ with

$$f^{(-1)}=\sum_{\alpha\in\hat{G}}\frac{d_\alpha^2}{o(G)^2f_\alpha}\chi^{(\alpha)}, \tag{6.38}$$

which satisfies $f*f^{(-1)}=\delta_e$.

We now specialize these notions to the group $\mathcal{S}_k$, of size $o(\mathcal{S}_k)=k!$.

For $k=3$, there are three classes $\{\pi_1\}$, $\{\pi_2,\pi_3\}$, and $\{\pi_4,\pi_5,\pi_6\}$, whose portraits are given by $\mathbf{i}(\pi_1)=(3,0,0)$, $\mathbf{i}(\pi_2)=(0,0,1)$ and $\mathbf{i}(\pi_4)=(1,1,0)$, abbreviated as (3), (0,0,1) and (1,1) respectively.

The functions from $\mathcal{S}_k$ to $\mathbf{C}$ given by $\pi \mapsto r_\pi(\sigma)(h_1, h_2, \ldots, h_k)$ are examples of elements of $\mathcal{A}(\mathcal{S}_k)$. Note here an important fact: when $h_1 = \ldots = h_k = h$ the function $\pi \mapsto r_\pi(\sigma)(h, \ldots, h)$ becomes a central function, i.e. an element of $\mathcal{Z}(\mathcal{S}_k)$. Other important examples of central functions are given by $\pi \mapsto m(\pi)$, p^m and $(-1)^k(-q)^m$. Since m is a class function, it can also be written $m(\pi) = m(\mathbf{i}) = i_1 + \cdots + i_k$ where $\mathbf{i}$ is the portrait of π. In terms of convolution, one can observe that Theorem 2 and Theorem 3 can be rewritten

$$\mathbb{E}(r(S)) = p^m * r(\sigma), \qquad r(\sigma^{-1}) = (-1)^k(-q)^m * \mathbb{E}(r(S^{-1})).$$

We consider now the set $\hat{G}$ of irreducible representations. It is certainly not possible to describe them all here (see [S] chap VI) but it is important to say that they are naturally indexed by the set M_k of sequences $\mathbf{m} = (m_1, \ldots, m_k)$ of integers such that $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and such that $m_1 + \cdots + m_k = k$. For this reason, we choose to denote by $\chi^{\mathbf{m}}$ the character $\chi^{(\alpha)}$ of the corresponding irreducible representation. Since $\chi^{\mathbf{m}}$ is constant on the classes, we denote its value on π by $\chi^{\mathbf{m}}(\mathbf{i})$ when $\mathbf{i}$ is the portrait of π. A theoretical knowledge of the matrix (6.35) for the group $\mathcal{S}_k$, i.e. of

$$C_k = (\chi^{\mathbf{m}}(\mathbf{i}))_{\mathbf{m} \in M_k, \mathbf{i} \in I_k} \tag{6.39}$$

is obtained by the introduction of the *Schur polynomials* with k variables defined for $\mathbf{m}$ in M_k by

$$s_{\mathbf{m}}(z_1, \ldots, z_k) = \frac{\det(z_i^{m_j+k-j})_{1 \leq i,j \leq k}}{\det(z_i^{k-j})_{1 \leq i,j \leq k}}. \tag{6.40}$$

The characters are now given by *Frobenius formula* (see [S], Th.VI 5.1): for all $\mathbf{i}$ in I_k one has

$$\prod_{j=1}^{k} (z_1^j + \cdots + z_k^j)^{i_j} = \sum_{\mathbf{m} \in M_k} \chi^{\mathbf{m}}(\mathbf{i}) s_{\mathbf{m}}(z_1, \ldots, z_k). \tag{6.41}$$

The expression for the dimension $d_{\mathbf{m}}$ of the corresponding irreducible representation has been given in (2.14).

Let us illustrate these notions for $k = 3$. We have

$$M_3 = \{(3,0,0),\ (2,1,0),\ (1,1,1)\} = \{(3),\ (2,1),\ (1,1,1)\}.$$

The Schur functions are

$$\begin{aligned}
s_{(3)}(z_1, z_2, z_3) &= z_1^3 + z_2^3 + z_3^3 + (z_1 z_2^2 + 5 \text{ terms}) + z_1 z_2 z_3 \\
s_{(2,1)}(z_1, z_2, z_3) &= (z_1 z_2^2 + 5 \text{ terms}) + 2 z_1 z_2 z_3 \\
s_{(1,1,1)}(z_1, z_2, z_3) &= z_1 z_2 z_3
\end{aligned}$$

Ways to compute these functions for lower values of k are given in Fulton (1996).
The table of characters of $\mathcal{S}_3$, together with the dimensions of the representations, is

$$\begin{array}{llccc}
\mathbf{m} & : & (3) & (1,1,1) & (2,1) \\
d_{\mathbf{m}} & : & 1 & 1 & 2 \\
\chi^{\mathbf{m}}((3)) & : & 1 & 1 & 2 \\
\chi^{\mathbf{m}}((1,1)) & : & 1 & -1 & 0 \\
\chi^{\mathbf{m}}((0,0,1)) & : & 1 & 1 & -1
\end{array} \tag{6.42}$$

We can now explain the form of $(M(q))^{-1}$, as given in (6.34) at the beginning of this section, by applying the above theory to $G = \mathcal{S}_3$ and to the element f of $\mathcal{Z}(\mathcal{S}_3)$ defined by $f((3)) = q^3$, $f((1,1)) = -q^2$ and $f((0,0,1)) = q$. Recall that (6.34) is the representative matrix of $L(f)^{-1}$. Following (6.36), we write $f = \sum_{\mathbf{m}\in M_3} f_{\mathbf{m}}\chi^{\mathbf{m}}$ with

$$f_{\mathbf{m}} = \frac{1}{6}\sum_{\mathbf{i}\in I_3}(-1)^3(-q)^{m(\mathbf{i})}\chi^{\mathbf{m}}(\mathbf{i})\sharp\mathbf{i}.$$

and using the table of characters (6.42), we obtain

$$(f_{(3)}, f_{(1,1,1)}, f_{(2,1)}) = (\frac{1}{6}(q^3-3q^2+2q), \frac{1}{6}(q^3+3q^2+2q), \frac{1}{3}(q^3-q)).$$

According to (6.37) the eigenvalues of $M(q)$ are therefore the ones given at the beginning of this section. We next want to compute $f^{(-1)}$. To do so, we use (6.38). The vector $(f^{(-1)}(\mathbf{i}))_{\mathbf{i}\in I_3}$ is obtained with the help of (6.42):

$$\begin{bmatrix} f^{(-1)}((3)) \\ f^{(-1)}((1,1)) \\ f^{(-1)}((0,0,1)) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/36 & 0 & 0 \\ 0 & 1/36 & 0 \\ 0 & 0 & 4/36 \end{bmatrix} \begin{bmatrix} \frac{6}{q^3-3q^2+2q} \\ \frac{6}{q^3+3q^2+2q} \\ \frac{3}{q^3-q} \end{bmatrix} = \begin{bmatrix} \frac{q^2-2}{P(q)} \\ \frac{q}{P(q)} \\ \frac{2}{P(q)} \end{bmatrix},$$

with $P(q) = q(q^2-1)(q^2-4)$.

