ON THE SKITOVICH-DARMOIS THEOREM FOR COMPACT ABELIAN GROUPS

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Abstract

Let X be a separable compact Abelian group, Aut(X) the group of topological automorphisms of X, $f_n : X \to X$ a homomorphism $f_n(x) = nx$, and $X^{(n)} = Im\ f_n$. Denote by $I(X)$ the set of idempotent distributions on X and by $\Gamma(X)$ the set of Gaussian distributions on X. Consider linear statistics $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$ and $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2)$, where ξ_j are independent random variables taking on values in X and with distributions μ_j, and $\alpha_j,\ \beta_j \in \mathrm{Aut}(X)$. The following results are obtained.

Let X be a totally disconnected group. Then the independence of L_1 and L_2 implies that $\mu_1,\ \mu_2 \in I(X)$ if and only if X possesses the property: for each prime p the factor-group $X/X^{(p)}$ is finite.

If X is connected, then there exist independent random variables ξ_j taking on values in X and with distributions μ_j, and $\alpha_j,\ \beta_j \in \mathrm{Aut}(X)$ such that L_1 and L_2 are independent, whereas $\mu_1,\ \mu_2 \notin \Gamma(X) * I(X)$.

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We continue with this article a series of researches devoted to group analogues of the classical Skitovich-Darmois theorem ([14], [1], see also [12, Ch. 3]), where a Gaussian distribution is characterized by the independence of linear statistics. Namely, linear forms of n independent random variables ξ_j with values in a locally compact Abelian group X are considered (coefficients of the forms are topological automorphisms of X) and the following problem is solved: for which groups X the independence of linear forms implies that the distributions of random variables are Gaussian up to shifts by the Haar distributions of compact subgroups of X (the shifts by the Haar distributions appear in a natural way in the characterization problems on groups (see [3, Ch. 3]).

This problem was solved by Ghurye and Olkin in [9] for the group $X = \mathbb{R}^n$, and in [2], [4], and [5] for the classes of finite Abelian groups, compact Abelian groups and discrete torsion Abelian groups respectively. It was discovered in [6] that for $n = 2$ the class of finite Abelian groups for which the Skitovich-Darmois theorem takes place coincides with all finite Abelian groups! If one requires the Skitovich-Darmois theorem to be valid for any n, then this class of finite Abelian groups is very poor (see [2]). The situation is in contrast to the classical case, where there is not a distinction between $n = 2$ and arbitrary n.

In this article we consider the case of two random variables with values in a compact Abelian group and investigate completely the cases of totally disconnected groups and connected groups.

Let X be a locally compact Abelian separable metric group, $Y = X^*$ the character group of the group X, and (x, y) the value of a character $y \in Y$ on an element $x \in X$. If G is a subgroup of X, we denote by $A(Y, G) = \{y \in Y : (x, y) = 1 \text{ for all } x \in G\}$ its annihilator. Let $\mathrm{Aut}(X)$ be the group of topological automorphisms of X. If $\alpha \in \mathrm{Aut}(X)$, we define the conjugate automorphism $\tilde{\alpha} \in \mathrm{Aut}(Y)$ by the rule $(x, \tilde{\alpha}(y)) = (\alpha(x), y)$ for all $x \in X,\ y \in Y$. Denote by I the identity automorphism of the group Y. Let $M^1(X)$ be the convolution semigroup of probability distributions on X. If $\mu \in M^1(X)$, we denote by $\hat{\mu}(y) = \int_X (x, y) d\mu(x)$ its characteristic function. If $\mu \in M^1(X)$, we define the distribution $\bar{\mu} \in M^1(X)$ by the rule $\bar{\mu}(E) = \mu(-E)$ for all Borel sets $E \subset X$. Observe that $\hat{\bar{\mu}}(y) = \overline{\hat{\mu}(y)}$.

A distribution $\mu \in M^1(X)$ is called Gaussian ([13]) if its characteristic function can be represented in the form

$$\hat{\mu}(y) = (x, y) \exp\{-\varphi(y)\},$$

where $x \in X$ and $\varphi(y)$ is a continuous nonnegative function which satisfies the equation

$$\varphi(y_1 + y_2) + \varphi(y_1 - y_2) = 2[\varphi(y_1) + \varphi(y_2)], \quad y_1, y_2 \in Y.$$

Denote by $\Gamma(X)$ the set of Gaussian distributions on X. The support of a distribution $\gamma \in \Gamma(X)$ is a coset of the connected component of zero of X.

