The product formula for the spherical functions on symmetric spaces in the complex case

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Abstract

In this paper, we prove the existence of the product formula for the spherical functions in the complex case and we study properties of the integral kernel of this formula.

1 Introduction

Let G be a semisimple noncompact Lie group with finite center and K a maximal compact subgroup of G and $X=G/K$ the corresponding Riemannian symmetric space of noncompact type. We have a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and we choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. In what follows, Σ corresponds to the root system of $\mathfrak{g}$ and Σ^+ to the positive roots. We have the root space decomposition $\mathfrak{g}=\mathfrak{n}_0+\sum_{\alpha\in\Sigma}\mathfrak{n}_\alpha$. Let $\mathfrak{n}=\sum_{\alpha\in\Sigma^+}\mathfrak{n}_\alpha$. Denote the groups corresponding to the Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ by A and N respectively. We have the Cartan decomposition $G=K\,A\,K$ and the Iwasawa decomposition $G=K\,A\,N$. Let $\mathfrak{a}^+=\{H\in A\colon\ \alpha(H)>0\ \forall\ \alpha\in\Sigma^+\}$ and $A^+=\exp(\mathfrak{a}^+)$.

If λ is a complex-valued functional on $\mathfrak{a}$, the corresponding spherical function is

$$\phi_\lambda(e^H)=\int_K e^{(i\,\lambda-\rho)(\mathcal{H}(e^H\,k))}\,dk$$

where $g=k\,e^{\mathcal{H}(g)}\,n\in K\,A\,N$. A spherical function, like any K-biinvariant function, can also be considered as a K-invariant function on the Riemannian symmetric space of noncompact type $X=G/K$. Naturally, such a function is completely determined by its values on A (or on A^+). The books [4, 5] constitute a standard reference on these topics.

Suppose $X\neq 0$, $Y\neq 0$ and $Y\notin W\cdot\{-X\}$ (or equivalently that $X\notin W\cdot\{-Y\}$). Suppose also that the symmetric space G/K is irreducible. We will assume these hypotheses throughout the paper.

In [5, (32), page 480], Helgason shows that a Weyl-invariant measure $\mu_{X,Y}$ exists on the Lie algebra $\mathfrak{a}$ such that

$$\phi_\lambda(e^X)\,\phi_\lambda(e^Y)=\int_{\mathfrak{a}}\phi_\lambda(e^H)\,d\mu_{X,Y}(H)$$

(unlike us, Helgason states his results at the group level).

It is known [5] that

$$\phi_\lambda(e^X)\,\phi_\lambda(e^Y)=\int_K\phi_\lambda(e^X\,k\,e^Y)\,dk.$$

The measure $\mu_{X,Y}$ is then to satisfy

$$\int_K f(e^X\,k\,e^Y)\,dk=\int_{\mathfrak{a}} f(e^H)\,d\mu_{X,Y}(H)$$

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for all functions f which are biinvariant under the action of K.

The support of the measure $\mu_{X,Y}$ is shown to be included in $C(X)+C(Y)$ where $C(H)$ is the convex hull of the orbit of H under the action of the Weyl group W.

The natural question is whether the measure $\mu_{X,Y}$ is absolutely continuous with respect to the Lebesgue measure on $\mathfrak{a}$ *i.e.* whether we have a "product formula"

$$\phi_\lambda(e^X)\,\phi_\lambda(e^Y) = \int_{\mathfrak{a}} \phi_\lambda(e^H)\,k(H,X,Y)\,dH \tag{1}$$

where $k(H,X,Y)$ is Weyl invariant in each of the variables. Helgason also discusses this measure and some partial results in [6].

Koornwinder gives explicit formulae for the rank one case in [9]. In fact, he gives a product formula for a larger class of special functions, namely the Jacobi functions. The formulae given can be derived using an addition formula which is not currently available in higher rank situations. The reader may wish to consult also [1, 2, 7, 8, 9, 10, 11].

In this paper, we show the product formula (1) for symmetric spaces in the complex case and we give some information on the kernel k.

Our formula has applications in special functions theory and multivariate statistics because it may be equivalently expressed in terms of the Schur or zonal polynomials on Hermitian positive definite matrices.

There are also important relations between product formulae for spherical functions and arithmetic of probability measures. Ostrovskii ([12]) and Trukhina ([13]) showed that the only measures without indecomposable factors (in the sense of convolution product), respectively in the set of radial measures on R^n and in the set of K-invariant measures on real hyperbolic spaces, are the Gaussian measures. Also Voit ([14]) studied this question on some hypergroups. The main tool of all this research is a product formula (1) with some information on its kernel. We think that our formula will give similar characterization of Gaussian measures on symmetric spaces with G complex.

Two more intrinsic applications of (1) are given in the end of Section 2.

We thank Tom Koornwinder for helpful remarks and Amos Nevo for pointing out to us the application of the product formula given in the Corollary 6.

2 The product formula on complex Lie groups

We consider the spherical functions on complex groups.

We require some preliminaries.

We first note that there exists a function $K(X,H)$ which is Weyl-invariant in both of its arguments such that

$$\phi_\lambda(e^X) = \int_{C(X)} e^{<i\,\lambda,H>}\,K(X,H)\,dH \tag{2}$$

(K is defined for $X \neq 0$).

The existence of the kernel $K(X,H)$ in (2) is shown in [5, Page 479]. It is simply the kernel of the Abel transform. This is valid for every symmetric space of noncompact type.

If we use the Cartan decomposition, the integration on G can be written in polar coordinates. With suitable normalization, we have

$$\int_G f(g)\,dg = \int_K \int_K \int_{\mathfrak{a}^+} f(k_1\, e^H\, k_2)\, \delta(H)\, dH\, dk_1\, dk_2$$

where $\delta(H) = \prod_{\alpha\in\Sigma^+} \sinh^{m_\alpha} \alpha(H)$ and m_α denotes the multiplicity of the root α.

In the complex case $m_\alpha = 2$ for each α and we have

$$\delta^{1/2}(X) = \sum_{w\in W} \epsilon(w)\, e^{<w\rho, X>}. \tag{3}$$

It is worthwhile to mention that as it is written in (3), the function $\delta^{1/2}$ is skew Weyl-invariant *i.e.* $\delta^{1/2}(w\cdot H) = \epsilon(w)\, \delta^{1/2}(H)$.

