# INDEPENDENT LINEAR STATISTICS ON FINITE ABELIAN GROUPS

**G.M. FELDMAN, [1] and P. GRACZYK**

AMS 1991 SUBJECT CLASSIFICATION: Primary 60 B 15; secondary 62 E 10.

KEY WORDS AND PHRASES: Characterization of probability distributions, independence of linear statistics, finite Abelian group.

**Abstract**

Let $X$ be a finite Abelian group, $\xi_j,\ j = 1, 2, 3$ independent random variables taking on values in $X$ and with distributions $\mu_j$, and $\alpha_j,\ \beta_j$ automorphisms of $X$. We obtain the complete description of groups $X$ for which the independence of linear statistics $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2) + \alpha_3(\xi_3)$ and $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2) + \beta_3(\xi_3)$ implies that either one or two or three of the distributions $\mu_j$ are idempotent.

[1]This research was supported in part by Programme de Bourses de Recherche Scientifique et Technique de l'OTAN (France)

It is well known that if $\xi_j,\ j=1,2,...,n,\ n\geq 2$, are independent random variables and linear statistics $L_1=\alpha_1\xi_1+\cdots+\alpha_n\xi_n$ and $L_2=\beta_1\xi_1+\cdots+\beta_n\xi_n$, where coefficients $\alpha_j,\beta_j$ are different from zero, are independent, then $\xi_j$ are Gaussian. This theorem proved by Skitovich ([15]) and Darmois ([1]) summarizes a series of researches begun by Bernstein who proved that the independence of the sum and the difference of two independent random variables implies that they are Gaussian. The Skitovich-Darmois theorem was generalized by Ghurye and Olkin. They received the analogous theorem, where instead of independent random variables, independent random vectors are considered, and coefficients $\alpha_j,\beta_j$ are nonsingular matrices ([9], see also [10, Ch. 3]). Let us consider the following general statement of the problem.

Let $X$ be a locally compact Abelian separable group, $\mathrm{Aut}(X)$ the group of topological automorphisms of $X$, $\Gamma(X)$ the set of Gaussian distributions on the group $X$. (We note that on totally disconnected groups, in particular, on finite groups, the Gaussian distributions are degenerate (see [14])). Let $I(X)$ be the set of idempotent distributions $X$, i.e. the set of shifts of Haar distributions $m_K$ of compact subgroups $K$ of $X$. Let $\xi_j,\ j=1,2,...,n,\ n\geq 2$, be independent random variables taking on values in $X$ and with distributions $\mu_j$. Consider the linear statistics $L_1=\alpha_1(\xi_1)+\cdots+\alpha_n(\xi_n)$ and $L_2=\beta_1(\xi_1)+\cdots+\beta_n(\xi_n),\ n\geq 2$, where $\alpha_j,\beta_j\in\mathrm{Aut}(X)$.

P R O B L E M. *For which groups $X$ the independence of $L_1$ and $L_2$ implies that all $\mu_j\in\Gamma(X)*I(X)$.*

(The shifts on the idempotent distributions appear in the natural way in the characterization problems on groups (see [3, Ch. 3]))

This problem was solved in the classes of finite Abelian groups ([2]), compact Abelian groups ([4]) and discrete torsion Abelian groups ([5]). [2] It was discovered in [6] that for $n=2$ the Skitovich-Darmois theorem is valid for all finite Abelian groups, whereas the class of finite Abelian groups for which the Skitovich-Darmois theorem is valid for any $n\geq 2$ is poor enough. (The situation is in contrast to the classical case, where there is not a distinction between $n=2$ and arbitrary $n$.) Denote by $\mathbb{Z}(n)$ the group residues modulo $n$. The following theorem has been proved in [2].

**Theorem A.** *Let $X$ be a finite Abelian group of the form*

$$\mathbb{Z}(2^{m_1})+...+\mathbb{Z}(2^{m_l}),\quad 0\leq m_1<...<m_l. \tag{1}$$

*Then the independence of $L_1$ and $L_2$ implies that all $\mu_j$ are degenerate distributions. If $X$ is of the form*

$$\mathbb{Z}(3)+G, \tag{2}$$

*where a group $G$ is the same as in (1), then there exist shifts $\mu_j'$ of $\mu_j$ such that the supports of all $\mu_j'$ are in $\mathbb{Z}(3)$ and either all $\mu_j$ are degenerate distributions*

[2]We remark that in the last time a series of articles was published where the analogues of the characterization theorems by Bernstein and Skitovich-Darmois were considered in the non Abelian case, namely on Lie groups, quantum groups, symmetric spaces (see [8], [11]-[13]).