7 Proof of Theorem 4.

For $r \le k$, denote by $M_{k,r}$ the set of sequences of integers $\mathbf{m} = (m_1, \ldots, m_r)$ such that $m_1 \ge m_2 \ge \cdots \ge m_r \ge 0$ and such that $m_1 + \cdots + m_r = k$.

We now extend the definition of Schur polynomials $s_{\mathbf{m}}$, as given in (6.40) to the case of r variables and $\mathbf{m} \in M_{k,r}$ as follows

$$s_{\mathbf{m}}(z_1, \ldots, z_r) = s_{\mathbf{m}}(z_1, \ldots, z_r, 0, \ldots, 0) \tag{7.43}$$

(see Fulton (1997), Section 6.2). Note that for $\mathbf{m} \in M_k \setminus M_{k,r}$ we have

$$s_{\mathbf{m}}(z_1, \ldots, z_r, 0, \ldots, 0) = 0. \tag{7.44}$$

We are now going to prove part 1) of Theorem 4 by showing the following, which is an identity between two polynomials with respect to the variable t. For all $\mathbf{i} \in I_k$, we have

$$t^{i_1+\ldots+i_k} = \sum_{\mathbf{m}\in M_k}\chi^{\mathbf{m}}(\mathbf{i})\frac{d_{\mathbf{m}}}{k!}\prod_{j=1}^{k}\prod_{i=1}^{m_j}(t+i-j). \tag{7.45}$$

It is immediate to verify that, for $t = -q$, (7.45) is equivalent to (2.16). To prove (7.45), it suffices to fix $r \le k$ and let $z_1 = \cdots = z_r = 1$ and $z_{r+1} = \cdots = z_k = 0$ in Frobenius formula (6.41). We are very grateful to William Fulton for giving us this idea. We thus obtain

$$
\begin{aligned}
r^{i_1+\ldots+i_k} &\overset{(1)}{=} \sum_{\mathbf{m}\in M_k} \chi^{\mathbf{m}}(\mathbf{i}) s_{\mathbf{m}}(1,\ldots,1,0,\ldots,0) \\
&\overset{(2)}{=} \sum_{\mathbf{m}\in M_{k,r}} \chi^{\mathbf{m}}(\mathbf{i}) s_{\mathbf{m}}(1,\ldots,1) \\
&\overset{(3)}{=} \sum_{\mathbf{m}\in M_{k,r}} \chi^{\mathbf{m}}(\mathbf{i}) \prod_{1\le i<j\le r} \frac{m_i - m_j + j - i}{j-i} \\
&\overset{(4)}{=} \sum_{\mathbf{m}\in M_{k,r}} \chi^{\mathbf{m}}(\mathbf{i}) \prod_{j=1}^{k} \frac{d_{\mathbf{m}}}{k!} \prod_{i=1}^{m_j} (r+i-j) \\
&\overset{(5)}{=} \sum_{\mathbf{m}\in M_k} \chi^{\mathbf{m}}(\mathbf{i}) \frac{d_{\mathbf{m}}}{k!} \prod_{j=1}^{k} \prod_{i=1}^{m_j} (r+i-j)
\end{aligned}
$$

In the above sequence of equalities, (1) comes from (6.41), (2) from (7.43) and (7.44), (3) from Fulton (1997), Exercise 6, p. 76, completed with formula (7) p. 75 and Fulton's definition of Schur polynomials, p. 3. Now, (4) comes from Stanley (1971), as indicated by Fulton (1997), formula (9), p. 55. Finally (5) comes from the easily verified fact that $\prod_{i=1}^{m_j}(r+i-j) = 0$ if $\mathbf{m}$ is in $M_k \setminus M_{k,r}$.

Thus (7.45) is true for $t = r = 1, 2, \ldots, k$. Furthermore, it is trivially true for $t = 0$. Since both members of (7.45) are polynomials of degree $\le k$, (7.45) is proved, as well as (2.16).

The proof of part 2) of Theorem 4 is now immediate. Since $q > k$, then for all $\mathbf{m}$ in M_k, we have $f_{\mathbf{m}} \neq 0$. Thus $f^{(-1)}$ as defined in (2.17) exists. From (6.38), and with the notations of Section 6 we have $f^{(-1)} * (-1)^k(-q)^m = \delta_e$. This proves (2.18) and Theorem 4.

Comment:

For a given $\mathbf{m}$ in M_k, the polynomial in q equal to $\prod_{j=1}^{k}\prod_{i=1}^{m_j}(q+j-i)$ is called the content polynomial. Many of its properties (but not (7.45)) appear in Macdonald (1995), in particular on p. 15-16.

8 The role of the Jordan algebra.

In this section, Theorem 9 provides an alternative method for computing $f^{(-1)}(\pi)$ appearing in the inverse formula (2.18). We consider the problem of the computation of moments within the framework of Jordan algebras. Though not essential to the problem of moments, Jordan algebras provide a convenient framework. The statement of Proposition 7 does not necessitate any knowledge of Jordan algebras but its proof does. For the properties of Jordan algebras used in this proof, we will refer to Faraut and Koranyi, hereafter abbreviated [FK]. We thus temporarily depart from our effort of self containment in this paper but as we can read it in Fulton (1997) page ix, 'Here the goal of self-containement works against that of building bridges.'

The space V as defined in Section 1 is considered as a Jordan algebra, and our complex Wishart distributions $\gamma_{p,\sigma}$ are slight extensions (since our p may not be an integer) of the Wishart distributions described in [FK] p. 349-350, for this particular V. An important feature of this V is the fact its spherical polynomials $\phi_{\mathbf{m}}$ appearing on p. 228 of [FK] can be written explicitly and are related to the Schur polynomials (7.43) by

$$\phi_{\mathbf{m}}(x) = c_{\mathbf{m},r} s_{\mathbf{m}}(z_1, \ldots, z_r), \tag{8.46}$$

where $z_1, \ldots, z_r$ are the eigenvalues of $x \in V$ and where $c_{\mathbf{m},r}$ is a proportionality factor (see [FK] page 238). If one wants to ignore Jordan algebras, (8.46) can be taken as a definition of spherical polynomials. An important object is the space $\mathcal{P}$ of polynomials $P(\sigma, h)$ on V which are such that for all (r, r) complex matrices a one has $P(a\sigma a^*, h) = P(\sigma, a^* h a)$. For $\sigma \in \Omega$, they are necessarily of the form $P(\sigma, h) = Q(\sigma^{1/2} h \sigma^{1/2})$ where Q is some polynomial on V (see [FK] page 291). On $\mathcal{P}$, one defines two linear operators T_p and T_p^* by

$$T_p(P)(\sigma, h) = \int_\Omega P(s, h)\gamma_{p,\sigma}(ds),$$

and

$$T_p^*(P)(\sigma^{-1}, h) = \int_\Omega P(s^{-1}, h)\gamma_{p,\sigma}(ds).$$

Note that $T_p^*(P)$ exists only if p is large enough with respect to the degree of P. These operators are linked to the idea of representations of the Wishart distributions: see Letac and Massam (2000), (2.1). Proposition 7 below will show that actually the $\phi_{\mathbf{m}}$'s are associated with the eigenvectors of T_p and T_p^*.