Still in the complex case, we have

$$\phi_\lambda(e^X) = \frac{\pi(\rho)}{\pi(i\lambda)} \frac{\sum_{w\in W} \epsilon(w)\, e^{<iw\cdot\lambda, X>}}{\delta^{1/2}(X)}.$$

Theorem 1 *Suppose G is a complex Lie group. Then we have the following product formula*

$$\phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_{\mathfrak{a}} \phi_\lambda(e^H)\, k(H, X, Y)\, \delta(H)\, dH$$

where

$$k(H, X, Y) = \frac{1}{\delta^{1/2}(H)\, \delta^{1/2}(Y)} \frac{1}{|W|} \sum_{w\in W} \epsilon(w)\, K(X, w\cdot H - Y). \tag{4}$$

Proof: We observe first that

$$\begin{aligned}
&\int_{C(X)} \phi_\lambda(e^{H+Y})\, \frac{K(X,H)\, \delta^{1/2}(H+Y)}{\delta^{1/2}(Y)}\, dH \\
&= \frac{\pi(\rho)}{\pi(i\lambda)} \sum_{w\in W} \epsilon(w) \int_{C(X)} e^{<iw\cdot\lambda, H+Y>} \frac{K(X,H)\, \delta^{1/2}(H+Y)}{\delta^{1/2}(H+Y)\, \delta^{1/2}(Y)}\, dH \\
&= \frac{\pi(\rho)}{\pi(i\lambda)} \sum_{w\in W} \epsilon(w) \frac{e^{<iw\cdot\lambda, Y>}}{\delta^{1/2}(Y)} \int_{C(X)} e^{<iw\cdot\lambda, H>}\, K(X,H)\, dH \\
&= \frac{\pi(\rho)}{\pi(i\lambda)} \sum_{w\in W} \epsilon(w) \frac{e^{<iw\cdot\lambda, Y>}}{\delta^{1/2}(Y)}\, \phi_{w\cdot\lambda}(e^X) \\
&= \phi_\lambda(e^Y)\, \phi_\lambda(e^X)
\end{aligned}$$

(we note first that $\phi_{w\cdot\lambda} = \phi_\lambda$ and then we add over w).

Hence,

$$\begin{aligned}
&\int_{C(X)+Y} \phi_\lambda(e^H)\, \frac{K(X, H-Y)}{\delta^{1/2}(H)\, \delta^{1/2}(Y)}\, \delta(H)\, dH \\
&= \int_{C(X)} \phi_\lambda(e^{H+Y})\, \frac{K(X,H)\, \delta^{1/2}(H+Y)}{\delta^{1/2}(Y)}\, dH = \phi_\lambda(e^Y)\, \phi_\lambda(e^X).
\end{aligned}$$

We finish by ensuring that the kernel is Weyl-invariant in every argument. ∎

Corollary 2 *Suppose G is a complex group.*

1. *The support of the measure $\mu_{X,Y}$ is contained in $(\cup_{w\in W}\, w\cdot(C(X)+Y))\cap(\cup_{w\in W}\, w\cdot(C(Y)+X))\subset C(X)+C(Y)$.*

2. *$0\notin$ support($\mu_{X,Y}$).*

Proof:

1. We note that $K(X,H)$ is strictly positive for $H\in C(X)^\circ$ and 0 on the complement of $C(X)$ and we use the symmetry of the product formula in X and Y.

2. Suppose that $0\in\mathrm{support}(\mu_{X,Y})$. Then $0\in C(Y)+X$ and $0\in C(X)+Y$ which means that $-X\in C(Y)$ and $X\in -C(Y)=C(-Y)$. In the same way, $Y\in C(-X)$. This is only possible when Y belongs to the W-orbit of $-X$.

∎

Corollary 3 *$X+Y\in$ support($\mu_{X,Y}$).*

Proof: Without loss of generality we suppose that $X,\,Y\in\mathfrak{a}^+$.

Naturally, $X+Y\in C(X)+Y$. Suppose that $X+Y\in C(X)+w\cdot Y$ for $w\in W$. This means that $X-v=w\cdot Y-Y$ for a vector $v\in C(X)$. Let ${}^+\mathfrak{a}=\{H\in\mathfrak{a}\colon\ H=\sum_{i=1}^n c_i\,\alpha_i,\ c_i>0\}$ where $\alpha_1,\,\ldots,\,\alpha_n$ are the simple roots. Recall that if $H\in\mathfrak{a}^+$ and $w\in W$ then $H-w\,H\in\overline{{}^+\mathfrak{a}}$ ([5, Chapter IV]). It follows that $X-v\in\overline{{}^+\mathfrak{a}}$ and $w\cdot Y-Y\in-\overline{{}^+\mathfrak{a}}\cap\overline{{}^+\mathfrak{a}}=\{0\}$, so $w\cdot Y=Y$. As $Y\in\mathfrak{a}^+$, we deduce that $w=\mathrm{id}$.

The sets $C(X)+w\cdot Y$ being closed and bounded, it follows that a non-empty neighbourhood U of $X+Y$ is disjoint with all $C(X)+w\cdot Y$ except for $w=\mathrm{id}$.

By Theorem 1, for any $H\in U\cap(C(Y)+X)^\circ$ the function

$$k(H,X,Y)=\frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(Y)}\,\frac{1}{|W|}\,K(X,H-Y)>0.$$

Hence $X+Y\in\mathrm{support}(k(\cdot,X,Y))$. ∎

Remark 4 *The measures δ_{e^X} and δ_{e^Y} are not K-invariant on G, except the excluded cases $X,\,Y=0$. If δ_K denotes the Haar measure on K then define the K-biinvariant probability measures $\delta^\sharp_{e^X}$ and $\delta^\sharp_{e^Y}$ by convolving the Dirac masses with δ_K on both sides. Comparing the spherical Fourier transforms we see that*

$$\mu_{X,Y}=\delta^\sharp_{e^X}*\delta^\sharp_{e^Y}.$$

If we convolve two uniform distributions on centered spheres of radii $0<r<s$ in $\mathbf{R}^n$, we obtain an absolutely continuous measure supported by the annulus of radii $s-r$ and $s+r$. Our results show that a similar property holds on symmetric spaces with G complex; however the description of the support of $\delta^\sharp_{e^X}*\delta^\sharp_{e^Y}$, the symmetric space analogue of the annulus, is more complicated.