*or* $\mu'_{j_1} = \mu'_{j_2} = m_{\mathbb{Z}(3)}$ *for at least two distributions* $\mu_{j_1}$ *and* $\mu_{j_2}$ *and the rest of* $\mu_j$ *are arbitrary.*

*If* $X$ *is not isomorphic to the groups of the form either* (1) *or* (2)*, then for any* $n \geq 4$ *there exist independent random variables* $\xi_j,\ j = 1, 2, ..., n$ *taking on values in* $X$ *and with distributions* $\mu_j$*, and* $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ *such that* $L_1$ *and* $L_2$ *are independent, whereas all* $\mu_j \notin I(X)$.

On the other hand for $n = 2$ the following statement takes place.

**Theorem B** ([6]). *Let* $\xi_1, \xi_2$ *be independent random variables taking on values in a finite Abelian group* $X$ *and with distributions* $\mu_1, \mu_2$*. Then the independence of* $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$ *and* $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2)$*, where* $\alpha_j, \beta_j \in \mathrm{Aut}(X)$*, implies that* $\mu_j = m_{K_j} * E_{x_j}$*, where* $K_j$ *is a subgroup of* $X$*, and* $x_j \in X,\ j = 1, 2$.

Thus the case when the number of random variables $n = 3$ for finite Abelian groups was not investigated yet. Its study is the aim of this work. We will prove the following statement.

**Theorem 1.** *Let* $\xi_j,\ j = 1, 2, 3$ *be independent random variables taking on values in a finite Abelian group* $X$ *and with distributions* $\mu_j$*. If* $X$ *is of the form*

$$\mathbb{Z}(5) + G, \tag{3}$$

*where a group* $G$ *is the same as in (1), then the independence of* $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2) + \alpha_3(\xi_3)$ *and* $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2) + \beta_3(\xi_3)$*, where* $\alpha_j, \beta_j \in \mathrm{Aut}(X)$*, implies that there exist shifts* $\mu'_j$ *of* $\mu_j$ *such that the supports of all* $\mu'_j$ *are in* $\mathbb{Z}(5)$*, and one of the following statements is valid:*

*(i) all* $\mu_j$ *are degenerate distributions;*

*(ii)* $\mu'_{j_1} = \mu'_{j_2} = m_{\mathbb{Z}(5)}$ *for two distributions* $\mu_{j_1}$ *and* $\mu_{j_2}$*, and* $\mu_{j_3}$ *is arbitrary;*

*(iii)* $\mu'_{j_1} = m_{\mathbb{Z}(5)}$ *for one distribution* $\mu_{j_1}$ *and characteristic functions of distributions* $\mu_{j_2}$ *and* $\mu_{j_3}$ *vanish.*

*If* $X$ *is not isomorphic to the groups of the form* $(1)-(3)$*, then there exist independent random variables* $\xi_j,\ j = 1, 2, 3$ *taking on values in* $X$ *and with distributions* $\mu_j$*, and* $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ *such that* $L_1$ *and* $L_2$ *are independent, whereas all* $\mu_j \notin I(X)$.

Before proving the main result we shall agree about notation. Denote by $Y = X^*$ the character group of $X$, by $(x, y)$ the value of a character $y \in Y$ on an element $x \in X$. For any $\alpha \in \mathrm{Aut}(X)$ define the conjugate automorphism $\tilde{\alpha} \in \mathrm{Aut}(Y)$ by the rule $(x, \tilde{\alpha}(y)) = (\alpha(x), y)$ for all $x \in X,\ y \in Y$. The convolution semigroup of probability distributions on $X$ denote by $M^1(X)$. If $\mu \in M^1(X)$, then denote by $\hat{\mu}(y) = \int_X (x, y) d\mu(x)$ its characteristic function. Observe that if $X$ is a compact group, then the characteristic function of Haar distribution $m_X$ is of the following form

$$\hat{m}_X(y) = \begin{cases} 1, & if \quad y = 0, \\ \\ 0, & if \quad y \neq 0. \end{cases}$$

Denote by $\{0, 1, , ..., n-1\}$ the elements of the groups $\mathbb{Z}(n) \approx (\mathbb{Z}(n))^*$.

