Some convexity results for the Cartan decomposition

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Abstract

In this paper, we consider the set $\mathcal{S} = a(e^X\,K\,e^Y)$ where $a(g)$ is the abelian part in the Cartan decomposition of g. This is exactly the support of the measure intervening in the product formula for the spherical functions on symmetric spaces of noncompact type. We give a simple description of that support in the case of $\mathbf{SL}(3,\mathbf{F})$ where $\mathbf{F}=\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. In particular, we show that $\mathcal{S}$ is convex.

We also give an application of our result to the description of singular values of a product of two arbitrary matrices with prescribed singular values.

1 Introduction

Let G be a semisimple noncompact connected Lie group with finite center and K a maximal compact subgroup of G and $X = G/K$ the corresponding Riemannian symmetric space of noncompact type. We have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and we choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. In what follows, Σ corresponds to the root system of $\mathfrak{g}$ and Σ^+ to the positive roots. This implies that we have chosen a set of simple positive roots $\alpha_1, \ldots, \alpha_r$ where $r = \dim\mathfrak{a}$ is the rank of the symmetric space. We have the root space decomposition $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$. Recall that $\mathfrak{k}$, the Lie algebra of K, can be described as

$$\mathfrak{k} = \operatorname{span}\{X_\alpha + \theta(X_\alpha)\colon\ X_\alpha \in \mathfrak{g}_\alpha,\ \alpha\in\Sigma^+\cup\{0\}\}$$

where θ is the Cartan automorphism. Let $\mathfrak{n} = \sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}} = \sum_{\alpha\in\Sigma^+}\mathfrak{g}_{-\alpha} = \sum_{\alpha\in\Sigma^+}\theta(\mathfrak{g}_\alpha)$. Denote the groups corresponding to the Lie algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ by A, N and $\bar{N}$ respectively. We have the Cartan decomposition $G = K\,A\,K$ and the Iwasawa decomposition $G = K\,A\,N$. Let $\mathfrak{a}^+ = \{H\in A\colon\ \alpha(H) > 0\ \forall\ \alpha\in\Sigma^+\}$ and $A^+ = \exp(\mathfrak{a}^+)$. In particular, for any $g\in G$, $g = k_1\,e^{a(g)}\,k_2$ where $a(g)\in\overline{\mathfrak{a}^+}$ is uniquely determined by g.

If λ is a complex-valued functional on $\mathfrak{a}$, the corresponding spherical function is

$$\phi_\lambda(e^H) = \int_K e^{(i\,\lambda-\rho)(\mathcal{H}(e^H\,k))}\,dk$$

where $g = k\,e^{\mathcal{H}(g)}\,n \in K\,A\,N$ and $\rho = (1/2)\sum_{\alpha\in\Sigma^+} m_\alpha\,\alpha$ (m_α denotes the multiplicity of the root α). A spherical function, like any K-biinvariant function, can also be considered as a K-invariant function on the Riemannian symmetric space of noncompact type $X = G/K$. Naturally, such a function is completely determined by its values on A (or on A^+). The books [5, 6] constitute a standard reference on these topics.

In [6, (32), page 480], Helgason shows that a Weyl-invariant measure $\mu_{X,Y}$ exists on the Lie algebra $\mathfrak{a}$ such that

$$\phi_\lambda(e^X)\,\phi_\lambda(e^Y) = \int_{\mathfrak{a}} \phi_\lambda(e^H)\,d\mu_{X,Y}(H)$$

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(unlike us, Helgason states his results at the group level).

It is known [6] that

$$\phi_\lambda(e^X)\,\phi_\lambda(e^Y) = \int_K \phi_\lambda(e^X\,k\,e^Y)\,dk.$$

The measure $\mu_{X,Y}$ satisfies then

$$\int_K f(a(e^X\,k\,e^Y))\,dk = \int_{\mathfrak{a}} f(e^H)\,d\mu_{X,Y}(H) \tag{1}$$

for all continuous functions f which are biinvariant under the action of K.

The support of the measure $\mu_{X,Y}$ is included in $C(X)+C(Y)$ where $C(H)$ is the convex hull of the orbit of H under the action of the Weyl group W.

The natural question whether the measure $\mu_{X,Y}$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{a}$, *i.e.* whether we have a "product formula"

$$\phi_\lambda(e^X)\,\phi_\lambda(e^Y) = \int_{\mathfrak{a}} \phi_\lambda(e^H)\,k(H,X,Y)\,dH \tag{2}$$

was answered positively when $X \in \mathfrak{a}^+$ or $Y \in \mathfrak{a}^+$ by Flensted-Jensen and Koornwinder ([1, 7]) in the rank one case and by the authors ([3]) in the general case. Very little is known about the properties of this density, in particular its support, except the rank one case and the complex case.

In rank 1 case the support of $\mu_{X,Y}$ was computed by Flensted-Jensen and Koornwinder ([1, 7]). This is the union of the segment $[|X-Y|, X+Y]$ and its reflection with respect to 0 $(X, Y \geq 0)$.

In [2], we found the support of $\mu_{X,Y}$ in the case of $\mathbf{SL}(3,\mathbf{C})$:

$$\operatorname{supp}(\mu_{X,Y}) \cap \overline{\mathfrak{a}^+} = (C(X)+Y) \cap (X + C(Y)) \cap \{H\colon H_3 \leq X_2 + Y_2 \leq H_1\}.$$

This was obtained by using an explicit expression for the density of the measure $\mu_{X,Y}$.

The objective of this paper is to study the support of the W-invariant measure $\mu_{X,Y}$ or, equivalently, the intersection of $\operatorname{supp}\mu_{X,Y}$ with the closed positive Weyl chamber $\overline{\mathfrak{a}^+}$.

It is clear by (1) that the support of $\mu_{X,Y}$ is included in the union of the translates of $a(e^X\,K\,e^Y)$ under the action of the Weyl group. These sets are in fact equal. We recall first a result of [3].

Lemma 1 *Suppose X, $Y \in \mathfrak{a}^+$. Let $F\colon K \mapsto \overline{\mathfrak{a}^+}$ defined by $F(k) = a(e^X\,k\,e^Y)$. Then there exists a closed set $C \subset K$ of Haar measure 0 such that F is analytic and dF is surjective on $K \setminus C$.*

Proof: This is a consequence of [3, Lemma 8]. ∎

Theorem 2 *Suppose X, $Y \in \mathfrak{a}^+$. Then $supp(\mu_{X,Y}) \cap \overline{\mathfrak{a}^+} = a(e^X\,K\,e^Y)$. Consequently, supp $(\mu_{X,Y}) = W \cdot a(e^X\,K\,e^Y)$.*

Proof: Let F and C be as in Lemma 1. Suppose that $H \in a(e^X\,K\,e^Y)$ and let U be any neighbourhood of H in $\overline{\mathfrak{a}^+}$. Then $V = F^{-1}(U)$ is an open set which cannot be included in C so there is a nonempty open set $V_0 \subset V$ such that F is analytic and dF is surjective on V_0. If we refer to (1), it follows easily that $H \in \mathrm{supp}\,(\mu_{X,Y})$. ∎

In this paper, we compute $\mathcal{S} = a(e^X\, K\, e^Y)$ (and therefore, by Theorem 2, the support of $\mu_{X,Y}$), for some non-exceptional rank 2 Riemannian symmetric spaces. We aim to gain a better comprehension of harmonic analysis on these spaces and we believe that our results provide useful indications for the general symmetric space case (see [4, 9]). In Section 2, we give a simple geometric description of the set $a(e^X\, K\, e^Y)$ for $G = \mathbf{SL}(3,\mathbf{F})$ where $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ (the quaternions).

In all these cases $\mathcal{S} = a(e^X\, K\, e^Y)$ is the convex hull of the set I described in Definition 9. Our result for the Cartan decomposition is a counterpart of the Kostant convexity theorem for the Iwasawa decomposition.

We end with an application of our result which gives necessary and sufficient inequalities on the singular values of the product of two complex (or real) 3×3 matrices. Only some necessary conditions (Gelfand-Naimark inequality, see [8]) were known before.

2 The set $\mathcal{S} = a(e^X\, K\, e^Y)$ on $\mathbf{SL}(3,\mathbf{F})$

Definition 3 *Let $W = M'/M$ be the Weyl group ($M' \subset K$ is the normalizer of $\mathfrak{a}$ in K while $M \subset K$ is its centralizer). If α is a root then $s_\alpha \in W$ is the reflection with respect to the hyperplane $\{\alpha = 0\}$.*

When appropriate we will not distinguish between $w \in W$ and $w \in M' \subset K$. On the other hand, to denote the action of w on $X \in \mathfrak{a}$, we will write $w\,X$. We then have $e^{w\,X} = w\, e^X\, w^{-1}$ ([5, VII, Propoposition 2.2]).

We will write $T = W\,X + W\,Y = \{w_1\,X + w_2\,Y\colon\ w_1, w_2 \in W\}$ and $T_0 = T \cap \overline{\mathfrak{a}^+}$.

We define $\mathcal{S} = a(e^X\, K\, e^Y)$.

If α is a nonzero root then K_α will denote the subgroup of K with Lie algebra $\mathfrak{k}_\alpha = \{X_\alpha + \theta(X_\alpha)\colon\ X_\alpha \in \mathfrak{g}_\alpha\}$.