Let us give two simple applications of our product formula.

Corollary 5 *Let μ, ν be two K-biinvariant finite measures on G semisimple and complex. Then the measure $\mu * \nu$ is absolutely continuous.*

Proof: We identify K-biinvariant measures on G with W-invariant measures on $\mathfrak{a}$. Observe that the spherical Fourier transform of $\mu * \nu$ is equal to

$$\int_{\mathfrak{a}} \int_{\mathfrak{a}} \phi_\lambda(X)\phi_\lambda(Y) d\mu(X) d\nu(Y) = \hat{\gamma}(\lambda)$$

where γ is a K-biinvariant measure with density

$$d\gamma(H) = \int_{\mathfrak{a}} \int_{\mathfrak{a}} k(H, X, Y)\, d\mu(X)\, d\nu(Y).$$

The use of the Fubini theorem is justified by

$$\int_{\mathfrak{a}} k(H, X, Y)\, \delta(H)\, dH = 1$$

which is the product formula for $\lambda = -i\,\rho$ and by the boundedness of ϕ_λ. ∎

Corollary 6 *Let G be a simple complex Lie group and $g \notin K$. Then the orbit KgK generates G.*

Proof: Let $g = k_1\, e^X\, k_2$ with $X \in \mathfrak{a}$. The existence of a continuous density of $\delta_X^\natural * \delta_X^\natural = \mu_{X,X}$ implies that $K\,g\,K\,g\,K$ contains a non-empty K-biinvariant open set. ∎

3 An explicit product formula for the complex groups

The result [3, Proposition 2] give us a method to construct the Abel kernel K in (2) and therefore the product formula kernel k in (1).

Suppose $\alpha_1, \ldots, \alpha_q$ are the positive roots and $\alpha_1, \ldots, \alpha_n$ are the simple positive roots. We have integers $a_{kj} \geq 0$ such that

$$\alpha_k = \sum_{j=1}^n a_{kj}\, \alpha_j$$

for $k = n+1, \ldots, q$. For $y_1 \geq 0, \ldots, y_n \geq 0$, define

$$\Delta(y_1, \ldots, y_n) = \{(y_{n+1}, \ldots, y_q)\colon\ y_{n+1}, \ldots, y_q \geq 0 \text{ and } \sum_{k=1}^n a_{kj}\, y_k \leq y_j,\ j = 1, \ldots, n\}.$$

We then define $\Psi(y_1, \ldots, y_n) = \int_{\Delta(y_1,\ldots,y_n)} dy_{n+1} \cdots dy_q$ and $T(y_1\,\alpha_1 + \ldots + y_n\,\alpha_n) = \Psi(y_1, \ldots, y_n)$. The support of T is $\overline{^+\mathfrak{a}} = \{H \in \mathfrak{a}\colon\ H = y_1\,\alpha_1 + \ldots + y_n\,\alpha_n,\ y_i \geq 0,\ i = 1, \ldots, n\}$. If the rank is 1, then T jumps from 1 (inside its support) to 0 (outside its support). When the rank is greater than 1, T is continuous.

It is not difficult to see that Ψ will be locally a polynomial of degree $q - n$ in $y_1, \ldots, y_n$.

Note that T is the distribution on $\mathfrak{a}$ which satisfies

$$(T,f) = \int_{\mathbf{R}^q_+} f(\sum_{\alpha\in\Sigma+} x_k\,\alpha_k)\,dx_1\,\cdots\,dx_q.$$

We have $\partial(\pi)\,T = \delta_0$ and, in particular, $\mathcal{L}(T)(\lambda) = \frac{1}{\pi(\lambda)}$.
Then

$$K(X,H) = \frac{\pi(\rho)}{\delta^{1/2}(X)} \sum_{w\in W} \epsilon(w)\,T(w\,X - H). \tag{5}$$

One of the drawbacks of the formula (4) is that it is not immediately clear that $k(H,X,Y) = k(H,Y,X)$ for every X and $Y \in \mathfrak{a}$ (it is clear from (1) that this should be the case). The following result makes this symmetry explicit.

Proposition 7 *Suppose G is a complex Lie group. Then the kernel $k(H,X,Y)$ of Theorem 1 can be written as*

$$k(H,X,Y) = \frac{\pi(\rho)}{|W|}\frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(X)\,\delta^{1/2}(Y)} \sum_{v,w\in W} \epsilon(v)\,\epsilon(w)\,T(v\,X + w\,Y - H).$$

Proof: We have

$$\begin{aligned}
&k(H,X,Y)\\
&= \frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(Y)}\,\frac{1}{|W|} \sum_{w\in W} \epsilon(w)\,K(X, w\cdot H - Y)\\
&= \frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(Y)}\,\frac{1}{|W|} \sum_{w\in W} \epsilon(w)\,K(X, H - w^{-1}\cdot Y)\\
&= \frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(Y)}\,\frac{1}{|W|} \sum_{w\in W} \epsilon(w)\,\frac{\pi(\rho)}{\delta^{1/2}(X)} \sum_{v\in W} \epsilon(v)\,T(v\,X - (H - w^{-1}\cdot Y))\\
&= \frac{\pi(\rho)}{|W|}\frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(X)\,\delta^{1/2}(Y)} \sum_{v,w\in W} \epsilon(v)\,\epsilon(w)\,T(v\,X + w\,Y - H).
\end{aligned}$$

∎

Definition 8 *We will say that the function F is piecewise polynomial if there is a finite partition of support(F) into domains P satisfying $\overline{P^\circ} = P$ on which F is given by a fixed polynomial.*

We will say that the function F is piecewise continuous if there is a finite partition of support(F) into domains P satisfying $\overline{P^\circ} = P$ on which F is given by a continuous function.