If $\xi_j, \ j = 1, 2, ..., n, \ n \geq 2$ are independent random variables taking on values in a group $X$ and with distributions $\mu_j$, in complete analogy with the classical case, the independence of linear statistics $L_1 = \alpha_1(\xi_1) + \cdots + \alpha_n(\xi_n)$ and $L_2 = \beta_1(\xi_1) + \cdots + \beta_n(\xi_n), \ \alpha_j, \beta_j \in \mathrm{Aut}(X)$ is equivalent to the fact that the characteristic functions of the distributions $\mu_j$ satisfy the equation

$$\prod_{j=1}^{n} \hat{\mu}_j(\tilde{\alpha}_j u + \tilde{\beta}_j v) = \prod_{j=1}^{n} \hat{\mu}_j(\tilde{\alpha}_j u) \prod_{j=1}^{n} \hat{\mu}_j \tilde{\beta}_j v) \quad u, v, \in Y. \tag{4}$$

We break down the proof of Theorem 1 into some lemmas.

**Lemma 1.** *Let $\xi_j, \ j = 1, 2, 3$ be independent random variables taking on values in $X = \mathbb{Z}(5)$ and with distributions $\mu_j$. Then the independence of $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \delta_1(\xi_1) + \delta_2(\xi_2) + \delta_3(\xi_3)$, where $\delta_j \in \mathrm{Aut}(X)$, implies that one of the following statements is valid:*

*$(i)$ all $\mu_j$ are degenerate distributions;*

*$(ii)$ $\mu_{j_1} = \mu_{j_2} = m_{\mathbb{Z}(5)}$ for two distributions $\mu_{j_1}$ and $\mu_{j_2}$, and $\mu_{j_3}$ is arbitrary;*

*$(iii)$ $\mu_{j_1} = m_{\mathbb{Z}(5)}$ for one distribution $\mu_{j_1}$ and characteristic functions of distributions $\mu_{j_2}$ and $\mu_{j_3}$ vanish.*

**Proof.** We assume at first that not all $\delta_j$ are different, say $\delta_2 = \delta_3$. Put $\xi_2' = \xi_2 + \xi_3$. Then the random variables $\xi_1$ and $\xi_2'$ are independent, and $\xi_2'$ has the distribution $\mu_2 * \mu_3$. Obviously the linear statistics $L_1 = \xi_1 + \xi_2'$ and $L_2 = \delta_1(\xi_1) + \delta_2(\xi_2')$ are independent. It follows easily from Theorem B that either all $\mu_j$ are degenerate distributions and hence $(i)$ is fulfilled or $\mu_1 = \mu_2 * \mu_3 = m_{\mathbb{Z}(5)}$ and then either $(ii)$ or $(iii)$ is fulfilled.

Let us assume now that all $\delta_j$ are different. It is easily seen that $\mathrm{Aut}(\mathbb{Z}(5)) \approx \mathbb{Z}(4)$ and each automorphism $\delta \in \mathrm{Aut}(\mathbb{Z}(5))$ has the form $\delta(x) = kx, \ k = 1, 2, 3, 4, \ x \in \mathbb{Z}(5)$. Since $L_1$ and $L_2$ are independent if and only if $L_1$ and $\delta(L_2)$ are independent, we can assume without restricting generality, that $L_2 = \xi_1 + 2\xi_2 + 3\xi_3$. Denote $f(y) = \hat{\mu}_1(y), \ g(y) = \hat{\mu}_2(y), \ h(y) = \hat{\mu}_3(y)$. Then equation (4) becomes

$$f(u+v)g(u+2v)h(u+3v) = f(u)g(u)h(u)f(v)g(2v)h(3v), \quad u, v, \in Y. \tag{5}$$

Let us assume that all $\hat{\mu}_j(y)$ do not vanish. Denote

$$\mathrm{F} = \prod_{y \in Y} f(y), \quad \mathrm{G} = \prod_{y \in Y} g(y), \quad \mathrm{H} = \prod_{y \in Y} h(y).$$

Put $u = v = y$ in (5), consider the obtained equality for each $y \in Y$ and multiply them. We receive $\mathrm{FGH} = \mathrm{F}^2\mathrm{G}^2\mathrm{H}^2$. Hence, $\mathrm{FGH} = 1$ and for this reason $|f(y)| = |g(y)| = |h(y)| \equiv 1$. It implies that all $\mu_j$ are degenerate distributions, i.e. $(i)$ is fulfilled.