Recall that r is the size of the Hermitian matrices which form the linear space V. We introduce the so called Gamma function of the cone Ω : for $\mathbf{s} = (s_1, \ldots, s_r)$ with $s_j > j - 1$, $j = 1, \ldots, r$, we define

$$\Gamma_\Omega(\mathbf{s}) = \prod_{j=1}^r \Gamma(s_j - j + 1).$$

If t is a real number which is large enough, we write $\Gamma_\Omega(\mathbf{s} + t)$ and $\Gamma_\Omega(t)$ for $\Gamma_\Omega(s_1 + t, \ldots, s_r + t)$ and $\Gamma_\Omega(t, \ldots, t)$, respectively. Note that if $\mathbf{m} \in M_{k,r}$ and if $q = p - r$, with $q > m_1 - 1$, then we have

$$\frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)} = \prod_{j=1}^{r} \frac{\Gamma(q+j-m_j)}{\Gamma_\Omega(q+j)} = \frac{1}{f_{\mathbf{m}}}, \tag{8.47}$$

where $f_{\mathbf{m}}$ is given in (2.15). An important fact in this formula is that the right hand side, as a function of q, does not depend on r.

We now have the following proposition.

Proposition 7: Let σ be in the cone Ω of positive definite elements of V. Consider a random variable S in V with the complex Wishart distribution $\gamma_{p,\sigma}$ and let $\mathbf{m}$ be in $M_{k,r}$. Then, for all $h \in V$,

$$\mathbb{E}(\phi_{\mathbf{m}}(S^{1/2}hS^{1/2})) = \frac{\Gamma_\Omega(p+\mathbf{m})}{\Gamma_\Omega(p)}\phi_{\mathbf{m}}(\sigma^{1/2}h\sigma^{1/2}). \tag{8.48}$$

Furthermore, if $q = p - r > m_1 - 1$ for all $j = 1, \ldots, r$, one has

$$\mathbb{E}(\phi_{\mathbf{m}}(S^{-1/2}hS^{-1/2})) = \frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)}\phi_{\mathbf{m}}(\sigma^{-1/2}h\sigma^{-1/2}). \tag{8.49}$$

Proof: Equation (8.48) is a consequence of [FK] (1994), Prop. VII.1.2 or Lemma XI.2.3. Equation (8.49) does not follow immediately from [FK]. To show that (8.49) is true, we need the following lemma, whose proof is given in Section 11.

Lemma 8: Let $\mathbf{m} = (m_1, \ldots, m_r)$ with $m_j \in \mathbb{R}$ and $p > r-1$ such that $p - m_j > r - j$ for $j = 1, \ldots, r$. Let $m^* = (m_r, \ldots, m_1)$ and for $s = (s_{lk})_{1\le l,k\le r}$ in Ω, and for $j = 1, \ldots, r$, let $\Delta_j(s)$ be the determinant of the (j, j) Hermitian matrix $(s_{lk})_{1\le l,k\le j}$. We write

$$\Delta_{\mathbf{m}}(s) = (\Delta_1(s))^{m_1-m_2}(\Delta_2(s))^{m_2-m_3}\cdots(\Delta_r(s))^{m_r}.$$

Then, if the random variable S has the Wishart distribution $\gamma_{p,\sigma}$, we have

$$\mathbb{E}(\Delta_{\mathbf{m}}(S^{-1})) = \frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)}\Delta_{\mathbf{m}}(\sigma^{-1}).$$

Lemma 8 enables us to complete the proof of Proposition 7. Denote by $\mathbf{U}(r)$ the group of unitary (r, r) matrices, i.e. the compact group of matrices u such that $uu^* =$ identity. We equip it with its Haar measure du. Then

$$\begin{aligned}
\mathbb{E}(\phi_{\mathbf{m}}(S^{-1/2}hS^{-1/2})) &\overset{(1)}{=} \mathbb{E}\Big(\int_{\mathbf{U}(r)} \Delta_{\mathbf{m}}((uSu^*)^{-1})du\Big)\\
&\overset{(2)}{=} \mathbb{E}\Big(\int_{\mathbf{U}(r)} \Delta_{\mathbf{m}}(uS^{-1}u^*)du\Big)\\
&\overset{(3)}{=} \int_{\mathbf{U}(r)} \mathbb{E}(\Delta_{\mathbf{m}}(uS^{-1}u^*))du\\
&\overset{(4)}{=} \frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)}\int_{\mathbf{U}(r)} \Delta_{\mathbf{m}}(u\sigma^{-1}u^*)du\\
&\overset{(5)}{=} \frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)}\phi_{\mathbf{m}}(\sigma^{-1}).
\end{aligned}$$

In this sequence of equalities, (1) and (5) comes from the definition of spherical functions, (2) is clear, (3) comes from Fubini and (4) from Lemma 8. Thus (8.49) is established for h = identity. The case $h \in \Omega$ is obtained by considering the new Wishart variable $S' = S^{1/2}h^{-1}S^{1/2}$. Since (8.48) and (8.49) are polynomial identities, which are true on the open subset Ω of V, they are also true on V and the general case is proved. The proof of Proposition 7 is complete.

Proposition 7, as used in the proof of Theorem 9, will allow us to shed light on how the coefficients in the right hand side of (2.19) are obtained.

It is useful to think that I_k indexes the rows and that M_k indexes the columns of the matrix C_k of characters of the group $\mathcal{S}_k$ considered in (6.39) and that C_k is a linear map from $\mathbb{R}^{M_k}$ to $\mathbb{R}^{I_k}$, and that C_k^{-1} is a linear map from $\mathbb{R}^{I_k}$ to $\mathbb{R}^{M_k}$.