Lemma 4 *If $w_1\,X + w_2\,Y \in \overline{\mathfrak{a}}^+$ then $F(w_1^{-1} w_2) = w_1\,X + w_2\,Y$. In particular we have $T_0 \subset \mathcal{S}$.*

Proof: If $w_1\,X + w_2\,Y \in T_0$ then $a(e^X\, w_1^{-1}\, w_2\, e^Y) = a(e^{w_1\,X} e^{w_2\,Y}) = w_1\,X + w_2\,Y$. ∎

Denote by $B(\cdot,\cdot)$ the Killing form on $\mathfrak{g}$. Define $A_\alpha \in \mathfrak{a}$ by $B(H, A_\alpha) = \alpha(H)$ for all $H \in \mathfrak{a}$. Denote $A'_\alpha = A_\alpha/\alpha(A_\alpha)$ so that $\alpha(A'_\alpha) = 1$.

Lemma 5 *Let α be a nonzero root. Write $Z_\alpha = X_\alpha + \theta(X_\alpha)$ for $X_\alpha \neq 0 \in \mathfrak{g}_\alpha$. Then we have*

$$e^{x\,A'_\alpha}\, e^{t\,Z_\alpha}\, e^{y\,A'_\alpha} = k_1(t)\, e^{s\,A'_\alpha}\, k_2(t), \qquad t \in \mathbf{R}$$

with $k_1(t)$, $k_2(t) \in K_\alpha$ and s taking all values from the closed interval between $|x-y|$ and $|x+y|$.

Proof: This is a rank-one reduction (the algebra generated by A_α, X_α and $\theta(X_\alpha)$ being isomorphic to $\mathfrak{sl}(2,\mathbf{R})$). We use then [1, page 256]. ∎

Definition 6 *Let $C_1, \ldots, C_{|W|} \subset \mathfrak{a}$ be pairwise disjoint open Weyl chambers, $C_1 = \mathfrak{a}^+$. For each i, there exists a unique element $w_i \in W$ such that $w_i(C_i) = \mathfrak{a}^+$ ([5, Ch.VII]). We define the projection π of $\mathfrak{a}$ to $\overline{\mathfrak{a}^+}$ by*

$$\pi(H) = w_i(H) \quad \text{when } H \in \overline{C_i}.$$

Note that the definition still holds when $H \in \overline{C_i} \cap \overline{C_j}$ ([5, Ch.VII]). When $X \in \mathfrak{a}$, we have $a(e^X) = \pi(X)$.

As in [6], let us denote by $\{H_1, H_2\}$ the closed segment connecting $H_1 \in \mathfrak{a}$ and $H_2 \in \mathfrak{a}$. When it does not lead to a misunderstanding, we will also write $\{H_1, H_2\} = H_1\, H_2$.

Proposition 7 *Suppose $w_1 X + w_2 Y \in T_0$ and let α be a positive root. Then the image I_{α,w_1,w_2} of $t \mapsto a(e^{w_1 X}\, e^{t Z_\alpha}\, e^{w_2 Y})$ is the projection π of:*

1. *the segment $\mathcal{I} = \{w_1 X + w_2 Y\ ,\ w_1 X + s_\alpha(w_2 Y)\}$ if $\alpha(w_1 X) \geq \alpha(w_2 Y)$,*
2. *the segment $\mathcal{I} = \{w_1 X + w_2 Y\ ,\ s_\alpha(w_1 X) + w_2 Y\}$ if $\alpha(w_1 X) \leq \alpha(w_2 Y)$.*

Proof: Note that $\alpha(w_1 X + w_2 Y) \geq 0$ since $w_1 X + w_2 Y \in \overline{\mathfrak{a}^+}$. Now using Lemma 5 and the fact that K_α centralizes the elements of $\mathfrak{a}$ which are in the hyperplane $\alpha = 0$, we have

$$\begin{aligned}
a(e^{w_1 X}\, e^{t Z_\alpha}\, e^{w_2 Y}) &= a(e^{w_1 X - \alpha(w_1 X) A'_\alpha}\, e^{\alpha(w_1 X) A'_\alpha}\, e^{t Z_\alpha}\, e^{\alpha(w_2 Y) A'_\alpha}\, e^{w_2 Y - \alpha(w_2 Y) A'_\alpha}) \\
&= a(e^{w_1 X - \alpha(w_1 X) A'_\alpha} k_1(t)\, e^{s A'_\alpha} k_2(t)\, e^{w_2 Y - \alpha(w_2 Y) A'_\alpha}) \\
&= a(k_1(t)\, e^{w_1 X - \alpha(w_1 X) A'_\alpha} e^{s A'_\alpha} e^{w_2 Y - \alpha(w_2 Y) A'_\alpha}\, k_2(t)) \\
&= a(e^{w_1 X - \alpha(w_1 X) A'_\alpha + w_2 Y - \alpha(w_2 Y) A'_\alpha + s A'_\alpha})
\end{aligned}$$

with s between $\alpha(w_1 X) + \alpha(w_2 Y)$ and $|\alpha(w_1 X) - \alpha(w_2 Y)|$. ∎

Remark 8 *The image I_{α,w_1,w_2} belongs to $\mathcal{S}$ since $a(e^{w_1 X}\, e^{t Z_\alpha}\, e^{w_2 Y}) = a(e^X\, w_1^{-1}\, e^{t Z_\alpha}\, w_2\, e^Y)$. It does not depend on the choice of $0 \neq Z_\alpha \in \mathfrak{k}_\alpha$. The set I_{α,w_1,w_2} is a segment or is a connected finite union of segments, the original segment starting at $w_1 X + w_2 Y$ being reflected each time it meets a wall of $\mathfrak{a}^+$.*

Definition 9 *Let $I \subset \mathcal{S}$ be defined as*

$$I = \bigcup_{\alpha > 0, w_1, w_2 \in W} I_{\alpha,w_1,w_2},$$

the network of all segments composing the images I_{α,w_1,w_2} created according to Proposition 7.

Definition 10 *Let $K_0 = \cup_{\alpha>0} W\, K_\alpha\, W$.*

Remark 11 *Note that when the Weyl group acts transitively over the roots (which is true in the case of the root system A_n) then $W\, K_\alpha\, W$ does not depend on the choice of nonzero root,* i.e. *$K_0 = W\, K_\alpha\, W$ for any fixed root α. Actually, when α and β are two different roots and $Ad(w)\,\alpha = \beta$, then $Ad(w) Z_\alpha \in \mathfrak{k}_\beta$.*

Note also that $a(e^X\, k\, e^Y) \in I$ if $k \in K_0$. Indeed, $a(e^{w_1 X}\, e^{t Z_\alpha}\, e^{w_2 Y}) = a(e^X\, w_1^{-1}\, e^{t Z_\alpha}\, w_2\, e^Y)$.

A "typical" example of the network I is given in Figure 1.

If $\mathcal{I} = \{H_1, H_2\}$ is a closed segment in $\mathfrak{a}$ we denote by $\mathcal{I}^\circ = \mathcal{I} \setminus \{H_1\, H_2\}$, the segment $\mathcal{I}$ deprived of its endpoints. We extend the same notation for a π-projection of a segment:

$$\pi(\mathcal{I})^\circ := \pi(\mathcal{I}^\circ).$$

The projections of H_1 and H_2 (the vertices of $\mathcal{I}$) by π will be called the vertices of $\pi(\mathcal{I})$. Given $Z \in \mathfrak{k}_\alpha$, we denote by w_{Z,w_1,w_2} the Weyl group element such that

$$a(e^{w_1 X}\, e^Z\, e^{w_2 Y}) = \pi(H) = w_{Z,w_1,w_2} H$$

with $H \in \mathcal{I}$ as in Proposition 7.

Figure 1: The network I inside $\overline{\mathfrak{a}^+}$ (the points of T_0 are shown as $\circ$)

Lemma 12 *Let $0 \neq X_\alpha \in \mathfrak{g}_\alpha$, $H_\alpha = [X_\alpha, \theta(X_\alpha)]$ and $Z_\alpha = X_\alpha + \theta(X_\alpha)$. Suppose that $w_1\, X + w_2\, Y \in \mathfrak{a}^+$.*

1. *If Z_α is such that $a(e^{w_1 X}\, e^{Z_\alpha}\, e^{w_2\, Y}) \in I^\circ_{\alpha, w_1, w_2} \cap \mathfrak{a}^+$ then for $|t|$ small enough,* $a(e^{w_1\, X}\, e^{t\, Z_\alpha}\, e^{Z_\alpha}\, e^{w_2\, Y}) = a(e^{w_1\, X}\, e^{Z_\alpha}\, e^{w_2\, Y}) + \gamma_1\, t\, w_{Z_\alpha, w_1, w_2}\, H_\alpha + O(t^2)$ *with $\gamma_1 \neq 0$.*

2. *For $|t|$ small enough, $a(e^{w_1\, X}\, e^{t\, Z_\alpha}\, e^{w_2\, Y}) = w_1\, X + w_2\, Y + \gamma_2\, t^2\, H_\alpha + O(t^3)$ with $\gamma_2 \neq 0$.*

Proof: Similarly as in the proof of [3, Corollary 14] or in the proof of Proposition 7 above, we may write $w_1 X = a\, H_\alpha + X'$ and $w_2\, Y = b\, H_\alpha + Y'$ where $\alpha(X') = \alpha(Y') = 0$. The fact that $\alpha(w_1\, X) \neq 0$ and $\alpha(w_2\, Y) \neq 0$ implies that $a \neq 0$ and $b \neq 0$. Using again the fact that $e^{X'}$ and $e^{Y'}$ commute with elements of K_α, we see that in order to prove the 1. and 2. of the lemma, it is enough to compute the limited Taylor expansion of $e^{a\, H_\alpha}\, e^{t\, Z_\alpha}\, e^{b\, H_\alpha}$ at $t = 1$ and at $t = 0$ respectively.