Specifically, the set of Gaussian distributions on a totally disconnected group coincides with the set of degenerate distributions. Denote by $I(X)$ the set of idempotent distributions on X, i.e. the set of shifts of Haar distributions m_K of compact subgroups K of X. Observe that the characteristic function of the Haar distribution m_K is of the following form

$$\hat{m}_K(y) = \begin{cases} 1, & if \quad y \in A(Y,K), \\ \\ 0, & if \quad y \notin A(Y,K). \end{cases}$$

For each integer n let $f_n : X \to X$ be a homomorphism $f_n(x) = nx$, and $X^{(n)} = \mathrm{Im}\, f_n$. Denote by $I\!N$ the set of natural numbers. Let $Z\!\!Z(n) = \{0, 1, ..., n-1\}$ be a finite cyclic group of order n with addition modulo n as a group operator. Given a prime number p denote by $Z\!\!Z(p^\infty)$ the group of p^n-th complex roots of unity, with n running over all $I\!N$, equipped with discrete topology. It is known that $(Z\!\!Z(p^\infty))^* \approx \Delta_p$, where Δ_p is a group of p-adic integers. Observe that the group Δ_p is totally disconnected (see [10, §10]).

Let ξ_1, ξ_2 be independent random variables taking on values in a locally compact Abelian group X and with distributions μ_1, μ_2, and $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, $j = 1, 2$. Taking into account that $\hat{\mu}_j(y) = \mathbf{E}[(\xi_j, y)]$ it is easily seen that the independence of linear statistics $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$ and $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2)$ is equivalent to the fact that the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\hat{\mu}_1(\tilde{\alpha}_1 u + \tilde{\beta}_1 v)\hat{\mu}_2(\tilde{\alpha}_2 u + \tilde{\beta}_2 v) = \hat{\mu}_1(\tilde{\alpha}_1 u)\hat{\mu}_2(\tilde{\alpha}_2 u)\hat{\mu}_1(\tilde{\beta}_1 v)\hat{\mu}_2(\tilde{\beta}_2 v), \ u, v \in Y. \quad (1)$$

Theorem 1. *Let ξ_1, ξ_2 be independent random variables taking on values in a separable compact totally disconnected Abelian group X and with distributions μ_1, μ_2. Then the independence of linear statistics $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$ and $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2)$, where $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, implies that $\mu_j \in I(X)$, $j = 1, 2$ if and only if the group X possesses the property:*

(i) for each prime p the factor-group $X/X^{(p)}$ is finite.

To prove the theorem we need two lemmas.

Lemma 1. *Let a group X be a direct product of cyclic groups*

$$\mathrm{P}_{m \in I\!N}\ Z\!\!Z(p^{k_m}), \quad k_m \le k_{m+1}, \quad m \in I\!N, \quad (2)$$

where p is a prime number. Then there exist independent random variables ξ_1, ξ_2 taking on values in X and with distributions $\mu_1, \mu_2 \notin I(X)$ and an automorphism $\alpha \in \mathrm{Aut}(X)$, such that linear statistics $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha(\xi_2)$ are independent.

Proof. The character group $Y = X^*$ is isomorphic to a week direct product of cyclic groups

$$\mathrm{P}^*_{m \in I\!N}\ Z\!\!Z(p^{k_m}), \quad k_m \le k_{m+1}, \quad m \in I\!N. \quad (3)$$

Elements of the group Y are sequences $y = \{l_m\}_{m=1}^{\infty}$, $l_m \in \mathbb{Z}(p^{k_m})$, where a finite number of $l_m \neq 0$. Denote by $e_m \in Y$ a generator of the subgroup $\mathbb{Z}(p^{k_m})$ of the form $e_m = \{l_i\}_{i=1}^{\infty}$, where $l_i = \delta_{im}$ (δ_{ij} is the Kronecker symbol). Define a homomorphism $a : Y \to Y$ on the generators e_m, $m \in \mathbb{N}$ by the rule

$$a(e_m) = \begin{cases} e_m + p^{k_{m+1}-k_m}\ e_{m+1} + p^{k_{m+2}-k_m}\ e_{m+2}, & m = 2j-1, \\ e_m + p^{k_{m+1}-k_m}\ e_{m+1}, & m = 2j. \end{cases} \qquad (4)$$