Theorem 9: Let C_k be the matrix (6.39) of characters of $\mathcal{S}_k$. Consider the following diagonal matrix

$$\Delta_k = \mathrm{Diag}(\sharp\mathbf{i};\ \mathbf{i} \in I_k)$$

acting on $\mathbb{R}^{I_k}$ and the diagonal matrix

$$D_k = \mathrm{Diag}(\mathbf{a_m};\ \mathbf{m} \in M_k)$$

acting on $\mathbb{R}^{M_k}$ defined by $a_{\mathbf{m}} = \frac{\Gamma_\Omega(p-\mathbf{m}^*)}{\Gamma_\Omega(p)}$ if $\mathbf{m} \in M_{k,r}$ and $a_{\mathbf{m}} = 0$ if $\mathbf{m} \in M_k \setminus M_{k,r}$. Then the line of the matrix $\Delta_k^{-1}C_kD_kC_k^{-1}$ corresponding to $\mathbf{i} = (k, 0, \dots, 0)$, the portrait of the identity is $(f^{-1}(\mathbf{i})|\mathbf{i} \in I_k)$, where $f(\pi) = (-1)^k(-q)^k$.

Proof: Let h be a Hermitian (r, r) matrix with eigenvalues $(z_1, \dots, z_r)$. From Frobenius formula (6.41) we get, for all $\mathbf{i} \in I_k$,

$$r_{\mathbf{i}}(e)(h) = \sum_{\mathbf{m} \in M_k} \chi^{\mathbf{m}}(\mathbf{i})\phi_{\mathbf{m}}(h), \tag{8.50}$$

by applying (2.13), the definition of the spherical functions and (7.44) . Using the matrix of characters C_k, we derive from (8.50)

$$(r_{\mathbf{i}}(S^{-1})(h))_{\mathbf{i} \in I_k} = C_k(\phi_{\mathbf{m}}(S^{-1/2}hS^{-1/2}))_{\mathbf{m} \in M_k}, \tag{8.51}$$

and also

$$(\phi_{\mathbf{m}}(\sigma^{-1/2}h\sigma^{-1/2}))_{\mathbf{m} \in M_k} = C_k^{-1}(r_{\mathbf{i}}(\sigma^{-1})(h))_{\mathbf{i} \in I_k}. \tag{8.52}$$

We take now the expectation of both sides of (8.51), apply (8.49) and (8.52) and get

$$(\mathbb{E}(r_{\mathbf{i}}(S^{-1})(h)))_{\mathbf{i} \in I_k} = C_kD_kC_k^{-1}(r_{\mathbf{i}}(\sigma^{-1})(h))_{\mathbf{i} \in I_k}. \tag{8.53}$$

Now recall that, from the definition of $f^{(-1)}(\pi)$, we have

$$\mathbb{E}(r_\pi(S^{-1})(h_1, \dots, h_k)) = \sum_{\pi' \in \mathcal{S}_k} f^{(-1)}(\pi')r_{\pi'^{-1}\circ\pi}(\sigma^{-1})(h_1, \dots, h_k).$$

In this formula, if we take π = identity and $h_1 = \ldots = h_k = h$ we obtain

$$\mathbb{E}(r_{(k)}(S^{-1})(h)) = \sum_{\mathbf{i} \in I_k} \sharp\mathbf{i} f^{(-1)}(\mathbf{i})r_{\mathbf{i}}(\sigma^{-1})(h). \tag{8.54}$$

Now let us compare the line of (8.53) corresponding to $\mathbf{i} = (k)$ with (8.54). The family of polynomials in h defined by $(r_{\mathbf{i}}(e)(h))_{\mathbf{i}\in I_k}$ is an independent family if the dimension r of V is $\geq k$: it is sufficient to express these polynomials in terms of the eigenvalues of h to verify this. Thus the result is proved for $r \geq k$. However, as we have seen, $f^{(-1)}(\mathbf{i})$ does not depend on r and the proof is complete.

At this point the reader might wonder why we do not explain the coefficients for our results in (2.18), when π is an arbitrary permutation. Things are a bit more delicate then. It is clear that $\prod_j \mathrm{trace}\,(S^{-1}h_j)$ is the symmetric k multilinear form which is the polarization of the homogeneous polynomial with degree k defined by $h \mapsto (\mathrm{trace}\,(S^{-1}h))^k$. However, the polarization of $r_{\mathbf{i}}(S^{-1})(h)$ has a more complicated form.

9 Examples for $4 \leq k \leq 9$.

While in Section 6 we simply rediscovered results which had been previously obtained by brute force in the case $k = 3$, in this section, we illustrate the usage of group theory to compute $\mathbb{E}(r_\pi(S^{-1})(h_1, \ldots, h_k))$ for $4 \leq k \leq 9$. It involves square matrices of size 5, 7, 11, 15, 22, 30 , instead of 24, 120, ... and 9! respectively. We will use tables of characters which can be easily be found in the literature (for $k = 4$ and 5, see for instance [S] p. 83 and 86, James and Kerber (1981) p. 349, or Fulton and Harris (1991) p. 19 and 28. In this last reference, on p. 28, lines W and W' should be exchanged). Let f denote the element of $\mathcal{Z}(\mathcal{S}_k)$ defined by $f(\pi) = (-1)^k(-q)^{m(\pi)}$. For $k = 4$ the table of characters is

$$\begin{array}{lcccccc}
\mathbf{m} & : & (4) & (1,1,1,1) & (3,1) & (2,1,1) & (2,2) \\
\chi^{\mathbf{m}}((4)) & : & 1 & 1 & 3 & 3 & 2 \\
\chi^{\mathbf{m}}((2,1)) & : & 1 & -1 & 1 & -1 & 0 \\
\chi^{\mathbf{m}}((1,0,1)) & : & 1 & 1 & 0 & 0 & -1 \\
\chi^{\mathbf{m}}((0,0,0,1)) & : & 1 & -1 & -1 & 1 & 0 \\
\chi^{\mathbf{m}}((0,2)) & : & 1 & 1 & -1 & -1 & 2
\end{array} \qquad (9.55)$$

Let C_4 be this $(5,5)$ matrix. Recall that the $d_{\mathbf{m}}$ are nothing but $\chi^{\mathbf{m}}((k))$, i.e. the first row of the above matrix. The size $\sharp\mathbf{i}$ of classes is given by (2.12). For $k = 4$ we have

$$\sharp(4) = 1 \ , \sharp(2,1) = 6 \ , \sharp(1,0,1) = 8 \ , \sharp(0,0,0,1) = 6 \ , \sharp(0,2) = 3$$

and the components of f in the basis of characters are given by

$$\begin{aligned}
&(f_{(4)}, f_{(1,1,1,1)}, f_{(3,1)}, f_{(2,1,1)}, f_{(2,2)}) \\
&= (q^4, -q^3, q^2, -q, q^2)\frac{1}{24}\mathrm{Diag}(1,6,8,6,3)C_4 \\
&= (\frac{1}{24}q(q-1)(q-2)(q-3), \frac{1}{24}q(q+1)(q+2)(q+3), \\
&\quad \frac{1}{8}q(q^2-1)(q-2), \frac{1}{8}q(q^2-1)(q+2), \frac{1}{12}q^2(q^2-1))
\end{aligned}$$