The Lie algebra generated by H_α, X_α and $\theta(X_\alpha)$ is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. Indeed, X_α corresponds to the matrix $c\, E_{1,2}$ with $c = \sqrt{-\alpha(H_\alpha)/2} \neq 0$, $\theta(X_\alpha)$ to $-c\, E_{2,1}$ and H_α to $-c^2\, (E_{1,1} - E_{2,2})$ (see [3, Proposition 13]). Note that the constant c may take any strictly positive value when Z_α (and therefore X_α) varies.

We now work in $\mathbf{SL}(2, \mathbf{R})$. Let $e^{a\, H_\alpha}\, e^{t\, Z_\alpha}\, e^{b\, H_\alpha}$ correspond to $g_t = k_1(t)\, e^{\mathrm{a}(t)\, c^2\, (E_{1,1} - E_{2,2})}\, k_2(t)$ (the Cartan decomposition in $\mathbf{SL}(2, \mathbf{R})$). We basically want the limited expansion of $\mathrm{a}(t)$ since

$$a(e^{w_1\, X}\, e^{(t+h)\, Z_\alpha}\, e^{w_2\, Y}) = a(e^{w_1\, X}\, e^{t\, Z_\alpha}\, e^{w_2\, Y}) + \mathrm{a}'(t)\, h\, w\, H_\alpha + \mathrm{a}''(t)\, h^2\, w\, H_\alpha + O(h^3)$$

where $w \in W$ comes from an eventual projection to $\overline{\mathfrak{a}^+}$.

Note that $\mathrm{a}(0) = |a + b| > 0$. We compute

$$\begin{aligned}
f(t) &:= \operatorname{tr} g_t\, g_t^T = 2\, \sin^2(c\, t)\, \cosh(2(a - b)c^2) + 2\, \cos^2(c\, t)\, \cosh(2(a + b)c^2), \\
f'(t) &= -4\, c\, \sin(2\, c\, t)\, \sinh(2\, a\, c^2)\, \sinh(2\, b\, c^2), \\
f''(t) &= -8\, c^2\, \cos(2\, ct)\, \sinh(2\, a\, c^2)\, \sinh(2\, b\, c^2)
\end{aligned}$$

and note that $f(t) = 2\, \cosh(2\, c^2\, \mathrm{a}(t))$. This means that $f'(t) = -4\, \sinh(2\, c^2\, \mathrm{a}(t))\, \mathrm{a}'(t)$ and therefore that $\mathrm{a}'(0) = 0$. Similarly, computing $f''(0)$ shows that $\mathrm{a}''(0) \neq 0$. This proves 2.

Now, $\mathrm{a}'(t) = 0$ implies $f'(t) = 0$. On the other hand, $f'(t) = 0$ if and only if $2\,\sin(c\,t)\,\cos(c\,t) = 0$. This implies that $f(t) = 2\,\cosh(2\,(a-b)\,c^2)$ *i.e.* $\mathrm{a}(t) = |a-b|$ or $f(t) = 2\,\cosh(2\,(a+b)\,c^2)$ *i.e.* $\mathrm{a}(t) = |a+b|$.

It follows that the function $\mathrm{a}(t)$ has only two extremal values $|a-b|$ and $|a+b|$. We not only prove directly the result of [1] on the form of $\mathcal{S}$ in rank 1 case but we show that the values of the function $a(t)$ run over the whole projected segment I_{α,w_1,w_2} from one vertex to another, without any interior reflection points. ∎

Corollary 13 *Suppose $H_0 \in T_0 \cap \mathfrak{a}^+$. Then any point in any sector of less than π with vertex H_0 and edges in I which is close enough to H_0 belongs to $\mathcal{S}$ (refer to Figure 1).*

Proof: We have $H_0 = a(e^{w_1 X}\,e^{w_2 Y})$ and consider $g(t_1,t_2) = a(e^{w_1 X}\,e^{t_1 Z_{\alpha_1}}\,e^{t_2 Z_{\alpha_2}}\,e^{w_2 Y})$ where α_1 and α_2 correspond to the sides of the sector. We have $g(t_1,t_2) = H_0 + \psi_1\,H_{\alpha_1}\,t_1^2 + \psi_2\,H_{\alpha_2}\,t_2^2 + O(\|t\|^3)$ with $\psi_1 \neq 0$ and $\psi_2 \neq 0$. The absence of a mixed term in $t_1\,t_2$ follows from the invariance $g(\pm t_1, \pm t_2) = g(t_1,t_2)$ as shown in [3, Lemma 17]. ∎

We will say that R is an **intersection point** in I if $R \in I^\circ_{\gamma,w_1,w_2} \cap I^\circ_{\gamma',w_1',w_2'}$ with $I^\circ_{\gamma,w_1,w_2} \neq I^\circ_{\gamma',w_1',w_2'}$ (recall that I°_{γ,w_1,w_2} is equal to I_{γ,w_1,w_2} without its extremities). In particular, $R \notin T_0$.

Lemma 14 *Given any X, $Y \in \mathfrak{a}^+$, the set $X + W\,Y$ intersect at most 3 Weyl chambers or the set $W\,X + Y$ intersect at most 3 Weyl chambers (the Weyl chambers $\mathfrak{a}^+$, $s_\alpha\,\mathfrak{a}^+$ and $s_\beta\,\mathfrak{a}^+$).*

Proof: Let A_1, ..., A_6 be the points of $X + W\,Y$ starting from $A_1 = X + Y$ and going clockwise.

Then $X + W\,Y$ intersect more than 3 Weyl chambers if and only if A_4 is below $\alpha + \beta = 0$ *i.e.* if and only if $(\alpha+\beta)(A_4) < 0$. Noting that $A_4 = [X_1, X_2, X_3] + [Y_3, Y_2, Y_1]$, $(\alpha+\beta)(A_4) < 0$ means that $X_1 + Y_3 < X_3 + Y_1$ *i.e.* $X_1 - X_3 < Y_1 - Y_3$.

Applying the same reasoning to $Y + W\,X$, we can conclude that the corresponding vertex is above $\alpha + \beta = 0$ *i.e.* that $Y + W\,X$ intersects at most 3 Weyl chambers. ∎

Let the points A_1, ..., A_6 be as in the proof of Lemma 14 in the case $X_1 - X_3 > Y_1 - Y_3$ and let it be the elements of $Y + W\,X$ otherwise.

Let $C(X,Y) = \begin{cases} \partial(X + C(Y)) \cup A_1\,A_4 \cup A_2\,A_5 \cup A_3\,A_6 & \text{if } X_1 - X_3 > Y_1 - Y_3, \\ \partial(Y + C(X)) \cup A_1\,A_4 \cup A_2\,A_5 \cup A_3\,A_6 & \text{otherwise} \end{cases}$.

In the following Proposition we explain in which way, by a simple geometric transformation, it is possible to get the network I from the set $C(X,Y)$.

Proposition 15 *Let $C'(X,Y) = \pi(C(X,Y))$ and $D = \pi(\{A_i\}_{i=1,\ldots,6})$ (the projection π was defined in the Definition 6). Remove any segment of $C'(X,Y)$ joining a point of D and the wall $\{\alpha = 0\}$ or $\{\beta = 0\}$. Then the image of $C(X,Y)$ by all these transformations is equal to I and $T_0 = D$.*

Remark 16 *Suppose that the set $C(X,Y)$ intersects the wall $\{\alpha = 0\}$ of $\mathfrak{a}^+$. The projection π intervening in the Proposition 15 consists in "folding" symmetrically along $\alpha = 0$ the portion which is in $\{\alpha < 0\}$ into $\overline{\mathfrak{a}^+}$. The resulting set is $(C(X,Y) \cup s_\alpha\,C(X,Y)) \cap \overline{\mathfrak{a}^+}$.*

Apply the analogous "folding" operation if the set $C(X,Y)$ intersects the wall $\{\beta = 0\}$ of $\mathfrak{a}^+$. We obtain in this way the set $C'(X,Y)$.

For the geometrical meaning of the Proposition 15 refer to Figure 2 (the vertical axis $\alpha = 0$ and the axis $\beta = 0$ are shown as doted lines, the points of T_0 are marked by $\circ$ and the point $A_1 = X + Y$ is always the upper right vertex).