Observe now that it is impossible that only one of the characteristic functions vanishes. It follows from the fact that for any $u \in Y$, $u \neq 0$ and any $\tilde{\delta} \in \mathrm{Aut}(Y)$ there exists an element $v \in Y$ such that $u + \tilde{\delta} v = 0$, and $\hat{\mu}_j(0) \neq 0$ for all $j = 1, 2, 3$.

Let us assume that only two of the characteristic functions vanish, say $f(y)$ and $g(y)$. But then they must have common zeros. In fact, in the opposite case we have then $f(u_0) = g(2u_0) = 0$, $f(2u_0) \neq 0$, $g(u_0) \neq 0$ for some $u_0 \in Y$, and hence for $u = 3u_0$, $v = 4u_0$ the right-hand side of (5) vanishes, whereas the left-hand side does not vanish. The remaining cases, when either $f(y)$ and $h(y)$ or $g(y)$ and $h(y)$ vanish, can be considered similarly.

Thus, $f(y)$ and $g(y)$ have common zeros. Let $f(u_0) = g(u_0) = 0$. Substituting $u = u_0$, $v = 2u_0$ into (5) we receive that $f(3u_0) = 0$, i.e. $\mu_1 = m_{\mathbb{Z}(5)}$. Sustituting $u = 3u_0$, $v = 2u_0$ in (5) we obtain that $g(2u_0) = 0$, i.e. $\mu_2 = m_{\mathbb{Z}(5)}$. It follows from (5) that if $\mu_1 = \mu_2 = m_{\mathbb{Z}(5)}$, then the distribution $\mu_3$ can be arbitrary. In this case $(ii)$ is valid.

Let us assume now that all three characteristic functions $f(y)$, $g(y)$ and $h(y)$ vanish. Consider at first the case when there is a point $u_0 \in Y$ such that

$$f(u_0) = g(u_0) = h(u_0) = 0. \tag{6}$$

Put $u = u_0$, $v = 2u_0$ in (5), we receive $f(3u_0)h(2u_0) = 0$, i.e. either $\mu_1 = m_{\mathbb{Z}(5)}$ or $\mu_3 = m_{\mathbb{Z}(5)}$ and $(iii)$ is fulfilled.

Let us assume that (6) is not valid. The following three cases are possible.

1) $f(u_0) = g(u_0) = h(2u_0) = 0$, $h(u_0) \neq 0$ for some $u_0 \in Y$. Putting $u = 2u_0$, $v = 3u_0$ in (5), we obtain that $g(3u_0) = 0$, i.e. $\mu_2 = m_{\mathbb{Z}(5)}$ and $(iii)$ is fulfilled.

2) $f(u_0) = g(2u_0) = h(u_0) = 0$, $g(u_0) \neq 0$ for some $u_0 \in Y$. Put $u = 2u_0$, $v = u_0$ in (5), we receive $f(3u_0) = 0$, i.e. $\mu_1 = m_{\mathbb{Z}(5)}$. Put $u = u_0$, $v = 4u_0$ in (5), we obtain that $h(3u_0) = 0$, i.e. $\mu_3 = m_{\mathbb{Z}(5)}$. It is easily seen from (5) that if $\mu_1 = \mu_3 = m_{\mathbb{Z}(5)}$ then the distribution $\mu_2$ can be arbitrary. In this case $(ii)$ is valid.

3) $f(2u_0) = g(u_0) = h(u_0) = 0$, $f(u_0) \neq 0$ for some $u_0 \in Y$. Putting $u = 4u_0$, $v = 2u_0$ in (5), we receive that $g(3u_0) = 0$, i.e. $\mu_2 = m_{\mathbb{Z}(5)}$ and $(iii)$ is fulfilled.

Lemma 1 is proved completely.