From this we can compute $f^{(-1)}$, which gives the solution to our inversion problem by $\mathbb{E}(r_\pi(S^{-1})) = (f^{(-1)} * r(\sigma^{-1}))_\pi$. It is enough is to compute the column vector of the $(f^{-1}(\mathbf{i}))_{\mathbf{i}\in I_4}$. By formula (6.38) we get (we use t to indicate transposition)

$$\begin{aligned}
(f^{(-1)}(\mathbf{i}))_{\mathbf{i}\in I_4} &= C_4 \frac{1}{24^2}\mathrm{Diag}(1,1,9,9,4)\times \\
&\quad (\frac{24}{q(q-1)(q-2)(q-3)}, \frac{24}{q(q+1)(q+2)(q+3)}, \\
&\quad \frac{8}{q(q^2-1)(q-2)}, \frac{8}{q(q^2-1)(q+2)}, \frac{12}{q^2(q^2-1)})^t \\
&= \frac{1}{q^2(q^2-1)(q^2-4)(q^2-9)}(q^4-8q^2+6, q^3-4q, 2q^2-3, 5q, q^2+6)^t
\end{aligned}$$

For $k=5$ the character table is

$$\begin{array}{lcccccccc}
\mathbf{m} & : & (5) & (1,1,1,1,1) & (4,1) & (2,1,1,1) & (3,1,1) & (3,2) & (2,2,1) \\
\chi^{\mathbf{m}}((5)) & : & 1 & 1 & 4 & 4 & 6 & 5 & 5 \\
\chi^{\mathbf{m}}((3,1)) & : & 1 & -1 & 2 & -2 & 0 & 1 & -1 \\
\chi^{\mathbf{m}}((2,0,1)) & : & 1 & 1 & 1 & 1 & 0 & -1 & -1 \\
\chi^{\mathbf{m}}((1,0,0,1)) & : & 1 & -1 & 0 & 0 & 0 & -1 & 1 \\
\chi^{\mathbf{m}}((0,0,0,0,1)) & : & 1 & 1 & -1 & -1 & 1 & 0 & 0 \\
\chi^{\mathbf{m}}((1,2)) & : & 1 & 1 & 0 & 0 & -2 & 1 & 1 \\
\chi^{\mathbf{m}}((0,1,1)) & : & 1 & -1 & -1 & 1 & 0 & 1 & -1
\end{array} \tag{9.56}$$

Let C_5 be this $(7,7)$ matrix of characters. For $k=5$, using (2.12) yields the following size for the classes

$$\sharp(5) = 1 \ , \sharp(3,1) = 10 \ , \sharp(2,0,1) = 20 \ , \sharp(1,0,0,1) = 30$$

$$\sharp(0,0,0,0,1) = 24 \ , \sharp(1,2) = 15 \ , \sharp(0,1,1) = 20.$$

The components of f in the basis of characters are given by

$$\begin{aligned}
(f_{(\mathbf{m})})_{\mathbf{m}\in M_5} &= (q^5, -q^4, q^3, -q^2, q, q^3, -q^2)\frac{1}{120}\mathrm{Diag}(1,10,20,30,24,15,20)C_5 \\
&= (\frac{1}{120}q(q-1)(q-2)(q-3)(q-4), \frac{1}{120}q(q+1)(q+2)(q+3)(q+4), \\
&\quad \frac{1}{30}q(q^2-1)(q-2)(q-3), \frac{1}{30}q(q^2-1)(q+2)(q+3), \\
&\quad \frac{1}{20}q(q^2-1)(q^2-4), \frac{1}{24}q^2(q^2-1)(q-2), \frac{1}{24}q^2(q^2-1)(q+2))
\end{aligned}$$

The components of $f^{(-1)}$ are:

$$\begin{aligned}
(f^{(-1)}(\mathbf{i}))_{\mathbf{i}\in I_5} &= C_5 \frac{1}{120^2}\mathrm{Diag}(1,1,16,16,36,25,25)\times \\
&\quad (\frac{120}{q(q-1)(q-2)(q-3)(q-4)}, \frac{120}{q(q+1)(q+2)(q+3)(q+4)},
\end{aligned}$$

$$\frac{30}{q(q^2-1)(q-2)(q-3)}, \frac{30}{q(q^2-1)(q+2)(q+3)},$$
$$\frac{20}{q(q^2-1)(q^2-4)}, \frac{24}{q^2(q^2-1)(q-2)}, \frac{24}{q^2(q^2-1)(q+2)})^t$$
$$= \frac{1}{q^2(q^2-1)(q^2-4)(q^2-9)(q^2-16)}(q^5-20q^3+78q, q^4-14q^2+24,$$
$$2q^3-18q, 5q^2-24, 14q, q^3-2q, 2q^2+24)^t$$

For the cases $k = 6, 7, 8, 9$ treated below, we will simply give the results without any explanations. For convenience, $f^{(-1)}(\mathbf{i})$ will be written as a column vector. We adopt the same ordering of conjugacy classes of permutations in $\mathcal{S}_k$ as [James, Kerber]. The $p(k) = \sharp I_k$ sequences of portraits $\mathbf{i} = (i_1, \ldots, i_k)$ of permutations are ordered using the increasing lexicographical order:

$$\mathbf{i} < \jmath \;\text{ if }\; i_m < j_m \;\text{ for }\; m = \max\{n \mid i_n \neq j_n\}$$

We denote $P_k(q)$ the lowest common multiple of polynomials $f_{\mathbf{m}}$ given by formula (2.15) of Theorem 4.

In the cases $k = 5, 6, 7, 8, 9$ computations have been done using *XMaple*. For further computations, the tables of characters of the symmetric group $\mathcal{S}_k$ up to $k = 18$ are available on the webpage of GAP99 (1999).

Case $k = 6, p(k) = 11$.

Ordering of classes(using the shortened notation of portraits):

$$(6), (4,1), (2,2), (0,3), (3,0,1), (1,1,1), (0,0,2), (2,0,0,1),$$
$$(0,1,0,1), (1,0,0,0,1), (0,0,0,0,0,1).$$

$P_6(q) = q^2(q^2-1)^2(q^2-4)(q^2-9)(q^2-16)(q^2-25)$.

$$(f^{(-1)}(\mathbf{i}))_{\mathbf{i}\in I_6} = \frac{1}{P_6(q)}\begin{pmatrix} q^8-41q^6+458q^4-1258q^2+240 \\ q^7-33q^5+254q^3-342q \\ q^6-19q^4+58q^2-160 \\ q^5+q^3+358q \\ 2q^6-51q^4+229q^2-60 \\ 2q^5-5q^3-117q \\ 4q^4+116q^2-360 \\ 5q^5-93q^3+208q \\ 5q^4+75q^2+40 \\ 14q^4-154q^2+140 \\ 42q^3-42q \end{pmatrix}$$

Case $k = 7, p(k) = 15$**.**

Ordering of classes:

$$(7),(5,1),(3,2),(1,3),(4,0,1),(2,1,1),(0,2,1),(1,0,2),(3,0,0,1),(1,1,0,1),\\(0,0,1,1),(2,0,0,0,1),(0,1,0,0,1),(1,0,0,0,0,1),(0,0,0,0,0,0,1).$$