Figure 2: $C(X,Y)$, $C(X,Y) \cup s_\alpha(C(X,Y) \cap \{\alpha < 0\})$, $C'(X,Y)$ and I

Proof: We can assume without loss of generality that $X_1 - X_3 > Y_1 - Y_3$. Call the constructed set J. If we consider Proposition 7, it is clear that $J \subset I$.

Let $H = w_1 X + w_2 Y \in T_0 \in \overline{\mathfrak{a}^+}$. We only need to show that H is one of the A_i's or a reflection of one of the A_i's by s_α or s_β. In other words, we have to show that $w_1 \in \{\mathrm{id}, s_\alpha, s_\beta\}$. The possibilities to eliminate are $w_1 = s_\alpha\, s_\beta$, $w_1 = s_\beta\, s_\alpha$ and $w_1 = s_\alpha\, s_\beta\, s_\alpha$. In the first and third case, the first entry of H would be $H_1 = X_3 + Y_i$. There is one entry of H of the form $X_1 + Y_j$. Since $H \in \overline{\mathfrak{a}^+}$, we have $0 \leq X_3 + Y_i - X_1 - Y_j = Y_i - Y_j - (X_1 - X_3) \leq Y_1 - Y_3 - (X_1 - X_3) < 0$ which is absurd. In the second case, the last component of H is X_1. A similar reasoning also leads to a contradiction. ∎

Lemma 17 *All the intersection points belong to* $\pi(A_1\, A_4)$, $\pi(A_2\, A_5)$ *or* $\pi(A_3\, A_6)$.

Proof: We can assume without loss of generality that $X_1 - X_3 > Y_1 - Y_3$, that is, that $X + C(Y)$ intersects at most 3 Weyl chambers and therefore that $C(X,Y) = \partial(X + C(Y)) \cup A_1\, A_4 \cup A_2\, A_5 \cup A_3\, A_6$. We distinguish the following cases:

(1) $X + C(Y) \subset \bar{\mathfrak{a}}^+$.

(2) $X + C(Y)$ intersects $\mathfrak{a}^+$ and $\{\alpha < 0\}$ but not $\{\beta < 0\}$.

(2') $X + C(Y)$ intersects $\mathfrak{a}^+$ and $\{\beta < 0\}$ but not $\{\alpha < 0\}$.

(3) $X + C(Y)$ intersects $\mathfrak{a}^+$, $\{\alpha < 0\}$ and $\{\beta < 0\}$.

The Lemma is clear in case (1). The intersection points are then the vertices of the central triangle of $C(X,Y)$. They are given by $R_1 = [X_1 + Y_2, X_2 + Y_2, *]$, $R_2 = [*, X_2 + Y_2, X_3 + Y_2]$ and $R_3 = [X_1 + Y_2, *, X_3 + Y_2]$ (the coordinate $*$ is determined by the fact that the trace is zero) Refer to Figure 3 (on this and on the following figures the set I is drawn in **boldface**).

In the case (2), we verify that only one of the points R_i, say R_0, may not belong to $\mathfrak{a}^+$. In fact R_0 is equal to R_2 or R_3. By the construction of I given in the Proposition 15, it follows that $\pi(R_0)$ is not an intersection point.

The new intersection points may only appear when the reflected part $s_\alpha(C(X,Y) \cap \{\alpha < 0\})$ intersects some segments of $C(X,Y) \cap \mathfrak{a}^+$ different from the exterior edges of $C(X,Y)$. These segments are among $\pi(A_1\, A_4)$, $\pi(A_2\, A_5)$ and $\pi(A_3\, A_6)$. The case (2') is similar by symmetry. Refer to Figure 4.

Figure 3: Case (1)

Figure 4: Case (2)

Case (3) boils down to considering separately cases (2) and (2') because the parts of $C(X,Y)$ which are reflected by s_α and s_β while constructing I are disjoint. This follows from the fact that the hypothesis $X_1 - X_3 > Y_1 - Y_3$ is equivalent to the inequality $X_1 + Y_3 > X_3 + Y_1$ between the second entries of $A_5' = \pi(A_5) = s_\alpha(A_5)$ and $A_3' = \pi(A_3) = s_\beta(A_5)$, which in turn means that the point A_5' is situated above and the point A_3' below a line $H_2 = c$. Hence this line separates the sets $\pi(C(X,Y) \cap \{\alpha < 0\})$ and $\pi(C(X,Y) \cap \{\beta < 0\})$. Refer to Figure 5. ∎

Figure 5: Case (3)

Lemma 18 and Theorem 19 will only apply in the case of $\mathbf{F} = \mathbf{R}$. As we will see in Proposition 23, this will not be an impediment in proving our main result for $\mathbf{SL}(3, \mathbf{F})$.

Lemma 18 *We are now restricting ourselves to the case* $\mathbf{F} = \mathbf{R}$. *Let* $k = e^{\theta_1 Z_{\alpha+\beta}} e^{\theta Z_\alpha} e^{\theta_2 Z_{\alpha+\beta}}$. *Then* $k \in K_0$ *if and only if one of the following is true.*

1. $\sin\theta = 0$.

2. $\sin\theta = \pm 1$ *and* $\sin\theta_1 = 0$ *or* ± 1.

3. $\sin\theta = \pm 1$ *and* $\sin\theta_2 = 0$ *or* ± 1.

4. $\sin\theta_1 = \pm 1$ *or 0 and* $\sin\theta_2 = \pm 1$ *or 0.*

Proof: It is easy to check that these conditions give elements of K_0.

To prove the inverse statement, let us assume that $k \in K_0$ and $\sin\theta \neq 0$. Writing the product $k = e^{\theta_1 Z_{\alpha+\beta}} e^{\theta X_\alpha} e^{\theta_2 Z_{\alpha+\beta}}$ explicitly, we have

$$k = \begin{bmatrix} \cos\theta_1 \cos\theta \cos\theta_2 - \sin\theta_1 \sin\theta_2 & -\cos\theta_1 \sin\theta & -\cos\theta_1 \cos\theta \sin\theta_2 - \sin\theta_1 \cos\theta_2 \\ \sin\theta \cos\theta_2 & \cos\theta & -\sin\theta \sin\theta_2 \\ \sin\theta_1 \cos\theta \cos\theta_2 + \cos\theta_1 \sin\theta_2 & -\sin\theta_1 \sin\theta & -\sin\theta_1 \cos\theta \sin\theta_2 + \cos\theta_1 \cos\theta_2 \end{bmatrix}.$$

The elements of K_0 are obtained from the elements of K_α, K_β and $K_{\alpha+\beta}$ by permutations of rows and columns. It follows that any line and any column of k contains at least one zero. Also, there is one line and one column with elements $\{\pm 1, 0, 0\}$. It follows that there is at least a 0 on the second row.

Suppose that, say, $\cos\theta = 0$. Considering different possible situations of ± 1 we deduce that there is another 0 on the same column or row. This forces $\cos\theta_1 = 0$ or $\cos\theta_2 = 0$ or $\sin\theta_1 = 0$ or $\sin\theta_2 = 0$.

The other cases are handled in much the same way. ∎

The technical condition in the following theorem will be overcome in the final theorem on the form of S.

Theorem 19 *We are restricting ourselves to the case* $\mathbf{F} = \mathbf{R}$. *Suppose that* $X_i + Y_j \neq X_p + Y_q$ *whenever* $(i,j) \neq (p,q)$. *If* $R \in \mathfrak{a}^+$ *is an intersection point in* I *then* $R \in F(K \setminus K_0)$.

Proof: According to Lemma 17, we have 3 cases. We will assume first that $R \in \pi(A_1 A_4)$.

Let $F(\theta_1, \theta, \theta_2) = a(e^X e^{\theta_1 Z_{\alpha+\beta}} e^{\theta Z_\alpha} e^{\theta_2 Z_{\alpha+\beta}} e^Y) \in I$. Then $F(\theta_1,\theta,\theta_2) \in \pi(A_1 A_4)$ if and only if $e^{2X_2+2Y_2}$ is an eigenvalue of the matrix $e^X k e^{2Y} k^T e^X$ where $k = e^{\theta_1 Z_{\alpha+\beta}} e^{\theta Z_\alpha} e^{\theta_2 Z_{\alpha+\beta}}$. Setting $\det(e^X k e^{2Y} k^T e^X - e^{2X_2+2Y_2} I) = 0$, we see after tedious elementary computations (which may be very quickly done using for example Maple) that $F(\theta_1,\theta,\theta_2)$ belongs to $\pi(A_1 A_4)$ if and only if

$$\sin^2\theta\,(f_0(X,Y) - f_1(X,Y)\sin^2\theta_1 - f_2(X,Y)\sin^2\theta_2) = 0$$

where

$$f_0 = \left(e^{2X_2+2Y_2} - e^{2X_3+2Y_3}\right)\left(e^{2Y_1} - e^{2Y_2}\right)\left(e^{2X_1} - e^{2X_2}\right), \tag{3}$$
$$f_1 = e^{2X_2}\left(e^{2Y_1} - e^{2Y_2}\right)\left(e^{2Y_2} - e^{2Y_3}\right)\left(e^{2X_1} - e^{2X_3}\right), \tag{4}$$
$$f_2 = e^{2Y_2}\left(e^{2Y_1} - e^{2Y_3}\right)\left(e^{2X_1} - e^{2X_2}\right)\left(e^{2X_2} - e^{2X_3}\right). \tag{5}$$