Corollary 9 *The function $(H,X,Y) \to \delta^{1/2}(H)\,\delta^{1/2}(X)\,\delta^{1/2}(Y)\,k(H,X,Y)$ is a piecewise polynomial continuous function on its support.*

Remark 10 *It is interesting to note that $k(-H,X,-Y) = k(Y,X,H)$ (refer to (4)) and, in particular, that k is symmetric in H, X and Y if $-id \in W$ which is the case when $G = \mathbf{SL}(2,\mathbf{C})$. It is not difficult to find examples that show that this symmetry is not true when $G = \mathbf{SL}(3,\mathbf{C})$.*

Proposition 11 *Suppose G is a complex Lie group of rank greater than 1.*

1. *When $X \in \mathfrak{a}^+$, the function $H \to K(X,H)$ is continuous.*

   When $X \in \partial\mathfrak{a}^+ \setminus \{0\}$, the function $H \to K(X,H)$ is piecewise continuous. Moreover, if Δ_X denotes the set of all positive roots annihilating X then

$$K(X,H) = \frac{\pi(\rho)\,\prod_{\alpha\in\Delta_X} D_\alpha U(X,H)}{\prod_{\alpha\in\Delta_X}\|\alpha\|^2\,\prod_{\beta\in\Delta^+\setminus\Delta_X}\sinh\langle\beta,X\rangle} \tag{6}$$

   where $U(X,H) = \sum_{w\in W}\epsilon(w)\,T(w\,X - H)$ and D_α denotes the derivative in the direction of α.

2. *When $X,\ Y \in \mathfrak{a}^+$, the function $H \to k(H,X,Y)$ is continuous on $\mathfrak{a}^+$ and piecewise continuous on $\partial\mathfrak{a}^+ \setminus \{0\}$.*

   When $X \in \partial\mathfrak{a}^+ \setminus \{0\}$ or $Y \in \partial\mathfrak{a}^+ \setminus \{0\}$, the function $H \to k(H,X,Y)$ is piecewise continuous. Moreover, in this case, when $H \in \mathfrak{a}^+$

$$k(H,X,Y) = \frac{\pi(\rho)}{|W|}\,\frac{\prod_{\alpha\in\Delta_X} D^X_\alpha \prod_{\alpha\in\Delta_Y} D^Y_\alpha V(H,X,Y)}{\delta^{1/2}(H)\,\prod_{\alpha\in\Delta_X\cup\Delta_Y}\|\alpha\|^2\,\prod_{\beta\in\Delta^+\setminus\Delta_X}\sinh\langle\beta,X\rangle\,\prod_{\beta\in\Delta^+\setminus\Delta_Y}\sinh\langle\beta,Y\rangle}$$

   where $V(H,X,Y) = \sum_{v,w\in W}\epsilon(v)\,\epsilon(w)\,T(v\,X + w\,Y - H)$.

Proof:

1. The only case to be considered is $X \in \partial\mathfrak{a}^+ \setminus \{0\}$, *i.e.* X belongs to a wall of $\mathfrak{a}^+$. In the formula we have for K:

$$K(X,H) = \frac{\pi(\rho)}{\delta^{1/2}(X)}\,U(X,H),$$

   there is a singularity when $\delta^{1/2}(X) = 0$.

   As written in [3, (8)], the (ordinary) Fourier transform of $H \to U(X,H)$ is equal, up to a constant $\frac{1}{\pi(i\lambda)}$, to the numerator

$$\sum_{w\in W}\epsilon(w)\,e^{<iw\cdot\lambda,X>}$$

   of the formula for the spherical function ϕ_λ which is equal to $\frac{1}{\pi(\rho)}\,\delta^{1/2}(X)\,\phi_\lambda(e^X)$.

   The injectivity of Fourier transform and the properties of spherical functions imply that $U(X,H) = 0$ for all H if and only if $\alpha(X) = 0$ for a positive root α.

   We know that T is continuous and piecewise polynomial, and therefore, so is $U(X,H)$. From this, one may deduce that in a neighbourhood of X, the function $U(\cdot,H)$ is a product of $\prod_{\alpha\in\Delta_X}\langle\alpha,\cdot\rangle$ and a piecewise polynomial function. The formula (6) then follows.

2. The proof is similar, using Proposition 7 and Remark 10.

∎

The following examples are instructive.

1. Let $G = SL(3, \mathbf{C})$. For $X = A\alpha_1 + B\alpha_2 = [A, B - A, -B]$ and $H = u\,\alpha_1 + v\,\alpha_2 = [u, v-u, -v]$ in $\mathfrak{a}^+$, we have

$$K(X,H) = \frac{\min^+\{2\,A - B, A - u, B - v, 2\,B - A\}}{\sinh(2\,A - B)\,\sinh(2\,B - A)\,\sinh(A + B)}.$$

Note also that if $H \in C(X)^\circ$, we have $u < A$ and $v < B$ (see Lemma 12).

Now, take any $X \neq 0$ in $\{\alpha_1 = 0\} \cap \overline{\mathfrak{a}^+}$. We then have $X = x\alpha_1 + 2\,x\,\alpha_2$ with $x > 0$. If we fix $H \in \mathfrak{a}^+$ with $u < x$ and $v < 2\,x$, Proposition 11 tells us that

$$K(X,H) = \frac{1}{\sinh^2(3\,x)}.$$

That shows that $H \to K(X,H)$ is not continuous on $\partial C(X)$ since $K(X,H) = 0$ for H outside $C(X)$.

2. When X, $Y \in \mathfrak{a}^+$, $H \to k(H,X,Y)$ may not be continuous on $\overline{\mathfrak{a}^+}$ (consider for example $X = [4, 3, -7]$, $Y = [6, -2, -4]$ and $H = [2, 2, -4]$ on $\mathbf{SL}(3,\mathbf{C})/\mathbf{SU}(3)$).