$P_7(q) = q^2(q^2-1)^2(q^2-4)(q^2-9)(q^2-16)(q^2-25)(q^2-36).$

$$(f^{(-1)}(\mathbf{i}))_{\mathbf{i}\in I_7} = \frac{1}{P_7(q)}\begin{pmatrix} q^9 - 71q^7 + 1568q^5 - 11398q^3 + 15780q \\ q^8 - 61q^6 + 1058q^4 - 4958q^2 + 1440 \\ q^7 - 45q^5 + 488q^3 - 1284q \\ q^6 - 23q^4 + 382q^2 - 2880 \\ 2q^7 - 102q^5 + 1288q^3 - 2868q \\ 2q^6 - 52q^4 + 50q^2 \\ 2q^5 + 10q^3 + 1668q \\ 4q^5 + 32q^3 - 1716q \\ 5q^6 - 207q^4 + 1762q^2 - 720 \\ 5q^5 - 29q^3 - 816q \\ 10q^4 + 470q^2 - 2160 \\ 14q^5 - 434q^3 + 2100q \\ 14q^4 + 226q^2 + 1440 \\ 42q^4 - 762q^2 + 720 \\ 132q^3 - 132q \end{pmatrix}$$

Case $k = 8, p(k) = 22$**.**

Ordering of classes:

$$(7),(6,1),(4,2),(2,3),(0,4),(5,0,1),(3,1,1),(1,2,1),(2,0,2),(0,1,2),$$
$$(4,0,0,1),(2,1,0,1),(0,2,0,1),(1,0,1,1),(0,0,0,2),(3,0,0,0,1),(1,1,0,0,1),$$
$$(0,0,1,0,1),(2,0,0,0,0,1),(0,1,0,0,0,1),(1,0,0,0,0,0,1),(0,0,0,0,0,0,1).$$

$P_8(q) = q^2(q^2-1)^2(q^2-4)^2(q^2-9)(q^2-16)(q^2-25)(q^2-36)(q^2-49).$

$$(f^{(-1)}(\mathbf{i}))_{\mathbf{i}\in I_8} = \frac{1}{P_8(q)} \begin{pmatrix} q^{12} - 117q^{10} + 4792q^8 - 82644q^6 + 573772q^4 - 1337484q^2 + 771120 \\ q^{11} - 105q^9 + 3682q^7 - 50490q^5 + 247552q^3 - 331680q \\ q^{10} - 87q^8 + 2356q^6 - 23058q^4 + 76228q^2 - 5040 \\ q^9 - 63q^7 + 1374q^5 - 15712q^3 + 44640q \\ q^8 - 33q^6 + 1404q^4 + 23828q^2 + 65520 \\ 2q^{10} - 185q^8 + 5359q^6 - 54370q^4 + 167634q^2 - 128520 \\ 2q^9 - 131q^7 + 2197q^5 - 10084q^3 + 28176q \\ 2q^8 - 65q^6 + 1753q^4 - 34450q^2 + 83160 \\ 4q^8 - 116q^6 - 1814q^4 + 29646q^2 - 22680 \\ 4q^7 + 52q^5 + 8056q^3 - 63552q \\ 5q^9 - 401q^7 + 9322q^5 - 65494q^3 + 106968q \\ 5q^8 - 213q^6 + 960q^4 + 9328q^2 - 40320 \\ 5q^7 + 5q^5 + 6590q^3 + 3480q \\ 10q^7 + 127q^5 - 12605q^3 + 62868q \\ 25q^6 + 1975q^4 - 27200q^2 + 136080 \\ 14q^8 - 938q^6 + 16576q^4 - 76132q^2 + 65520 \\ 14q^7 - 250q^5 - 2404q^3 - 2400q \\ 28q^6 + 1645q^4 - 15533q^2 + 3780 \\ 42q^7 - 2154q^5 + 25008q^3 - 68256q \\ 42q^6 + 420q^4 + 12138q^2 - 57960 \\ 132q^6 - 4125q^4 + 17853q^2 - 13860 \\ 429q^5 - 2145q^3 + 1716q \end{pmatrix}$$

Case $k = 9, p(k) = 30$.

Ordering of classes:

$$(9),(7,1),(5,2),(3,3),(1,4),(6,0,1),(4,1,1),(2,2,1),(0,3,1),(3,0,2),(1,1,2),(0,0,3),$$
$$(5,0,0,1),(3,1,0,1),(1,2,0,1),(2,0,1,1),(0,1,1,1),(1,0,0,2),$$
$$(4,0,0,0,1),(2,1,0,0,1),(0,2,0,0,1),(1,0,1,0,1),(0,0,0,1,1),$$
$$(3,0,0,0,0,1),(1,1,0,0,0,1),(0,0,1,0,0,1),(2,0,0,0,0,0,1),(0,1,0,0,0,0,1),$$
$$(1,0,0,0,0,0,0,1),(0,0,0,0,0,0,0,0,1).$$

$P_9(q) = q^3(q^2-1)^2(q^2-4)^2(q^2-9)(q^2-16)(q^2-25)(q^2-36)(q^2-49)(q^2-64)$.

$(f^{(-1)}(\mathbf{i}))_{\mathbf{i}\in I_9} = \dfrac{1}{P_9(q)}\times$

$$\begin{pmatrix}
q^{14}-173q^{12}+11008q^{10}-317396q^8+4114204q^6-20599916q^4+29866032q^2-7902720\\
q^{13}-159q^{11}+8992q^9-218598q^7+2182372q^5-7063008q^3+5181120q\\
q^{12}-139q^{10}+6552q^8-124098q^6+874052q^4-1498928q^2-376320\\
q^{11}-113q^9+4284q^7-69868q^5+440000q^3+321216q\\
q^{10}-81q^8+2892q^6-22924q^4-534288q^2+1128960\\
2q^{12}-288q^{10}+14114q^8-275976q^6+1933844q^4-3526416q^2+1128960\\
2q^{11}-230q^9+8152q^7-99604q^5+406176q^3-495936q\\
2q^{10}-160q^8+4666q^6-94060q^4+707792q^2-376320\\
2q^9-78q^7+5388q^5+115648q^3+665280q\\
4q^{10}-304q^8+3458q^6+64354q^4-294312q^2\\
4q^9-128q^7+6604q^5-287712q^3+1052352q\\
8q^8+248q^6+48992q^4-825408q^2-1128960\\
5q^{11}-645q^9+26870q^7-405150q^5+1766600q^3-1538880q\\
5q^{10}-447q^8+10058q^6-37968q^4-75808q^2+376320\\
5q^9-219q^7+6926q^5-181656q^3+23744q\\
10q^9-316q^7-15766q^5+366856q^3-592704q\\
10q^8+230q^6+32240q^4-227360q^2+376320\\
25q^8+935q^6-89200q^4+736720q^2-376320\\
14q^{10}-1582q^8+54180q^6-595028q^4+1402576q^2-376320\\
14q^9-866q^7+7956q^5+47776q^3-266560q\\
14q^8-66q^6+26916q^4+288976q^2-376320\\
28q^8+483q^6-66843q^4+362012q^2+188160\\
70q^7+8050q^5-142520q^3+1041600q\\
42q^9-3966q^7+103620q^5-735744q^3+923328q\\
42q^8-1308q^6+618q^4-286632q^2\\
84q^7+5964q^5-41664q^3-538944q\\
132q^8-9438q^6+155298q^4-478632q^2\\
132q^7+572q^5+105248q^3-438592q\\
429q^7-18161q^5+81796q^3-64064q\\
1430q^6-7150q^4+5720q^2
\end{pmatrix}$$