We will also use

$$f_1 - f_0 = \left(e^{2X_2} - e^{2X_3}\right)\left(e^{2Y_1} - e^{2Y_2}\right)\left(e^{2X_2+2Y_2} - e^{2X_1+2Y_3}\right), \tag{6}$$
$$f_2 - f_0 = \left(e^{2X_1} - e^{2X_2}\right)\left(e^{2Y_2} - e^{2Y_3}\right)\left(e^{2X_2+2Y_2} - e^{2X_3+2Y_1}\right), \tag{7}$$
$$f_1 + f_2 - f_0 = \left(e^{2X_2} - e^{2X_3}\right)\left(e^{2Y_2} - e^{2Y_3}\right)\left(e^{2X_1+2Y_1} - e^{2X_2+2Y_2}\right). \tag{8}$$

We are interested in the condition

$$f_0(X,Y) - f_1(X,Y)\,\sin^2\theta_1 - f_2(X,Y)\,\sin^2\theta_2 = 0. \tag{9}$$

Let $\sigma = \{(x,y)\colon\ \ f_1\,x + f_2\,y = f_0,\ 0 \le x, y \le 1\}$. With the restriction that $\theta_1,\,\theta_2 \in [0,\pi/2]$, let

$$(\theta_1,\theta_2) = \psi(x,y) = (\arcsin\sqrt{x}, \arcsin\sqrt{y}).$$

Since ψ is continuous, the set $\psi(\sigma)$ is connected. It follows that the set Σ of points $(\theta_1,\theta_2) \in [0,\pi/2]^2$ such that (9) holds is connected.

The fact that $f_0 - f_1 - f_2 < 0$ implies that there is a solution of (9) with $\theta_1 = \theta_0 = \theta_2$, $\theta_0 \in (0,\pi/2)$. In particular, σ and $\psi(\sigma)$ are not empty. Note then that $F(\theta_0,\pi,\theta_0) = X+Y$.

Now we compute

$$\begin{aligned}
\mathrm{tr}\ e^{2\,F(\theta_1,\theta,\theta_2)} &= e^{2\,X_1}[\cos^2\theta_1(d\cos^2\theta + e^{2\,Y_2}\sin^2\theta) + \frac{1}{2}\cos\theta\sin(2\theta_1)\sin(2\theta_2)(e^{2\,Y_3} - e^{2\,Y_1}) + f\sin^2\theta_1] \\
&+ e^{2\,X_2}[d\sin^2\theta + e^{2\,Y_2}\cos^2\theta] \\
&+ e^{2\,X_3}[\sin^2\theta_1(d\cos^2\theta + e^{2\,Y_2}\sin^2\theta) - \frac{1}{2}\cos\theta\sin(2\theta_1)\sin(2\theta_2)(e^{2\,Y_3} - e^{2\,Y_1}) + f\cos^2\theta_1]
\end{aligned}$$

where we denote $d = e^{2\,Y_1}\cos^2\theta_2 + e^{2\,Y_3}\sin^2\theta_2$ and $f = e^{2\,Y_1}\sin^2\theta_2 + e^{2\,Y_3}\cos^2\theta_2$. It follows that

$$\begin{aligned}
\mathrm{tr}\ e^{2\,F(\theta_1,\theta,\theta_2)} = \mathrm{tr}\ e^{2\,F(\theta_1,0,\theta_2)} &+ \frac{1}{2}\,(e^{2\,X_1} - e^{2\,X_3})\,(e^{2\,Y_1} - e^{2\,Y_3})\ \sin(2\,\theta_1)\ \sin(2\,\theta_2)\,(1-\cos\theta) \\
&+ \big((e^{2\,Y_1} - e^{2\,Y_2}) - (e^{2\,Y_1} - e^{2\,Y_3})\ \sin^2\theta_2\big) \\
&\cdot \big(-(e^{2\,X_1} - e^{2\,X_2}) + (e^{2\,X_1} - e^{2\,X_3})\ \sin^2\theta_1\big)\ \sin^2\theta.
\end{aligned} \tag{10}$$

Let R be as in the hypothesis. Recall that $\mathbf{SO}(3) = K_{\alpha+\beta}\,K_\alpha\,K_{\alpha+\beta}$ (see [10]). Since R is an intersection point in I we can write $R = F(k)$ with $k = e^{\theta_1\,Z_{\alpha+\beta}}\,e^{\theta\,Z_\alpha}\,e^{\theta_2\,Z_{\alpha+\beta}}$ and $\sin\theta \neq 0$ (basically, because R belongs also to the segment emanating from another element of T_0). This means that (9) is satisfied. Condition (4) of Lemma 18 cannot be fulfilled since f_0, $f_0 - f_1$, $f_0 - f_2$ and $f_0 - f_1 - f_2$ are nonzero. Hence one of remaining conditions (2)–(3) of Lemma 18 is verified. Therefore, in addition to (9), one of the following holds:

1. $\sin\theta = \pm 1$ and (a) $\sin\theta_1 = 0$ or (b) $\sin\theta_1 \pm 1$.

2. $\sin\theta = \pm 1$ and (a) $\sin\theta_2 = 0$ or (b) $\sin\theta_2 = 0 \pm 1$.

In all possible cases, (10) becomes

$$\begin{aligned}
\mathrm{tr}\ e^{2\,F(\theta_1,\theta,\theta_2)} = \mathrm{tr}\ e^{2\,F(\theta_1,0,\theta_2)} &+ \big((e^{2\,Y_1} - e^{2\,Y_2}) - (e^{2\,Y_1} - e^{2\,Y_3})\ \sin^2\theta_2\big) \\
&\cdot \big(-(e^{2\,X_1} - e^{2\,X_2}) + (e^{2\,X_1} - e^{2\,X_3})\ \sin^2\theta_1\big)\ \sin^2\theta.
\end{aligned} \tag{11}$$

Using (9), these cases correspond to

1a. $\sin^2\theta_1 = 0$ and $\sin^2\theta_2 = f_0/f_2$

1b. $\sin^2\theta_1 = 1$ and $\sin^2\theta_2 = (f_0 - f_1)/f_2$

2a. $\sin^2\theta_1 = f_0/f_1$ and $\sin^2\theta_2 = 0$

2b. $\sin^2\theta_1 = (f_0 - f_2)/f_1$ and $\sin^2\theta_2 = 1$.

Using (3)–(7), we observe that the coefficient of $\sin^2\theta = 1$ in (11) is strictly positive in all these cases. This means that if $R = F(\theta_1, \theta, \theta_2)$ is an intersection point then for (θ_1, θ_2) fixed, then $\operatorname{tr} e^{2F(\theta_1,\theta,\theta_2)}$ is maximum at R and minimum at $R' = F(\theta_1, 0, \theta_2) \neq R$.

For $P_1,\ P_2 \in I_{\alpha+\beta,id,id}$, we have $\operatorname{tr} e^{P_1} > \operatorname{tr} e^{P_2}$ whenever, starting at $X+Y$ and running along $I_{\alpha+\beta,id,id}$, the point P_1 precedes P_2. This follows from the negativity of

$$f'(t) = -4\,c\,\sin(2\,ct)\,\sinh(2\,a\,c^2)\,\sinh(2\,b\,c^2), \qquad t > 0$$

where $f(t) = \operatorname{tr} g_t g_t^T$, according to the formula for $f'(t)$ in the proof of Lemma 12. Since $c > 0$, the sign of f' is that of $-a\,b$ where $a = \gamma(w_1 X)/\gamma(H_\gamma)$ and $b = \gamma(w_2 Y)/\gamma(H_\gamma)$ when running along the segment I_{γ,w_1,w_2} starting at $w_1 X + w_2 Y \in T_0$ (see the proof of [3, Corollary 14]). When starting from $X+Y$ with $\gamma = \alpha+\beta$ we have $a < 0$ and $b < 0$.

It follows that $R' \in \pi(A_1\,A_4)$ and $X+Y \in \pi(A_1\,A_4)$ are on different sides of the intersection point $R \in \pi(A_1\,A_4)$. Hence for $\epsilon > 0$ sufficiently small, the points $F(\theta_0, \pi-\epsilon, \theta_0) \in \pi(A_1\,A_4)$ and $F(\theta_1, \pi-\epsilon, \theta_2) \in \pi(A_1\,A_4)$ (recall that $R = F(\theta_1, \pi/2, \theta_2)$) are on different sides of the intersection point $R \in \pi(A_1\,A_4)$.

The sets $\psi(\sigma)$ and $\pi(A_1\,A_4)$ being connected it follows that there exist θ_1', θ_2' such that

$$R = F(\theta_1', \pi-\epsilon, \theta_2').$$

Since $(0,0)$, $(\pi/2, 0)$, $(0, \pi/2)$, $(\pi/2, \pi/2) \notin \psi(\sigma)$, it follows by Lemma 1 that $R \in F(K \setminus K_0)$.