Let us now consider an example where K and k are easy to compute. If $G = \mathbf{SL}(2,\mathbf{C})$, we have $T(X) = 1$ if $X \in \mathfrak{a}^+$ and 0 otherwise. This means that for X and $H \in \mathfrak{a}^+$, we have

$$\begin{aligned} K(X,H) &= \frac{\pi(\rho)}{\delta^{1/2}(X)}\,(T(X-H) - T(-X-H)),\\ &= \frac{\pi(\rho)}{\delta^{1/2}(X)} \qquad \text{if } X_2 \le H_1 < X_1 \text{ and 0 otherwise} \end{aligned}$$

and therefore if X, Y and $H \in \mathfrak{a}^+$,

$$k(H,X,Y) = \frac{\pi(\rho)}{\delta^{1/2}(H)\,\delta^{1/2}(X)\,\delta^{1/2}(Y)} \qquad \text{if } |X_1 - Y_1| < |H_1| \le X_1 + Y_1 \text{ and 0 otherwise.}$$

This formula is given in [6, Page 369].

However, even for $\mathbf{SL}(n,\mathbf{C})$, the computations become quickly onerous when $n > 3$. We will discuss the case $\mathbf{SL}(3,\mathbf{C})$ in the next section.

4 The support in the case of $\mathbf{SL}(3,\mathbf{C})$

In this section, we will assume throughout that $G = \mathbf{SL}(3,\mathbf{C})$. In this case, we have $T(X) = \min^+\{X_1, -X_3\}$ ($n = 2$ and $q = 3$) which brings

$$K(X,H) = \frac{\pi(\rho)}{\delta^{1/2}(X)} \times \\ \min^+\{X_1 - X_2, X_2 - X_3, X_1 - H_2, X_1 - H_2, X_1 - H_3, H_1 - X_3, H_2 - X_3, X_3 - Y_3\}.$$

The pictures of the support or the measure $\mu_{X,Y}$ is shown in Figure 1 (two cases are shown).

The following result will be used repeatedly in what follows to determine under which conditions an element H belongs to a set of the form $C(X) + Y$ with $X \in \mathfrak{a}^+$.

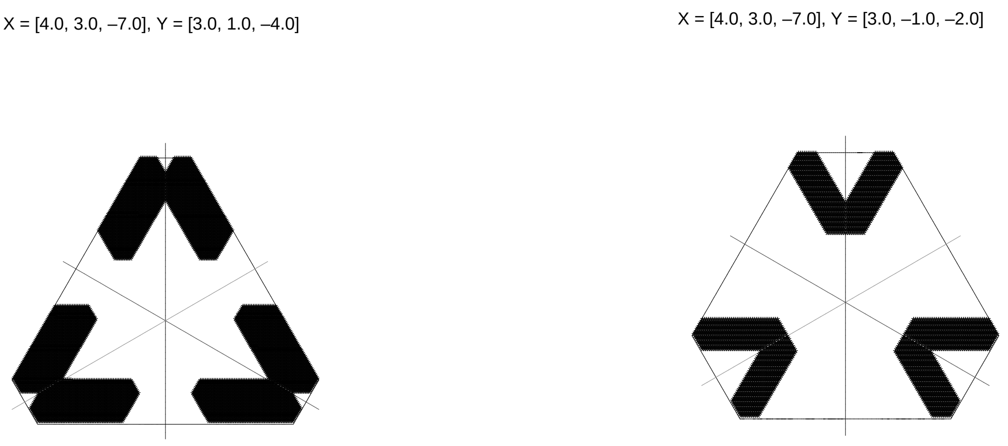

Figure 1: The support of $\mu_{X,Y}$

Lemma 12 *Suppose* $X \in \mathfrak{a}^+$*. Then* $C(X) = \{H \in \mathfrak{a}\colon\ X_3 \leq H_i \leq X_1,\ i = 1,2,3\}$ *and* $C(X)^\circ = \{H \in \mathfrak{a}\colon\ X_3 < H_i < X_1,\ i = 1,2,3\}$.

Proof: The sides of $C(X) \cap \mathfrak{a}^+$ which do not lie on the axes of symmetry belonging to W are given by $H_3 = X_3$ and $H_1 = X_1$. Since the coordinates of the origin satisfy $0 > X_3$ and $0 < X_1$, we have $\mathfrak{a}^+ \cap C(X) = \{H \in \mathfrak{a}^+\colon\ H_3 \geq X_3, H_1 \leq X_1\}$. The result follows by invariance under W; the elements of W act on $H = (H_1, H_2, H_3)$ by permuting the indices. ∎

Lemma 13 *Suppose* X *and* $Y \in \mathfrak{a}^+$*. Then*

$$(\cup_{w\in W}\, w \cdot (C(X) + Y)) \cap (\cup_{w\in W}\, w \cdot (C(Y) + X)) \cap \mathfrak{a}^+ = (C(X) + Y) \cap (C(Y) + X) \cap \mathfrak{a}^+.$$

Proof: Clearly, the set on the right hand side is included in the set on the left hand side.

Let $H \in ((\cup_{w\in W}\, w \cdot (C(X) + Y)) \cap (\cup_{w\in W}\, w \cdot (C(Y) + X))) \cap \mathfrak{a}^+$. We have

$$\begin{aligned} X_3 \leq \quad & H_i - Y_{w(i)} \quad \leq X_1, \\ Y_3 \leq \quad & H_i - X_{v(i)} \quad \leq Y_1 \end{aligned}$$

where $i = 1, \ldots, 3$ and w and $v \in W = S_3$. Recall that $H_1 > H_2 > H_3$, $X_1 > X_2 > X_3$ and $Y_1 > Y_2 > Y_3$. We have:

1. $H_1 - Y_1 \leq H_1 - Y_{v(1)} \leq X_1$.

2. $H_2 - Y_2 \leq H_2 - Y_{v(2)} \leq X_1$ if $v(2) = 2$ or 3. If $v(2) = 1$ then $v(1) = 2$ or 3. We then have $H_2 - Y_2 \leq H_1 - Y_{v(1)} \leq X_1$.

3. Let i be such that $v(i) = 3$. Then $H_3 - Y_3 \leq H_i - Y_{v(i)} \leq X_1$.

Using a similar approach, we show that $H_i - Y_i \geq X_3$ for each i and therefore, $H \in C(X) + Y$. In the same manner, $H \in C(Y) + X$. ∎

Note that $((C(X) + Y) \cap (C(Y) + X) \cap \mathfrak{a}^+)^\circ = (C(X)^\circ + Y) \cap (C(Y)^\circ + X) \cap \mathfrak{a}^+$.