**Lemma 2.** *For each of the groups $X = \mathbb{Z}(3) + \mathbb{Z}(3)$, $X = \mathbb{Z}(5) + \mathbb{Z}(5)$, $X = \mathbb{Z}(2^k) + \mathbb{Z}(2^k)$ there exist independent random variables $\xi_j$, $j = 1, 2, 3$ taking on values in $X$ and with distributions $\mu_j$ and $\alpha, \beta \in \mathrm{Aut}(X)$, such that $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \xi_1 + \alpha(\xi_2) + \beta(\xi_3)$ are independent, whereas all $\mu_j \notin I(X)$.*

**Proof.** Let $X = \mathbb{Z}(3) + \mathbb{Z}(3)$. We denote elements of the groups $X$ and $Y$ by $(p, q)$, $p, q \in \mathbb{Z}(3)$. Consider on the group $X$ the distribution $\mu(\{x\}) = \frac{1}{9}(1 + \mathrm{Re}(x, (0, 1)))$, $x \in X$. It is obvious that $\hat{\mu}(p, q) = 0$ if $p \neq 0$. Consider the automorphisms $\alpha, \beta \in \mathrm{Aut}(X)$ of the form

$$\alpha(p, q) = (p, p + q), \quad \beta(p, q) = (q, p), \quad p, q \in \mathbb{Z}(3).$$

Then

$$\tilde{\alpha}(p, q) = (p + q, q), \quad \tilde{\beta}(p, q) = (q, p), \quad p, q \in \mathbb{Z}(3).$$

Let $\xi_j$, $j = 1, 2, 3$ be independent identically distributed random variables taking on values in $X$ and with distribution $\mu$. We verify that $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \xi_1 + \alpha(\xi_2) + \beta(\xi_3)$ are independent. To this end it suffices to make sure that equation (4), which takes the form

$$\hat{\mu}(u + v)\hat{\mu}(u + \tilde{\alpha}v)\hat{\mu}(u + \tilde{\beta}v) = \hat{\mu}^3(u)\hat{\mu}(v)\hat{\mu}(\tilde{\alpha}v)\hat{\mu}(\tilde{\beta}v), \quad u, v \in Y, \qquad (7)$$

is valid for any $u, v \in Y$. Obviously equation (7) is fulfilled if either $u = 0$ or $v = 0$. So, we can verify (7) supposing that $u \neq 0$, $v \neq 0$. Denote by $a = (1, 0)$, $b = (0, 1)$ the generators of $Y$. Then $u = pa + qb$, $v = p'a + q'b$, $p, p', q, q' \in \{0, 1, 2\}$. If either $p \neq 0$ or $p' \neq 0$, then either $\hat{\mu}(u) = 0$ or $\hat{\mu}(v) = 0$ respectively, and the right-hand side of (7) vanishes. If $p = p' = 0$, then $v = q'b$, $q' \neq 0$ and $\hat{\mu}(\tilde{\alpha}v) = \hat{\mu}(\tilde{\alpha}q'b) = \hat{\mu}(q'a + q'b) = \hat{\mu}(q', q') = 0$. Thus, the right-hand side of (7) vanishes if $u \neq 0$, $v \neq 0$. To see that the left-hand side of (7) vanishes too assume at first that $p + p' \neq 0 (\mathrm{mod}\ 3)$. In this case $\hat{\mu}(u + v) = 0$. If $p + p' = 0 (\mathrm{mod}\ 3)$, and $q' \neq 0$, then $u + \tilde{\alpha}v = pa + qb + \tilde{\alpha}(p'a + q'b) = pa + qb + p'a + q'(a + b) = q'a + (q + q')b$. Therefore $\hat{\mu}(u + \tilde{\alpha}v) = 0$. If $q' = 0$, then $u + \tilde{\beta}v = pa + qb + \tilde{\beta}(p'a) = pa + (q + p')b$. We notice that $p \neq 0$, for otherwise we would have $p' = 0$, and hence $v = 0$. But if $p \neq 0$, then $\hat{\mu}(u + \tilde{\beta}v) = 0$. Thus (7) is fulfilled for all $u, v \in Y$, and it implies that $L_1$ and $L_2$ are independent. It is obvious that $\mu \notin I(X)$.

The proof for the group $X = \mathbb{Z}(5) + \mathbb{Z}(5)$ is the same up to notation.

If $X = \mathbb{Z}(2^k) + \mathbb{Z}(2^k)$, one should consider independent random variables with values in the subgroup $G = \mathbb{Z}(2) + \mathbb{Z}(2) \subset X$ and notice, that $G$ is invariant with respect to automorphisms $\alpha$ and $\beta$. The proof for the group $G$ coincides with the given above up to notation. Lemma 2 is proved.