For the other two cases, let us suppose without loss of generality that $X_1 - X_3 > Y_1 - Y_3$ so that the points A_i are the vertices of $X + C(Y)$. In the case $R \in \pi(A_3\,A_6)$, consider first the subcase $\pi(A_3) = A_3 = X + w_{\alpha+\beta}\,w_\beta\,Y \in \mathfrak{a}^+$, which is equivalent to

$$X_1 + Y_2 > X_2 + Y_3 > X_3 + Y_1.$$

Consider $F_1(\theta_1, \theta, \theta_2) = a(e^X\,e^{\theta_1 Z_\beta}\,e^{\theta Z_\alpha}\,e^{\theta_2 Z_\beta}\,e^{w_{\alpha+\beta}\,w_\beta\,Y})$. Then $F_1(\theta_1, \theta, \theta_2) \in I$ belongs to $\pi(A_3\,A_6)$ if and only if $\det(e^X\,k\,e^{2\,w_{\alpha+\beta} w\,\beta Y}\,k^T\,e^X - e^{2\,(X_1+Y_2)}\,I) = 0$ with $k = e^{\theta_1 Z_\beta}\,e^{\theta\,Z_\alpha}\,e^{\theta_2 Z_\beta}$.

Observe that $a(e^X\,k e^{w_{\alpha+\beta}\,w_\beta Y}) = a(e^{w_\alpha X}\,k_1\,e^{w_\alpha\,w_{\alpha+\beta}\,w_\beta\,Y})$ where $k_1 = e^{\theta_1 Z_{\alpha+\beta}}\,e^{-\theta\,Z_\alpha}\,e^{\theta_2 Z_{\alpha+\beta}}$ is as in the case $\pi(A_1\,A_4)$. Similarly

$$D = \det(e^X k e^{2\,w_{\alpha+\beta}\,w_\beta Y}\,k^T\,e^X - e^{2\,(X_1+Y_2)} I) = \det(e^{w_\alpha X}\,k_1 e^{2\,w_\alpha\,w_{\alpha+\beta}\,w_\beta\,Y}\,k_1^T\,e^{w_\alpha\,X} - e^{2\,(X_1+Y_2)} I)$$

and we observe that the determinant D is equal to the determinant computed in the case $\pi(A_1\,A_4)$ evaluated for variables X_2, X_1, X_3, Y_3, Y_2, Y_1, $-\theta$, θ_1, θ_2 instead of X_1, X_2, X_3, Y_1, Y_2, Y_3, θ, θ_1, θ_2. In particular we find without any new computation that $D = 0$ if and only if

$$\sin^2\theta\,(f_0(X,Y) - f_1(X,Y)\,\sin^2\theta_1 - f_2(X,Y)\,\sin^2\theta_2) = 0$$

where

$$f_0 = \left(e^{2\,X_1+2\,Y_2} - e^{2\,X_3+2\,Y_1}\right)\left(e^{2\,Y_3} - e^{2\,Y_2}\right)\left(e^{2\,X_2} - e^{2\,X_1}\right), \tag{12}$$
$$f_1 = e^{2\,X_1}\left(e^{2\,Y_3} - e^{2\,Y_2}\right)\left(e^{2\,Y_2} - e^{2\,Y_1}\right)\left(e^{2\,X_2} - e^{2\,X_3}\right), \tag{13}$$
$$f_2 = e^{2\,Y_2}\left(e^{2\,Y_3} - e^{2\,Y_1}\right)\left(e^{2\,X_2} - e^{2\,X_1}\right)\left(e^{2\,X_1} - e^{2\,X_3}\right) \tag{14}$$
$$f_1 - f_0 = \left(e^{2\,X_1} - e^{2\,X_3}\right)\left(e^{2\,Y_3} - e^{2\,Y_2}\right)\left(e^{2\,X_1+2\,Y_2} - e^{2\,X_2+2\,Y_1}\right), \tag{15}$$
$$f_2 - f_0 = \left(e^{2\,X_2} - e^{2\,X_1}\right)\left(e^{2\,Y_2} - e^{2\,Y_1}\right)\left(e^{2\,X_1+2\,Y_2} - e^{2\,X_3+2\,Y_3}\right), \tag{16}$$
$$f_1 + f_2 - f_0 = \left(e^{2\,X_1} - e^{2\,X_3}\right)\left(e^{2\,Y_2} - e^{2\,Y_1}\right)\left(e^{2\,X_2+2\,Y_3} - e^{2\,X_1+2\,Y_2}\right). \tag{17}$$

The inequality $X_1 + Y_2 > X_3 + Y_1$ implies that $f_0 > 0$ and the inequality $X_1 + Y_2 > X_2 + Y_3$ implies that $f_0 - f_1 - f_2 < 0$. It follows as in the first case that the inequality

$$f_0(X,Y) - f_1(X,Y)\,\sin^2\theta_1 - f_2(X,Y)\,\sin^2\theta_2 = 0 \tag{18}$$

admits a solution $\theta_1 = \theta_2 = \theta_0 \in (0, \pi/2)$. We have $F_1(\theta_0, \pi, \theta_0) = A_3$.

The computation of the trace of $e^{2F_1(\theta_1,\theta,\theta_2)}$ also boils down to the trace of $e^{2F(\theta_1,\theta,\theta_2)}$, in the same way as the computation of D boils down to the determinant of the first case. In particular, in the cases 1a–b and 2a–b as in the proof of the first case,

$$\begin{aligned}\operatorname{tr} e^{2F_1(\theta_1,\theta,\theta_2)} = \operatorname{tr} e^{2F_1(\theta_1,0,\theta_2)} + \big((e^{2Y_3} - e^{2Y_2}) - (e^{2Y_3} - e^{2Y_1})\,\sin^2\theta_2\big)\\ \cdot\big(-(e^{2X_2} - e^{2X_1}) + (e^{2X_2} - e^{2X_3})\,\sin^2\theta_1\big)\,\sin^2\theta.\end{aligned} \tag{19}$$

Observe that the second factor of $\sin^2\theta$ in (19) is always positive. The fact that $f_2 - f_0 > 0$ implies that the case 2b is impossible. It is trivial to see that in the case 2a the factor of $\sin^2\theta$ in (19) is negative. It is a matter of easy verifications using (12), (14) and (15) to see that the same is true in cases 1a and 1b.

Finally one checks using the same ideas as in the proof for $\pi(A_1\,A_4)$ that the trace of $g_t\,g_t^T$ increases when we start at A_3 and run along $\pi(A_3\,A_6)$. We conclude the proof in the same way as for $\pi(A_1\,A_4)$.

The subcase $A_3 \notin \mathfrak{a}^+$ (so $\pi(A_3) = w_\beta(A_3) \in \mathfrak{a}^+$) of the case $R \in \pi(A_3\,A_6)$ reduces to the first subcase if we consider $F_2(\theta_1,\theta,\theta_2) = a(e^{w_\beta X}\,e^{\theta_1 Z_\beta}\,e^{\theta\,Z_{\alpha+\beta}}\,e^{\theta_2\,Z_\beta}\,e^{w_\beta\,w_{\alpha+\beta}\,w_\beta Y}) = F_1(-\theta_1,\theta,-\theta_2)$. The inequalities $X_1 + Y_2 > X_3 + Y_1$ and $X_1 + Y_2 > X_2 + Y_3$ intervening in the case $A_3 \in \mathfrak{a}^+$ still hold true because $w_\beta(A_3) \in \mathfrak{a}^+$.

The proof in the case $R \in \pi(A_2\,A_5)$ goes along the same lines as that for $R \in \pi(A_3\,A_6)$ if we choose as a starting point $A_5 \in \mathfrak{a}^+$ (or $\pi(A_5) \in \mathfrak{a}^+$) and we consider $F_3(\theta_1,\theta,\theta_2) = a(e^X\,e^{\theta_1\,Z_\alpha}\,e^{\theta\,Z_{\alpha+\beta}}\,e^{\theta_2\,Z_\alpha}\,e^{w_{\alpha+\beta}\,w_\alpha\,Y})$ (or, respectively, $F_4(\theta_1,\theta,\theta_2) = a(e^{w_\alpha\,X}\,e^{\theta_1\,Z_\alpha}\,e^{\theta\,Z_\beta}\,e^{\theta_2\,Z_\alpha}\,e^{w_\alpha\,w_{\alpha+\beta}\,w_\alpha Y}) = F_3(-\theta_1,\theta,-\theta_2)$). In particular, the eigenvalues intervening in the study of $F_3 \in \pi(A_2\,A_5)$ are $e^{2(X_3+Y_2)}$, the case 2a is now impossible (since $f_0 > f_1 > 0$) and the trace of $g_t\,g_t^T$ increases when one starts at A_5 (or $\pi(A_5)$). The reader will verify easily this last case of the proof. ∎

Lemma 20 *We have $0 \in \mathcal{S}$ if and only if $0 \in T_0$.*

Proof: If $0 \in \mathcal{S}$ then $0 = a(e^X\,k\,e^Y)$ *i.e.* $e^X\,k\,e^Y = k'$ or $k\,e^Y = e^{-X}\,k$ which means that $Y = -w\,X$ for some $w \in W$ since the abelian component of the Cartan decomposition is unique modulo W: $0 = w\,X + Y \in T_0$. ∎

Lemma 21 *Let $\hat{K} = K \setminus K_0$. Then*

1. *If $H \in \mathcal{S} \setminus I$ then $H = a(e^X\,k\,e^Y)$ for some $k \in \hat{K}$.*
2. *If $k \in \hat{K}$ and $a(e^X\,k\,e^Y) \in \mathfrak{a}^+$ then $a(e^X\,k\,e^Y) \in \mathcal{S}^\circ$.*

Proof:

1. One notes that $I = a(e^X\,K_0\,e^Y)$.

2. Suppose that $k \in \hat{K}$ and $a(e^X\, k\, e^Y) \in \mathfrak{a}^+$ (the second condition ensures the analyticity of the map $k' \to a(e^X\, k'\, e^Y)$ in the neighbourhood of k).