Lemma 14 *Let X, $Y \in \mathfrak{a}^+$. Suppose $H \in (C(X)^\circ + Y) \cap (C(Y)^\circ + X) \cap \mathfrak{a}^+$. Then one of the following is true.*

1. *H belongs to no other $C(X)^\circ + w \cdot Y$.*
2. *H belongs to no other $C(Y)^\circ + v \cdot X$.*
3. *H belongs to exactly one other $C(X)^\circ + w \cdot Y$, $w \in W$.*
4. *H belongs to exactly one other $C(Y)^\circ + v \cdot X$, $v \in W$.*

Proof: Suppose the result is not true. This means that we can find $H \in (C(X)^\circ + Y) \cap (C(X)^\circ + w_1 \cdot Y) \cap (C(X)^\circ + w_2 \cdot Y) \cap (C(Y)^\circ + X) \cap (C(Y)^\circ + v_1 \cdot X) \cap (C(Y)^\circ + v_2 \cdot X) \cap \mathfrak{a}^+$ with $w_1 \neq e$, $w_2 \neq e$, $w_1 \neq w_2$ and $v_1 \neq e$, $v_2 \neq e$, $v_1 \neq v_2$.

In that case, we can find $i < 3$ such that $w_1(i) = 3$ or $w_2(i) = 3$ (aside from the identity, there is only one element of $W = S_3$ that fixes any given index). In the same way, we can find $j > 1$ such that $v_1(j) = 1$ or $v_2(j) = 1$. To simplify the notation, assume that $w_1(i) = 3$ and $v_1(j) = 1$. This means that $i \leq j$.

We have $H_i - Y_3 = H_i - Y_{w_1(i)} < X_1$ since $H \in C(X)^\circ + w_1 \cdot Y$ and $H_j - X_1 = H_j - X_{v_1(j)} > Y_3$ since $H \in C(Y)^\circ + v_1 \cdot X$. This means that $X_1 < H_j - Y_3$. Therefore $X_1 < H_j - Y_3 \leq H_i - Y_3 < X_1$ (recall that $i \leq j$) which is absurd. ∎

Proposition 15 *Suppose X, $Y \in \mathfrak{a}^+$. Let $S = (C(X)^0 + Y) \cap (C(Y)^\circ + X) \cap \mathfrak{a}^+$.*

Let $H \in \mathfrak{a}^+$. Then $k(H, X, Y)$ is nonzero (and therefore strictly positive) if and only if

$$H \in S \cap \{H_3 < X_2 + Y_2\} \cap \{H_1 > X_2 + Y_2\}.$$

*Note that if X and Y are both above ρ (*i.e.* $X_2 \geq 0$ and $Y_2 \geq 0$) then the condition $H_3 < X_2 + Y_2$ is automatically satisfied for $H \in \mathfrak{a}^+$. In the same manner, if X and Y are both below ρ (*i.e.* $X_2 \leq 0$ and $Y_2 \leq 0$) then the condition $H_1 > X_2 + Y_2$ is automatically satisfied for $H \in \mathfrak{a}^+$.*

Proof: If we refer to Corollary 2 and to Lemma 13, we can assume that $H \in (C(X) + Y) \cap (C(Y) + X) \cap \mathfrak{a}^+$ since otherwise $k(H, X, Y) = 0$.

Let S_0 be the set consisting of $H \in (C(X)^\circ + Y) \cap (C(Y)^\circ + X) \cap \mathfrak{a}^+$ such that H belongs to no other $C(X)^\circ + w \cdot Y$ or to no other $C(Y)^\circ + v \cdot X$, v, $w \in W$. For $i = 1$ and 2, let S_i be the set consisting of $H \notin S_0$ and $H \in (C(X) + Y)^\circ \cap (C(Y) + X)^\circ \cap (C(X)^\circ + w_i \cdot Y) \cap (C(Y)^\circ + w_i \cdot X) \cap \mathfrak{a}^+$ where $w_1 = (1 \to 1, 2 \to 3, 3 \to 2)$ and $w_2 = (1 \to 2, 2 \to 1, 3 \to 3) \in W$.

We will show that for $H \in \mathfrak{a}^+$, $k(H, X, Y) > 0$ if and only if

$$H \in S_0 \cup (S_1 \cap \{H_3 < X_2 + Y_2\}) \cup (S_2 \cap \{H_1 > X_2 + Y_2\}).$$

This will prove the result once we observe the following two facts:

1. If $H \in (C(X) + Y) \cap (C(Y) + X) \cap \mathfrak{a}^+$ does not belong to S_1 then $H_3 < X_2 + Y_2$.

   It is sufficient to prove that $H \in (C(X) + Y) \cap (C(Y) + X) \cap \mathfrak{a}^+$ and $H_3 \geq X_2 + Y_2$ imply that $H \in C(Y)^\circ + w_1 \cdot X$. Then, by symmetry of the above expressions in X and Y, we will also have $H \in C(X) + w_1 \cdot Y$ and therefore $H \in S_1$.

We note that $H \in C(Y)^\circ + w_1 \cdot X$ is equivalent to the inequalities: $Y_3 < H_1 - X_1 < Y_1$, $Y_3 < H_2 - X_3 < Y_1$ and $Y_3 < H_3 - X_2 < Y_1$.

The first inequality is obvious since $H \in C(Y)+X^\circ$, $Y_3 < H_2 - X_3$ is true since $H_2 > H_3 \geq X_2 + Y_2 > X_3 + Y_3$. Suppose $H_2 - X_3 < Y_1$ is false. Then $-H_1 - H_3 = H_2 \geq X_3 + Y_1$ and $H_3 \leq -X_3 - Y_1 - H_1$ which combined with $H_3 \geq X_2 + Y_2$ yields $X_2 + Y_2 \leq -X_3 - Y_1 - H_1$ or $X_2 + X_3 + Y_1 + Y_2 \leq -H_1$ *i.e.* $-X_1 - Y_3 \leq -H_1$ which contradicts $H_1 - X_1 > Y_3$ since $H \in C(Y) + X$. Finally, $H_3 - X_2 < Y_1$ holds because $H_3 - X_2 < H_2 - X_2 < Y_1$.