**Lemma 3.** *For each of the groups $X = \mathbb{Z}(2n-1)$, $n \geq 4$ and $X = \mathbb{Z}(3) + \mathbb{Z}(5)$ there exist independent random variables $\xi_j$, $j = 1, 2, 3$ taking on values in $X$ and with distributions $\mu_j$ and $\alpha, \beta \in \mathrm{Aut}(X)$, such that $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \xi_1 + \alpha(\xi_2) + \beta(\xi_3)$ are independent, whereas all $\mu_j \notin I(X)$.*

**Proof.** Consider the group $X = \mathbb{Z}(7)$ at first. Let $\xi_j$, $j = 1, 2, 3$ be independent random variables taking on values in $X$ and with distributions $\mu_1(\{x\}) = \frac{1}{7}(1 + \mathrm{Re}(x, 1))$ and $\mu_2(\{x\}) = \mu_3(\{x\}) = \frac{1}{7}(1 + \mathrm{Re}(x, 2))$, $x \in X$. Then $\hat{\mu}_1(y) = 0$ if $y \notin \{0, 1, 6\}$ and $\hat{\mu}_2(y) = 0$ if $y \notin \{0, 2, 5\}$. We verify that

$L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \xi_1 + 2\xi_2 + 5\xi_3$ are independent. Equation (4) takes the form

$$\hat\mu_1(u+v)\hat\mu_2(u+2v)\hat\mu_2(u+5v) = \hat\mu_1(u)\hat\mu_2^2(u)\hat\mu_1(v)\hat\mu_2(2v)\hat\mu_2(5v), \quad u, v \in Y. \tag{8}$$

It suffices to verify (8) supposing that $u \neq 0,\ v \neq 0$. It is obviously, that in this case the right-hand side of (8) vanishes. If the left-hand side (8) does not vanish, it follows from this, in particular, that $u + v \in \{0, 1, 6\},\ u + 2v \in \{0, 2, 5\}$. This implies that $(u, v) \in \{(5, 2),\ (2, 5),\ (2, 6),\ (4, 4),\ (5, 1),\ (3, 3)\}$. But in each of these cases we have $u + 5v \notin \{0, 2, 5\}$ and hence $\hat\mu_2(u + 5v) = 0$. Thus, (8) is fulfilled for or all $u, v \in Y$, and for this reason $L_1$ and $L_2$ are independent. It is obvious that $\mu_j \notin I(X)$.

The reasoning for the group $\mathbb{Z}(2n-1),\ n \geq 5$ is similar (see [6]). One should consider the distributions $\mu_1(\{x\}) = \mu_3(\{x\}) = \frac{1}{2n-1}(1 + \mathrm{Re}(x, 1))$, $\mu_2(\{x\}) = \frac{1}{2n-1}(1 + \mathrm{Re}(x, 2)),\ x \in X$ and the linear statistics $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \xi_1 + 2\xi_2 - \xi_3$.

Let us assume now that $X = \mathbb{Z}(3) + \mathbb{Z}(5)$. We denote elements of the groups $X$ and $Y$ by $(p, q),\ p \in \mathbb{Z}(3),\ q \in \mathbb{Z}(5)$. Let $\xi_j,\ j = 1, 2, 3$ be independent random variables taking on values in $X$ and with distributions $\mu_1(\{x\}) = \mu_3(\{x\}) = \frac{1}{15}(1 + \mathrm{Re}(x, (1, 1)))$ and $\mu_2(\{x\}) = \frac{1}{15}(1 + \mathrm{Re}(x, (2, 1))),\ x \in X$. Then $\hat\mu_1(y) = 0$ if $y \notin \{(0, 0),\ (1, 1),\ (2, 4)\}$ and $\hat\mu_2(y) = 0$ if $y \notin \{(0, 0),\ (2, 1),\ (1, 4)\}$. Reasoning as in the case of $X = \mathbb{Z}(7)$ it is easy to make sure that the linear statistics $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \xi_1 + 2\xi_2 - \xi_3$ are independent. It is obvious that $\mu_j \notin I(X)$. Lemma 3 is proved.