   Let $P = k\, e^{2Y}\, k^{-1}$ and for $Z = a_1\, Z_\alpha + a_2\, Z_\beta + a_3\, Z_{\alpha+\beta}$, let $R_t = e^X\, e^{t\,Z}\, P\, e^{-t\,Z}\, e^X$ and note that the eigenvalues of R_t determine $a(e^X\, e^{t\,Z}\, k\, e^Y)$. The functions $\operatorname{tr} R_t$ and $\operatorname{tr} R_t^{-1}$ in turn determine the eigenvalues of R_t.

   Now,

$$\operatorname{tr} R_t = \operatorname{tr}(e^X\, P\, e^X) + \operatorname{tr}(e^X\, [Z, P]\, e^X)\, t + O(t^2), \tag{20}$$

$$\operatorname{tr} R_t^{-1} = \operatorname{tr}(e^{-X}\, P^{-1}\, e^{-X}) + \operatorname{tr}(e^{-X}\, [Z, P^{-1}]\, e^{-X})\, t + O(t^2). \tag{21}$$

It suffices to show that, locally, these two functions can give any direction we want using the right choice of Z. The equations $\operatorname{tr}(e^X\, [Z, P]\, e^X) = 0$ and $\operatorname{tr}(e^{-X}\, [Z, P^{-1}]\, e^{-X}) = 0$ correspond to 2 equations of planes:

$$(e^{2\,x_1} - e^{2\,x_2})\, P_{1,2}\, a_1 + (e^{2\,x_2} - e^{2\,x_3})\, P_{2,3}\, a_2 + (e^{2\,x_1} - e^{2\,x_3})\, P_{1,3}\, a_3 = 0, \tag{22}$$

$$(e^{2\,x_1} - e^{2\,x_2})\, Q_{1,2}\, e^{2\,x_3}\, a_1 + (e^{2\,x_2} - e^{2\,x_3})\, Q_{2,3}\, e^{2\,x_1}\, a_2 + (e^{2\,x_1} - e^{2\,x_3})\, Q_{1,3}\, e^{2\,x_2}\, a_3 = 0 \tag{23}$$

where $Q = P^{-1}$. It suffices to show that the two planes do not coincide. To show that the two planes are not the same, it suffices to show that

$$(e^{2\,x_1} - e^{2\,x_2})\, P_{1,2}(e^{2\,x_1} - e^{2\,x_3})\, Q_{1,3}\, e^{2\,x_2} \neq (e^{2\,x_1} - e^{2\,x_3})\, P_{1,3}(e^{2\,x_1} - e^{2\,x_2})\, Q_{1,2}\, e^{2\,x_3}$$

i.e.

$$P_{1,2}\, Q_{1,3}\, e^{2\,x_2} \neq P_{1,3}\, Q_{1,2}\, e^{2\,x_3}.$$

Note that the format of k implies that P cannot have more than 2 zeros (which are then symmetric about the diagonal). If $P_{1,2} = 0$ and $P_{1,3} \neq 0$ then $P_{2,3} \neq 0$ and $P_{1,3}\, Q_{1,2} = -P_{1,3}^2\, P_{2,3} \neq 0$ and the result is clear. The case $P_{1,2} \neq 0$ and $P_{1,3} = 0$ is very similar. We can therefore assume that $P_{1,2}$ and $P_{1,3}$ are nonzero.

If $Q_{1,2} = 0$ or $Q_{1,3} = 0$, then the result is clear (we cannot have $Q_{1,2} = Q_{1,3} = 0$ since that would imply $P_{1,2} = P_{1,3} = 0$). Suppose therefore that $Q_{1,2} \neq 0$ and $Q_{1,3} \neq 0$. Now, using $P = k\, e^{2Y}\, k^{-1}$ and $Q = k\, e^{-2Y}\, k^{-1}$, we get

$$P_{1,2}\, Q_{1,3} - P_{1,3}\, Q_{1,2} = -k_{1,1}\, k_{1,2}\, k_{1,3}\, (e^{2\,y_1} - e^{2\,y_2})\, (e^{2\,y_2} - e^{2\,y_3})\, (e^{2\,y_1} - e^{2\,y_3}).$$

If $k_{1,1}\, k_{1,2}\, k_{1,3} = 0$ then $P_{1,2}\, Q_{1,3} = P_{1,3}\, Q_{1,2}$ and therefore $P_{1,2}\, Q_{1,3}\, e^{2\,x_2} \neq P_{1,3}\, Q_{1,2}\, e^{2\,x_3}$. If $k_{1,1}\, k_{1,2}\, k_{1,3} \neq 0$, multiplying the first two rows of K by -1 (taking for example $m\, k$ where $m = \operatorname{diag}[-1, -1, 1] \in M$) if necessary, we can assume that $k_{1,1}\, k_{1,2}\, k_{1,3} < 0$ which implies $P_{1,2}\, Q_{1,3} - P_{1,3}\, Q_{1,2} > 0$ and therefore $P_{1,2}\, Q_{1,3}\, e^{2\,x_2} > P_{1,3}\, Q_{1,2}\, e^{2\,x_3}$. ∎

Corollary 22 *The set $\mathcal{S} \setminus I \setminus \partial\mathfrak{a}^+$ is open in $\mathfrak{a}^+$.*

Proposition 23 *Let $Z = \{(X, Y) \in \mathfrak{a}^+ \times \mathfrak{a}^+ \colon\ W\, X + W\, Y \cap \partial\mathfrak{a}^+ \neq \emptyset\} \cup \bigcup_{i,j,p,q} \{(X, Y) \in \mathfrak{a}^+ \times \mathfrak{a}^+ \colon\ X_i + Y_j = X_p + Y_q\}$. If X, $Y \in \mathfrak{a}^+$ and $(X, Y) \notin Z$ then $\mathcal{S} = conv(I)$.*

Proof: Note that $w_1 X + w_2 Y \in \partial\mathfrak{a}^+$ if and only if $\alpha(w_1 X + w_2 Y) = 0$ for some nonzero root α. This implies that $\mathfrak{a}^+ \times \mathfrak{a}^+ \setminus Z$ is dense in $\overline{\mathfrak{a}^+} \times \overline{\mathfrak{a}^+}$ since we only remove a finite numbers of hyperplanes.

We prove first that $\mathrm{conv}(I) \subset \mathcal{S}$. We can assume without loss of generality that $\mathbf{F} = \mathbf{R}$ (indeed, it is clear that the set $\mathcal{S}$ corresponding to the real case is included in the others and that the network I is the same).

Assume that $\mathrm{conv}(I) \subset \mathcal{S}$ is not true. Divide $\mathrm{conv}(I) \setminus I$ into open connected components C_i. Pick C_j which contains $H_0 \notin \mathcal{S}$. We know that C_j is the interior of a polygon with, say, n vertices. Each edge of C_j is parallel to the direction A_γ of a root $\gamma \in \Delta_0$. It is important to recall that the segments composing I are reflected when they encounter a wall of $\mathfrak{a}^+$.

We claim first that $C_j \cap \mathcal{S} \cap \mathfrak{a}^+ \neq \emptyset$.

Suppose that more than two of the vertices of C_j belong to $\partial\mathfrak{a}^+$. We first note that C_j is a triangle. One sees this by taking into account the above observations concerning the edges of C_j. The same considerations then imply that at least one of its vertices belongs to T_0. The possibility is excluded by the hypothesis $(X, Y) \notin Z$.

We can therefore assume that no more than two vertices belong to $\partial\mathfrak{a}^+$. Since the sum of the angles inside C_j is $(n-2)\pi$, there must be an angle inside C_j less than π with a vertex $V \in \mathfrak{a}^+$. The vertex V is either an element of T_0 with a sector in C_j included in $\mathcal{S}$ (Corollary 13) or the intersection point of 2 different segments of I which is included in $\mathcal{S}^\circ$ by Theorem 19.

The claim therefore follows.

Let $H_1 \in \mathcal{S}^\circ \cap C_j$ and let $l(t)$, $t \in [0,1]$ be a continuous curve linking H_0 to H_1 contained in C_j ($l(0) = H_0$, $l(1) = H_1$, $l(s) \neq l(t)$ when $s \neq t$). One may choose l as a segment or a finite connected union of segments.

Let $t_0 = \sup\{t \in [0,1] \mid l(t) \notin \mathcal{S}\}$ and let $H_2 = l(t_0) \in C_j$. By the maximality of t_0, in any neighbourhood of H_2, there are points of $\mathcal{S}$ which is closed. It follows that $H_2 \in \mathcal{S}$ and $H_2 \in \mathcal{S} \setminus I \setminus \partial\mathfrak{a}^+$ which is open according to Corollary 22. This contradicts the definition of t_0 since $l(t_0 - \epsilon) \in \mathcal{S}$ for $\epsilon > 0$ small enough. The set $\{t \in [0,1] \mid l(t) \notin \mathcal{S}\}$ must be empty which contradicts the existence of H_0.