2. If $H \in (C(X) + Y) \cap (C(Y) + X) \cap \mathfrak{a}^+$ does not belong to S_2 then $H_1 > X_2 + Y_2$.

   The proof is similar.

Consider now Lemma 14. If cases 1. or 2. are verified, then $H \in S_0$. In that case, we either have $k(H, X, Y) = \frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(Y)} \frac{1}{|W|} K(X, H-Y) > 0$ or $k(H, X, Y) = \frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(Y)} \frac{1}{|W|} K(Y, H-X) > 0$.

If $H \notin S_0$ then H satisfies cases 3. and 4. of Lemma 14 and we have $H \in (C(X) + Y)^\circ \cap (C(Y) + X)^\circ \cap (C(X)^\circ + w \cdot Y) \cap (C(Y)^\circ + v \cdot X) \cap \mathfrak{a}^+$. Note that we cannot have $w(1) = 3$ or $w(3) = 1$ (and similarly for v). Indeed, if $w(1) = 3$ then $H_1 - Y_3 < X_1$ which means that $H_1 - X_1 < Y_3$ which is absurd while if $w(3) = 1$ then $H_3 - X_1 > Y_3$ which means that $H_3 - Y_3 > X_1$ which is absurd. Therefore, the only possibilities for w and v are w_1 and w_2. We also have $v = w$. Indeed, if we had $v \neq w$, it is not difficult to see by inspection (say by taking $w = w_1$ and $v = w_2$) that we would reach a contradiction by using a similar argument. We then have $k(H, X, Y) = \frac{1}{\delta^{1/2}(H)\,\delta^{1/2}(Y)} \frac{1}{|W|} (K(X, H - Y) - K(X, H - w_i \cdot Y)) > 0$ since $\epsilon(w_1) = \epsilon(w_2) = -1$.

It remains to show that for $H \in S_1$, $k(H, X, Y) > 0$ if and only if $H_3 < X_2 + Y_2$ and that for $H \in S_2$, $k(H, X, Y) > 0$ if and only if $H_1 > X_2 + Y_2$. Since the reasoning in the two cases are very similar, we will show only the first case.

Suppose $H \in S_1$. We deduce easily that $X_1 + Y_3 - H_2$ is strictly smaller than $X_1 - X_2$, $X_1 + Y_2 - H_2$, $X_1 + Y_3 - H_3$ and $X_1 + Y_2 - H_3$ while $H_3 - Y_2 - X_3$ is strictly smaller than $X_1 + Y_1 - H_1$, $H_2 - Y_2 - X_3$, $H_3 - Y_3 - X_3$ and $H_2 - Y_3 - X_3$. This implies that $K(X, H-Y) - K(X, H - w_1 \cdot Y) > 0$ is equivalent to

$$\min\{X_2 - X_3, H_1 - Y_1 - X_3\} > \min\{X_1 + Y_3 - H_2, H_3 - Y_2 - X_3\}. \tag{7}$$

Note that $X_2 - X_3 > H_3 - Y_2 - X_3$ and $H_1 - Y_1 - X_3 > X_1 + Y_3 - H_2$ are both equivalent to $H_3 < X_2 + Y_2$. The latter inequality is therefore sufficient for (7) to be true. It remains to show that it is necessary.

Now, we get down to several cases:

1. Suppose $X_1 - X_2 \leq X_2 - X_3$ and $Y_1 - Y_2 \leq Y_2 - Y_3$ (*i.e.* $X_2 \geq 0$ and $Y_2 \geq 0$). Since $H \in \mathfrak{a}^+$, the condition $H_3 < X_2 + Y_2$ is satisfied and there is nothing to prove.

2. Suppose $X_1 - X_2 \leq X_2 - X_3$ and $Y_1 - Y_2 > Y_2 - Y_3$ (*i.e.* $X_2 \geq 0$ and $Y_2 < 0$).

   Suppose $H_1 - Y_1 - X_3 > \min\{X_1 + Y_3 - H_2, H_3 - Y_2 - X_3\}$ and $H_3 \geq X_2 + Y_2$. That is only possible if $H_1 - Y_1 > H_3 - Y_2$.

   We have $H_1 - Y_1 > H_3 - Y_2 \geq X_2 + Y_2 - Y_2 = X_2 \geq 0$. Now, $H_3 \geq X_2 + Y_2$ if and only if $-Y_2 \geq -H_3 + X_2$ which implies $Y_1 + Y_3 = -Y_2 > -H_3 = H_1 + H_2$ which is equivalent to

$Y_1 > H_1 + (H_2 - Y_3) > H_1$ since $H_2 - Y_3 > X_2 \geq 0$ ($H_2 - X_2 > Y_3$ since $H \in C(Y)^\circ + X$). This contradicts $H_1 - Y_1 > 0$.

3. By symmetry, we do not have to consider the case $Y_1 - Y_2 \leq Y_2 - Y_3$ and $X_1 - X_2 > X_2 - X_3$.

4. Suppose $X_1 - X_2 > X_2 - X_3$ and $Y_1 - Y_2 > Y_2 - Y_3$ (*i.e.* $X_2 < 0$ and $Y_2 < 0$).

   Suppose that (7) is true and that $H_3 \geq X_2 + Y_2$. This means that $X_2 - X_3 \leq H_3 - Y_2 - X_3$ and $H_1 - Y_1 - X_3 \leq X_1 + Y_3 - H_2$.

   We consider two cases:

   (a) $H_1 - Y_1 - X_3 \geq X_2 - X_3 > X_1 + Y_3 - H_2$:
   We have $H_1 - Y_1 - X_3 \geq X_2 - X_3$ if and only if $H_1 - Y_1 \geq X_2$. On the other hand, $H_3 \geq X_2 + Y_2$ if and only if $-X_2 + Y_1 + Y_3 \geq H_1 + H_2$. To this last inequality, we apply $X_2 - X_3 > X_1 + Y_3 - H_2$ if and only if $H_2 > X_1 + Y_3 - X_2 + X_3$. We obtain $-X_2 + Y_1 + Y_3 > H_1 + X_1 + Y_3 - X_2 + X_3$ if and only if $Y_1 > H_1 + X_1 + X_3 = H_1 - X_2$ if and only if $X_2 > H_1 - Y_1$. This contradicts $H_1 - Y_1 \geq X_2$.