**Remark 1.** Lemmas 2 and 3 and their proofs remain true for any number $n \geq 3$ of independent random variables $\xi_j$. It suffices to consider the linear statistics $L_1 = \xi_1 + ... + \xi_n$ and $L_2 = \xi_1 + ... + \xi_{n-2} + \alpha(\xi_{n-1}) + \beta(\xi_n)$, where all $\xi_j$ are identically distributed with distribution $\mu$, in the case of Lemma 2, and $\xi_1, ..., \xi_{n-2}$ have distribution $\mu_1$, and $\xi_{n-1},\ \xi_n$ have distribution $\mu_2$ in the case of Lemma 3.

**Proof of Theorem 1.** Let $X$ be an arbitrary locally compact Abelian group. For each $\mu \in M^1(X)$, we define the distribution $\bar\mu \in M^1(X)$ by the rule $\bar\mu(E) = \mu(-E)$ for all Borel sets $E \subset X$. Then $\hat{\bar\mu}(y) = \overline{\hat\mu(y)}$. Denote by $\sigma(\mu)$ the support of $\mu \in M^1(X)$. It should be noticed that if $H$ is a close subgroup of $Y$ and $\hat\mu(y) \equiv 1,\ y \in H$, then $\hat\mu(y + h) = \hat\mu(y)$ for all $y \in Y,\ h \in H$ and $\sigma(\mu) \subset A(X, H) = \{x \in X : (x, y) = 1 \text{ for all } y \in H\}$.

Let $X$ be of the form (3). Consider the distributions

$$\nu_j = \mu_j * \bar\mu_j, \quad j = 1, 2, 3. \tag{9}$$

Then $\hat\nu_j(y) = |\hat\mu_j(y)|^2 \geq 0$ and the characteristic functions $\hat\nu_j(y)$ satisfy equation (4) too. Denote $H = G^*$. Since $a(H) = H$ for each $a \in \mathrm{Aut}(Y)$, we can consider the restriction of equation (4) for the characteristic functions $\hat\nu_j(y)$ on the subgroup $H$. Applying Theorem A we obtain, that $\hat\nu_j(y) = 1$

for $y \in H,\ j = 1,2,3$. It follows from this $\sigma(\nu_j) \subset A(X,H) = \mathbb{Z}(5)$ and (9) implies that distributions $\mu_j$ can be replaced by their shifts $\mu'_j$ such that the supports of all distributions $\mu'_j$ are in $\mathbb{Z}(5),\ j = 1,2,3$. Taking into account that the subgroup $\mathbb{Z}(5)$ is invariant with respect to any automorphism $\alpha \in \mathrm{Aut}(X)$, the statements $(i)-(iii)$ of Theorem 1 follows from Lemma 1.

Let as assume now that the finite group $X$ is not isomorphic to the groups of the form (1)-(3). Represent the group $X$ as a direct sum of its cyclic $p$-primary subgroups. Since the group $X$ is not isomorphic to the groups of the form (1)-(3) it is easily seen that $X$ contains as a direct summand $B$ one of the groups isomorphic to $\mathbb{Z}(2n-1)$ for $n \geq 4,\ \mathbb{Z}(3)+\mathbb{Z}(3),\ \mathbb{Z}(3)+\mathbb{Z}(5),\ \mathbb{Z}(5)+\mathbb{Z}(5),\ \mathbb{Z}(2^k)+\mathbb{Z}(2^k)$. Since $B$ is a direct summand of $X$ each automorphism $\alpha \in \mathrm{Aut}(B)$ can be extended to an automorphism of the group $X$. To complete the proof of the theorem it suffices to consider independent random variables $\xi_j,\ j = 1,2,3$ taking on values in $B,\ \alpha, \beta \in \mathrm{Aut}(B)$ and apply Lemmas 2 and 3.

**Remark 2.** Theorems A and B were generalized in particular for separable totally disconnected Abelian groups $X$. Denote by $K$ a group of the form (1), by $\Delta_p$ the group of $p$-adic integers, by $f_n : X \to X$ a homomorphism $f_n(x) = nx$, and $X^{(n)} = Im\ f_n$. Theorem A remains valid for totally disconnected groups if by $G$ we understand one of the groups $K,\ \Delta_2,\ \Delta_2 + K$. As for Theorem B for totally disconnected Abelian groups $X$ when $n = 2$, the independence of $L_1$ and $L_2$ implies that $\mu_1,\ \mu_2 \in I(X)$ if and only if for each prime $p$ the factor-group $X/X^{(p)}$ is finite (see [4], [7]). As to the case when the number of random variables $n = 3$ we can formulate the following

**Proposition 1.** *Theorem 1 is valid for separable totally disconnected Abelian groups, where we understand by $G$ one of the above mentioned groups $K,\ \Delta_2,\ \Delta_2 + K$.*

The proof of this statement follows the scheme of the proof of Theorem 1. Besides Lemmas 1-3 it is based on the following result.