10 Appendix: (1.3) is a probability

To show that, for $p > r-1$, (1.3) is a probability, we want to compute the integral

$$\int_\Omega (\det s)^{p-r} \exp(-\operatorname{trace} s)\lambda(ds). \tag{10.57}$$

Denote by T_r the real linear space of lower triangular complex (r,r) matrices t such that the diagonal elements of t are real, and denote by T_r^+ the open set $\{t \in T_r$ such that $t_{jj} > 0\}$. Denote by π the canonical projection on T_r of the space V of Hermitian (r,r) matrices, that is, if $t = \pi(s)$, then $t_{jk} = s_{jk}$ if $1 \leq k \leq j$ and $t_{jk} = 0$ if $j < k$. The Lebesgue measure $\lambda(dt)$ on T_r is the image of $\lambda(ds)$ by $s \mapsto t = \pi(s)$. To compute (10.57) we use the fact that $t \mapsto tt^*$ is a polynomial bijection from T_r^+ onto Ω (the Cholevsky decomposition for Hermitian matrices). Thus $t \mapsto \pi(tt^*)$ is a bijection between two open subsets of T_r, which are T_r^+ and $\pi(\Omega)$. Since we have a differentiable map $t \mapsto \pi(tt^*)$ from a part of the linear space T_r into itself, its differential is an endomorphism of T_r and it makes sense to compute its Jacobian. This Jacobian is given in the following lemma, which should be called Bartlett's lemma (see Goodman (1963)).

Lemma 10: Let $t = (t_{jk})$ be given in T_r^+. For simplicity, we denote $t_{jj} = t_j$. Then the determinant $D_r(t) = D_r$ of the linear endomorphism of T_r defined by $h \mapsto \pi(th^* + ht^*)$ is $D_r = 2^r \prod_{j=1}^{j=r} t_j^{2r-2j+1}$.

Proof . The proof is by induction on r. The result is obvious for $r = 1$. Suppose that it is true for r. We decompose T_{r+1} into the direct sum $T_r \oplus \mathbf{C^r} \oplus \mathbb{R}$ by writing arbitrary elements h and t of T_{r+1} according to blocks:

$$h = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \quad t = \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix}, \tag{10.58}$$

where a and α are in T_r, b and β are line vectors of $\mathbf{C^r}$ and c and $\gamma = t_{r+1}$ are real numbers. Thus we have

$$th^* + ht^* = \begin{bmatrix} a\alpha^* + \alpha a^* & \alpha b + a\beta \\ b\alpha^* + \beta a^* & 2c\gamma + b\beta^* + \beta b^* \end{bmatrix},$$

$$\pi(th^* + ht^*) = \begin{bmatrix} \pi(a\alpha^* + \alpha a^*) & 0 \\ b\alpha^* + \beta a^* & 2c\gamma + b\beta^* + \beta b^* \end{bmatrix}.$$

Introducing the three endomorphisms A, B and C of T_r, $\mathbf{C^r}$ and $\mathbb{R}$ respectively, which are defined by $a \mapsto \pi(a\alpha^* + \alpha a^*)$, $b \mapsto b\alpha^*$ and $c \mapsto 2c\gamma$, one can see that the initial endomorphism on $T_{r+1} = T_r \oplus \mathbf{C^r} \oplus \mathbb{R}$ defined by $h \mapsto \pi(ht^* + th^*)$ has the following (3,3) representative block matrix

$$\begin{pmatrix} \pi(a\alpha^* + \alpha a^*) \\ b\alpha^* + \beta a^* \\ b\beta^* + \beta b^* + 2c\gamma \end{pmatrix} = \begin{bmatrix} A & 0 & 0 \\ X & B & 0 \\ Y & Z & C \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where the X, Y, Z are linear maps of no interest here. Because this block matrix is triangular, $D_{r+1} = \det A \det B \det C$. Now by definition $\det A = D_r$ and one can see that $\det B = (\det \alpha^*)^2$. This comes from the rather subtle point that $\mathbf{C^r}$ is considered here as a real linear space of dimension $2r$. Indeed, if $m = a + ib$ is a complex (r, r) matrix, the determinant of the map from $\mathbf{C^r}$ into itself defined by $c \mapsto mc$ is $\det(a+ib)$ if $\mathbf{C^r}$ is considered as a complex linear space. But if $\mathbf{C^r}$ is considered as a real linear space, then a representative matrix of the endomorphism is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} I_r & iI_r \\ iI_r & I_r \end{pmatrix} \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix} \begin{pmatrix} I_r & iI_r \\ iI_r & I_r \end{pmatrix}^{-1},$$

whose determinant is $|\det(a+ib)|^2$. In the particular case $a + ib = \alpha^*$, the matrix α is triangular with a real diagonal and the result is indeed $(\det \alpha^*)^2$.

Using $\det \alpha^* = t_1 t_2 \ldots t_r$ and $\det C = 2t_{r+1}$ we get $D_{r+1} = D_r (t_1 t_2 \ldots t_r)^2 (2t_{r+1})$ and the induction follows.

Proposition 11. For $p > r - 1$, we have

$$\int_\Omega (\det s)^{p-r} \exp(-\operatorname{trace} s) \lambda(ds) = \pi^r \Gamma(p)\Gamma(p-1) \ldots \Gamma(p-r+1). \tag{10.59}$$

Proof. Using the notations above, in the left hand side of 10.59, we make the change of variable $s = tt^*$ and we use Bartlett's lemma to compute the Jacobian . The integral then becomes

$$\begin{aligned} &\int_{T_r^+} \exp(-\operatorname{trace} tt^*)(t_1 \ldots t_r)^{2p-2r} D_r(t) \lambda(dt) \\ = \;& 2^r \prod_{j=1}^r \int_0^\infty e^{-t_j^2} t_j^{2p-2j+1} dt_j \prod_{1 \le j < k \le r} \int_{\mathbf{C}} e^{\|t_{jk}\|^2} \lambda(t_{jk}) \\ = \;& \Gamma(p)\Gamma(p-1) \ldots \Gamma(p-r+1)\pi^r. \end{aligned}$$

We need one more lemma to complete the proof.