We now return to $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. Suppose now that there exists $H \in \mathcal{S} \setminus \mathrm{conv}(I)$. We can assume without loss of generality that $H \in \mathfrak{a}^+$. Otherwise, take an open neighbourhood U of $H \in \overline{\mathfrak{a}^+}$ which does not touch $\mathrm{conv}(I)$ and consider the open set $V = F^{-1}(U)$ in K. The set V cannot be included in the set C of Lemma 1 and therefore, there exists $k \in V$ such that $F(k) \in U \cap \mathcal{S}^\circ$ is in $\mathfrak{a}^+$ (and not in $\mathrm{conv}(I)$).

Consider the half line ℓ starting at 0 and passing through H. Note that by the hypothesis and by Lemma 20, $0 \notin \mathcal{S}$ and therefore $\mathcal{S} \cap \ell \subset \mathfrak{a}^+$. Let H_0 be the point on the compact $\mathcal{S} \cap \ell$ which is the furthest away from $\mathrm{conv}(I)$. It is plain using Corollary 22 that such a point cannot exist. This contradicts the existence of $H \in \mathcal{S} \setminus \mathrm{conv}(I)$. ∎

All that remains is to get rid of the technical condition "If $X, Y \in \mathfrak{a}^+$ and $(X, Y) \notin Z$" of Proposition 23. The following lemma is the tool we need to achieve this.

Lemma 24 *A "segment" of I has the form $a(e^X w_1 K_\alpha w_2 e^Y)$ where $w_i \in W$ and $\alpha \in \Delta_0$ (the set of positive roots). Every element $H \in conv(I)$ has the form*

$$H = \sum_{W \times W \times \Delta_0} \Gamma_{w_1, w_2, \alpha}\, a(e^X w_1 k_\alpha w_2 e^Y)$$

where $k_\alpha \in K_\alpha$, $\Gamma_{w_1,w_2,\alpha} \in [0,1]$ and $\sum_{W \times W \times \Delta_0} \Gamma_{w_1,w_2,\alpha} \leq 1$.

Proof: Clear. ∎

Theorem 25 *If $G = \mathbf{SL}(3, \mathbf{F})$ with $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ then $\mathcal{S} = conv(I)$.*

Proof: Let Z be as in Proposition 23. If $(X, Y) \notin Z$ then the result follows from Proposition 23.

Suppose then that $(X, Y) \in Z$. Let $\mathcal{S}_{X,Y} = a(e^X\, K\, e^Y)$ and let $I_{X,Y}$ be the network I associated to X and Y.

Take $(X_n, Y_n) \notin Z$ such that (X_n, Y_n) converges to (X, Y).

Let $H = a(e^X\, k\, e^Y)$ and note that $H = \lim_{n\to\infty} a(e^{X_n}\, k\, e^{Y_n})$. Now, for each n, we have by Lemma 24 and by Proposition 23 that

$$a(e^{X_n}\, k\, e^{Y_n}) = \sum_{W\times W} \Gamma_{w_1,w_2,\alpha}(n)\, a(e^{X_n}\, w_1\, k_\alpha(n)\, w_2\, e^Y)$$

with $\Gamma_{w_1,w_2,\alpha}(n) \in [0,1]$ and $\sum_{W\times W\times\Delta_0} \Gamma_{w_1,w_2,\alpha}(n) \leq 1$. Since $[0,1]$ and each K_α are compact, by taking a subsequence if necessary, we can assume without loss of generality that $\Gamma_{w_1,w_2,\alpha}(n)$ converges to $\Gamma_{w_1,w_2,\alpha}$ and that $k_\alpha(n)$ converges to k_α (for each α, w_1 and w_2). This means that

$$a(e^X\, k\, e^Y) = \sum_{W\times W} \Gamma_{w_1,w_2,\alpha}\, a(e^X\, w_1\, k_\alpha\, w_2\, e^Y) \in \mathrm{conv}(I_{X,Y}).$$

Suppose now that $H \in \mathrm{conv}(I_{X,Y})$ *i.e.* $H = \sum_{W\times W} \Gamma_{w_1,w_2,\alpha}\, a(e^X\, w_1\, k_\alpha\, w_2\, e^Y)$. Then $H = \lim_{n\to\infty} \sum_{W\times W} \Gamma_{w_1,w_2,\alpha}\, a(e^{X_n}\, w_1\, k_\alpha\, w_2\, e^{Y_n}) = \lim_{n\to\infty} a(e^{X_n}\, w_1\, k_n\, w_2\, e^{Y_n})$ with $\sum_{W\times W\times\Delta_0} \Gamma_{w_1,w_2,\alpha} \leq 1$ (using Proposition 23 and the fact that $\sum_{W\times W} \Gamma_{w_1,w_2,\alpha}\, a(e^{X_n}\, w_1\, k_\alpha\, w_2\, e^{Y_n})$ belongs to $\mathrm{conv}(I_{X_n,Y_n})$). As before, by taking a subsequence if necessary, we can assume that k_n converges to k. Hence, $H = \lim_{n\to\infty} a(e^{X_n}\, w_1\, k_n\, w_2\, e^{Y_n}) = a(e^X\, k\, e^Y) \in \mathcal{S}_{X,Y}$. ∎

Recall that the singular values of a complex square matrix A are the non-negative square roots of the eigenvalues of the Hermitian matrix AA^*. It is useful to note that if A is Hermitian (in particular if A is real symmetric) then its singular values are the absolute values of its eigenvalues. When A is Hermitian positive definite its eigenvalues and singular values coincide.

Theorem 26 *Let A and B be two complex matrices of size 3×3 with singular values $a_1 \geq a_2 \geq a_3 \geq 0$ and $b_1 \geq b_2 \geq b_3 \geq 0$ respectively. Let $\sigma_1 \geq \sigma_2 \geq \sigma_3$ be the singular values of the product $C = AB$. Then*

$$\max\{a_1b_3, a_3b_1\} \leq \sigma_1 \leq a_1b_1, \tag{24}$$
$$\max\{a_2b_3, a_3b_2\} \leq \sigma_2 \leq \min\{a_2b_1, a_1b_2\}, \tag{25}$$
$$a_3b_3 \leq \sigma_3 \leq \min\{a_3b_1, a_1b_3\}, \tag{26}$$
$$\sigma_3 \leq a_2b_2 \leq \sigma_1. \tag{27}$$

Conversely, for any square 3×3 matrix C with singular values satisfying equations (24)–(27), there exist matrices A and B with singular values $a_1 \geq a_2 \geq a_3 \geq 0$ and $b_1 \geq b_2 \geq b_3 \geq 0$ such that $AB = C$.

If C is real then A and B can be chosen to be real.

Proof: Suppose first that A and B are non-singular. In this case we can assume without loss of generality that $|A| = 1 = |B|$ so that A, $B \in \mathbf{SL}(3, \mathbf{C})$. We have then $A = k_1\, e^X\, k_2$ and $B = k_3\, e^Y\, k_4$ with $k_i \in \mathbf{SU}(3)$, $i \leq 4$ and $e^X = \mathrm{diag}[a_1, a_2, a_3]$, $e^Y = \mathrm{diag}[b_1, b_2, b_3]$. Therefore, $A\,B = k_1 e^X k_2 k_3 e^Y k_4$ and $e^{a(e^X k_2 k_3 e^Y)} = \mathrm{diag}[\sigma_1, \sigma_2, \sigma_3]$. The result follows by Theorem 25.

The general case follows by a continuity argument. ∎

Remark 27 *The conditions (24)–(27) in Theorem 26, when the considered matrices have the determinant 1, have a nice geometric interpretation*

$$[\ln \sigma_1, \ln \sigma_2, \ln \sigma_3] \in conv(I)$$

where I is constructed in the Proposition 15 starting from $X = [\ln a_1, \ln a_2, \ln a_3]$ and $Y = [\ln b_1, \ln b_2, \ln b_3]$.

3 Conclusion

Naturally, the network I can be defined for any symmetric space of noncompact type. In the other rank 2 cases, the difficulty is not so much with the the result corresponding to Proposition 23 but what to do with the "intersection points". When the rank is greater than 2, matters are even more complicated.

In [3] we showed that $\mu_{X,Y}$ is absolutely continuous with respect to the Haar measure on A whenever X, $Y \in \mathfrak{a}^+$. We described other situations when X or Y where in $\partial\mathfrak{a}^+$ where this still held and others when it did not. However, we did not find necessary and sufficient conditions on X and Y to settle this question. With a convexity theorem such as we have found in this paper, the situation becomes clearer. It is clear that $\mu_{X,Y}$ is absolutely continuous if and only if the network I does not live in a hyperplane of $\mathfrak{a}$. In fact, we can conclude that $\mu_{X,Y}$ is absolutely continuous if and only if the network $W\,X + W\,Y$ does not live in a hyperplane of $\mathfrak{a}$. A corresponding result for all ranks would be very useful and we conjecture it.

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