   (b) $X_2 - X_3 > H_1 - Y_1 - X_3 > H_3 - Y_2 - X_3$:
   We have $X_2 - X_3 > H_1 - Y_1 - X_3$ if and only if $X_2 > H_1 - Y_1$. On the other hand, $H_1 - Y_1 - X_3 > H_3 - Y_2 - X_3$ if and only if $-H_3 > -H_1 + Y_1 - Y_2$. We have $H_3 \geq X_2 + Y_2$ if and only if $-X_2 > -H_3 + Y_2 > -H_1 + Y_1 - Y_2 + Y_2 = -H_1 + Y_1$ which is equivalent to $X_2 < H_1 - Y_1$. This contradicts $X_2 > H_1 - Y_1$.

∎

Remark 16 *Let H, X and $Y \in \mathfrak{a}^+$. We note that computing $k(H, X, Y)$ requires one evaluation of K when $H \in S_0$ while it requires taking the difference of two values of K when $H \in S_1 \cap \{H_3 < X_2 + Y_2\}$ or $H \in S_2 \cap \{H_1 > X_2 + Y_2\}$.*

Remark 17 *When we refer to Figure 1, we can describe the support in a more informal and more concrete manner:*

$$\mathit{support}(\mu_{X,Y}) = C \setminus (D_1 \cup D_2)$$

where $C = (\cup_{w \in W}\, w \cdot (C(X) + Y)) \cap (\cup_{w \in W}\, w \cdot (C(Y) + X))$ and D_1, D_2 are either empty or equilateral triangles in the plane $\mathfrak{a}$ such that 0 is their centre and such that a side is, respectively, on the line $-v_H = H_3 = X_2 + Y_2 < 0$ ($u_H = H_1 = X_2 + Y_2 > 0$).

Naturally, $D_1 = W \cdot (\{H_3 > X_2 + Y_2\} \cap \mathfrak{a}^+$ and $D_2 = W \cdot (\{H_1 < X_2 + Y_2\} \cap \mathfrak{a}^+)$.

5 The function T in the case of $\mathbf{SL}(n, \mathbf{C})$

By Proposition 7, in order to know the kernel $k(H, X, Y)$ of the product formula, it is sufficient to know explicitly the function T defined in Section 3. We give here some more information available about the function T in the case $G = \mathbf{SL}(n, \mathbf{C})$.

Note that when writing $\lambda \in \mathfrak{a}^*$ in terms of the simple positive roots, *i.e.* $\lambda = \sum_{i=1}^{n-1} a_i\, \alpha_i$, we find that

$$\pi(\lambda) = \prod_{\alpha>0} \langle \lambda, \alpha \rangle = \prod_{i=1}^{n-1} \left[a_i \, (a_i + a_{i+1}) \cdots (a_i + \ldots + a_{n-1}) \right]. \tag{8}$$

Using Maple, it is possible to compute the function T for $\mathbf{SL}(4, \mathbf{C})$ since it is simply a matter of computing the Laplace inverse transform of $\frac{1}{\pi(\lambda)}$.

Recall that $T(y_1\, \alpha_1 + y_2\, \alpha_2 + y_3\, \alpha_3) = 0$ unless all y_i's are positive. Let $x_+ = \max\{0, x\}$. We find

$$T(y_1\, \alpha_1 + y_2\, \alpha_2 + y_3\, \alpha_3) = \begin{cases} y_2^3 & 0 \le y_2 \le \min\{y_1, y_3\} \\ -2\, y_1^3 + 3\, y_1^2\, y_2 & 0 \le y_1 \le y_2 \le y_3 \\ -2\, y_3^3 + 3\, y_3^2\, y_2 & 0 \le y_3 \le y_2 \le y_1 \\ -y_1^3 + 3\, y_1^2\, y_3 - (y_1 + y_2 - y_3)_+^3 & 0 \le y_1 \le y_3 \le y_2 \\ -y_3^3 + 3\, y_3^2\, y_1 - (y_1 + y_2 - y_3)_+^3 & 0 \le y_3 \le y_1 \le y_2 \end{cases}$$

A more general result for the function T:

Proposition 18 *The function T for $\mathbf{SL}(n, \mathbf{C})$ is given by*

$$\begin{aligned} & T(y_1\, \alpha_1 + \ldots + y_{n-1}\, \alpha_{n-1}) \\ & = \mathbf{1}_{\{0 \le y_1\}}\, \delta_0(dy_2, \ldots, dy_{n-1}) * \mathbf{1}_{\{y_1 \le y_2\}}\, \delta_0(dy_3, \ldots, dy_{n-1}) * \mathbf{1}_{\{y_1 \le y_2 \le y_3\}}\, \delta_0(dy_4, \ldots, dy_{n-1}) \\ & \qquad * \cdots * \mathbf{1}_{\{y_1 \le y_2 \le \ldots \le y_{n-2}\}}\, \delta_0(dy_{n-1}) * \mathbf{1}_{\{y_1 \le y_2 \le \ldots \le y_{n-1}\}}. \end{aligned}$$

Proof: If we consider (8), we can write $\frac{1}{\pi(\lambda)}$ as

$$\frac{1}{\pi(\lambda)} = \prod_{k=1}^{n-1} \left(\frac{1}{a_k\, (a_{k-1} + a_k)\, (a_{k-2} + a_{k-1} + a_k) \cdots (a_1 + a_2 + \ldots + a_{k-1} + a_k)} \right)$$

and then compute the inverse Laplace transform of each factor. ∎

Lemma 19 *If $X \in \mathfrak{a}^+$ then $C(X) = \{H\colon\ \sum_{i=1}^r H_{k_i} \le \sum_{i=1}^r X_i, (k_i) \in S_n, r \le n-1\}$.*

Proof: Similar to the proof of Lemma 12. ∎

Corollary 20 *On $\mathbf{SL}(n, \mathbf{C})$, the convex envelope of the support of $\mu_{X,Y}$ is $C(X+Y) = C(X) + C(Y)$.*

Proof: One observes easily that $C(X+Y) = C(X) + C(Y)$ using the above lemma. We then use Corollary 2, Corollary 3 and the fact that the support is Weyl invariant. ∎

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