For each of the groups $X = \Delta_2 + \Delta_2$ and $X = \Delta_p$ for a prime $p \geq 3$, there exist independent random variables $\xi_j,\ j = 1,2,3$ taking on values in $X$ and with distributions $\mu_j$ and $\alpha, \beta \in \mathrm{Aut}(X)$, such that $L_1 = \xi_1 + \xi_2 + \xi_3$ and $L_2 = \xi_1 + \alpha(\xi_2) + \beta(\xi_3)$ are independent, whereas all $\mu_j \notin I(X)$.

G.M. Feldman

Permanent address:
Institute for Low Temperature
Physics and Engineering
47, Lenin ave, Kharkov, 310164,
UKRAINE
e-mail: feldman@ilt.kharkov.ua

Current address:
Universite d'Angers
Departement de Mathematiques
2, bouleard Lavoisier, 49045 Angers Cedex 01,
FRANCE
e-mail: feldman@tonton.univ-angers.fr

P. Graczyk

Universite d'Angers
Departement de Mathematiques
2, bouleard Lavoisier, 49045 Angers Cedex 01,
FRANCE
e-mail: graczyk@tonton.univ-angers.fr

## REFERENCES


[1] Darmois, G. (1953). Analyse generale des liaisons stochastiques. *Rev. Inst. Intern. Stat.* **21** 2-8.

[2] Feldman, G.M. (1992). On the Skitovich-Darmois theorem on Abelian groups. *Theory Probab. Appl.* **37** 621-631.

[3] Feldman, G.M. (1993). Arithmetic of probability distributions and characterization problems on Abelian groups, Amer. Math. Soc., Providence, RI.

[4] Feldman, G.M. (1996). On the Skitovich-Darmois theorem on compact groups. *Theory Probab. Appl.* **41** 768-773.

[5] Feldman, G.M. (1997). The Skitovich-Darmois theorem for discrete periodic Abelian groups. *Theory Probab. Appl.* **42** .

[6] Feldman, G.M. (1998). More on the Skitovich-Darmois theorem for finite Abelian groups. *Prepublications Universite d'Angers,* 98/62.

[7] Feldman, G.M., and Graczyk, P. (1998). On the Skitovich-Darmois theorem on compact Abelian groups.*Prepublications Universite d'Angers,* 98/68.

[8] Franz, U., Neuenschwander, D., and R. Schott, R. (1997). Gauss laws in the sense of Bernstein and uniqueness of embedding into convolution semigroups on quantum groups and braided groups. *C.R. Acad. Sci. Paris* **324**, Serie I, 827-832.

[9] Ghurye, S.G., and Olkin, I. (1962). A characterization of the multivariate normal distribution. *Ann. Math. Stat.* **33** 553-541.

[10] Kagan, A.M., Linnik, Yu. V., and Rao, S.R. (1973). *Characterization problems of mathematical statistics.* Wiley, New York.

[11] Neuenschwander, D. (1993). Gauss measures in the sense of Bernstein on the Heisenberg group. *Probability and Math. Stat.* **14** 253-256.

[12] Neuenschwander, D., Roynette, B., and Schott, R. (1997). Characterization of Gauss measures on nilpotent Lie groups and symmetric spaces. *C.R. Acad. Sci. Paris* **324** Serie I, 87-92.

[13] Neuenschwander, D., and Schott, R. (1997). The Bernstein and Skitovic-Darmois characterization theorems for Gaussian distributions on groups, symmetric spaces, and quantum groups. *Expo. Math.* **15** 289-314.

[14] Parthasarathy, K.R., Ranga Rao, R., and Varadhan, S.R.S. (1963). Probability distributions on locally compact Abelian groups. *Illinois J. Math.* **7** 337-369 .

[15] Skitovich, V.P. (1953). On a property of a normal distribution. *Doklady Academii nauk SSSR* **89** 217-219 ( in Russian).