Lemma 12: Let a be a (r, r) complex matrix. Consider the endomorphism of V defined by $s \mapsto \varphi_a(s) = a^* s a$. Then $|\det \varphi_a| = |\det a|^{2r}$.

Proof: Assume first that $a = d$ is a diagonal matrix with real diagonal elements $(\lambda_1, \ldots \lambda_r)$. Then $\varphi_d(s) = (\lambda_j s_{jk} \lambda_k)_{1 \le j,k \le r}$. Thus any matrix with entries all equal to 0 except for the (j, j) entry is an eigenvector for the eigenvalue λ_j^2. The set of matrices with entries all equal to 0 except for the pair of entries (jk) and (kj) provides a two dimensional real linear space which is stable under φ_d and the restriction of φ_d to this space has determinant $(\lambda_j \lambda_k)^2$. Thus $\det \varphi_d = \prod_{j=1}^r (\lambda_j)^2 \prod_{1 \le j < k \le r} (\lambda_j \lambda_k)^2 = (\det d)^{2r}$.

Assume now that $a = u$ is a unitary matrix (thus $uu^* = u^* u = I_r$). It is useful to consider V as a Euclidean space with inner product defined by $\operatorname{trace}(ss_1) = \langle s, s_1 \rangle$. Then φ_u is an orthogonal endomorphism of V, since

$$\langle \varphi_u(s), \varphi_u(s) \rangle = \operatorname{trace}(u^* s u u^* s u) = \operatorname{trace} s^2 = \langle s, s \rangle.$$

This implies that $\det \varphi_u = \pm 1$ (we do not need the stronger result $\det \varphi_u = 1$ which can be shown by a connexity argument for the unitary group).

Finally, for the general case, we use a polar decomposition of a by writing $a = udv$ where u and v are unitary and d is diagonal, obtaining $\varphi_a = \varphi_v \circ \varphi_d \circ \varphi_u$ and

$$\det(\varphi_a) = \det(\varphi_v)\det(\varphi_d)\det(\varphi_u) = \pm(\det d)^{2r} = \pm(\det a)^{2r}.$$

Proposition 13. When $C(p,r)$ in (1.3) is such that

$$C(p,r)^{-1} = \pi^r\Gamma(p)\Gamma(p-1)\dots\Gamma(p-r+1),$$

then (1.3) is a probability.

Proof. Since $\mathrm{trace}\,\sigma^{-1/2}w\sigma^{-1/2} = \mathrm{trace}\,(\sigma^{-1}w)$, we are led to the change of variable $s = \sigma^{-1/2}w\sigma^{-1/2} = \varphi_{\sigma^{-1/2}}(w)$ in the integral

$$\int_\Omega \exp(-\mathrm{trace}\,(\sigma^{-1}w))(\det w)^{p-r}\lambda(dw).$$

With the help of the previous proposition and of the previous lemma we then obtain the result.

11 Appendix: Proof of Lemma 8

For $s = (s_{lk})_{1\le l,k\le r}$ in Ω, denote by $\Delta_j^*(s)$ the determinant of $(s_{lk})_{r-j+1\le l,k\le r}$ and

$$\Delta^*_{\mathbf{m}}(s) = \Delta_1^*(s)^{m_1-m_2}\Delta_2^*(s)^{m_2-m_3}\cdots\Delta_r^*(s)^{m_r}.$$

From [FK] VII.1.5, (ii), we know that

$$\Delta_{\mathbf{m}}(s^{-1}) = \Delta^*_{-\mathbf{m}^*}(s). \tag{11.60}$$

We also introduce the (r,r) matrix $u = (u_{ij})_{1\le i,j\le r}$ defined by $u_{i,r+1-i} = 1$ for all $i = 1,\dots,r$ and $u_{ij} = 0$ otherwise. Thus $usu = (s_{r+1-i,r+1-j})$, $\mathrm{trace}\,(\sigma^{-1}usu) = \mathrm{trace}\,((u\sigma u)^{-1}s)$ and

$$\Delta^*_{\mathbf{m}}(s) = \Delta_{\mathbf{m}}(usu) \tag{11.61}$$

Therefore, since $\Delta_r^*(s) = \Delta_r(s) = \det s$, we have

$$\begin{aligned}
&\mathbb{E}(\Delta_{\mathbf{m}}(S^{-1}))\\
&\overset{(1)}{=} \int_\Omega \Delta_1^*(s)^{-m_r+m_{r-1}}\Delta_2^*(s)^{-m_{r-1}+m_{r-2}}\cdots\Delta_r^*(s)^{-m_1}\Delta_r^*(s)^{p-r}e^{-\mathrm{trace}\,(\sigma^{-1}s)}\frac{\lambda(ds)}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(2)}{=} \int_\Omega \Delta_1^*(s)^{p-m_r-(p-m_{r-1})}\Delta_2^*(s)^{p-m_{r-1}-(p-m_{r-2})}\cdots\Delta_r^*(s)^{p-m_1}\Delta_r^*(s)^{-r}e^{-\mathrm{trace}\,(\sigma^{-1}s)}\frac{\lambda(ds)}{(\det\sigma)^p\Gamma_\Omega(p)}\\
&\overset{(3)}{=} \int_\Omega \Delta^*_{p-m^*}(s)(\det s)^{-r}\frac{\lambda(ds)}{(\det\sigma)^p\Gamma_\Omega(p)}
\end{aligned}$$

$$\overset{(4)}{=} \int_\Omega \Delta_{p-m^*}(s')(\det s')^{-r} e^{-\mathrm{trace}((u\sigma u)^{-1}s')} \frac{\lambda(ds')}{(\det\sigma)^p \Gamma_\Omega(p)}$$

$$\overset{(5)}{=} \Delta_{p-m^*}(u\sigma u) \frac{\Gamma(p-m^*)}{(\det\sigma)^p\Gamma_\Omega(p)}$$

$$\overset{(6)}{=} \Delta^*_{p-m^*}(\sigma) \frac{\Gamma(p-m^*)}{(\det\sigma)^p\Gamma_\Omega(p)}$$

$$\overset{(7)}{=} (\det\sigma)^p \Delta^*_{-m^*}(\sigma) \frac{\Gamma(p-m^*)}{(\det\sigma)^p\Gamma_\Omega(p)}$$

$$\overset{(8)}{=} \frac{\Gamma(p-m^*)}{\Gamma_\Omega(p)} \Delta_{\mathbf{m}}(\sigma^{-1})$$

Equalities (1), (2) and (7) are due to definitions, equalities (3) and (8) are due to (11.60), equality (4) is obtained by the change of variables $s' = usu$ and Lemma 12 to insure that $\lambda(ds) = \lambda(ds')$, (5) is true by [FK] VII.1.2 while equality (6) is due to (11.61). The proof of the lemma is thus complete.

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