It follows from this that $a\{l_m\}_{m=1}^{\infty} = \{h_m\}_{m=1}^{\infty}$, where

$$h_m = \begin{cases} l_1, & m = 1, \\ l_{m-1}\ p^{k_m-k_{m-1}} + l_m \pmod{p^{k_m}}, & m = 2j, \\ l_{m-2}\ p^{k_m-k_{m-2}} + l_{m-1}\ p^{k_m-k_{m-1}} + l_m \pmod{p^{k_m}}, & m = 2j+1. \end{cases} \qquad (5)$$

Using (5) it is easily verified by induction that a is a monomorphism. It follows immediately from (4) that for any $j \in \mathbb{N}$

$a(e_{2j-1} + (p^{k_{2j}} - p^{k_{2j}-k_{2j-1}})\ e_{2j}) = e_{2j-1};$

and

$a(e_{2j} + (p^{k_{2j+1}} - p^{k_{2j+1}-k_{2j}})\ e_{2j+1} + p^{k_{2j+2}-k_{2j}}\ e_{2j+2}) = e_{2j}.$

This proves that a is an epimorphism, and hence $a \in \mathrm{Aut}(Y)$. On the other hand (5) implies that

$$Im(I-a) \cap \mathbb{Z}(p^{k_1}) = \{0\} \qquad (6)$$

and

$$Ker(I-a) = \{0\} \qquad (7).$$

Consider on X the positive function

$$\rho(x) = 1 + \frac{1}{4}(x, e_1) + \frac{1}{4}(x, (p^{k_1}-1)\ e_1).$$

Let $\mu_1 = \mu_2$ be distributions on the group X with the density $\rho(x)$ with respect to the Haar distribution m_X. The characteristic functions $\hat{\mu}_j(y)$ have the form

$$\hat{\mu}_j(y) = \begin{cases} 1, & if \quad y = 0, \\ \frac{1}{4}, & if \quad y \in \{e_1,\ (p^{k_1}-1)e_1\}, \\ 0, & if \quad y \notin \{0,\ e_1,\ (p^{k_1}-1)e_1\}, \end{cases}$$

so that $\hat{\mu}_j(y) = 0$, when $y \notin \mathbb{Z}(p^{k_1})$. We check that the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\hat{\mu}_1(u+v)\hat{\mu}_2(u+av) = \hat{\mu}_1(u)\hat{\mu}_2(u)\hat{\mu}_1(v)\hat{\mu}_2(av), \quad u, v \in Y. \qquad (8)$$

This is trivially true for either $u = 0$ or $v = 0$, so that it suffices to verify (8) assuming that $u \neq 0$ and $v \neq 0$. It is obvious that if $v \neq 0$, then the points

v and av do not belong to $\mathbb{Z}(p^{k_1})$ simultaneously. Hence the right-hand side of (8) vanishes. If the left-right side of (8) does not vanish, this implies that $u+v \in \mathbb{Z}(p^{k_1}),\ u+av \in \mathbb{Z}(p^{k_1})$. It follows from this that $(I-a)v \in \mathbb{Z}(p^{k_1})$. Taking into account (6), we obtain that $(I-a)v = 0$. But then (7) implies that $v = 0$. We received contradiction and hence the left-hand side of equation (8) vanishes as well. Thus equation (8) is valid for any $u, v \in Y$.

Let ξ_1, ξ_2 be independent random variables taking on values in X and with distributions μ_1, μ_2. Put $\alpha = \tilde{a},\ \alpha \in \mathrm{Aut}(X)$. It follows from (1) that the fulfillment of equation (8) for any $u, v \in Y$ is equivalent to the fact that $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha(\xi_2)$ are independent. Evidently that $\mu_1, \mu_2 \notin I(X)$ as required. Lemma 1 is proved.

Lemma 2. *Let X be Δ_p^∞, where p is a prime number. Then there exist independent random variables ξ_1, ξ_2 taking on values in X and with distributions $\mu_1, \mu_2 \notin I(X)$ and an automorphism $\alpha \in \mathrm{Aut}(X)$, such that linear statistics $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha(\xi_2)$ are independent.*

Proof. Let us represent the character group $Y = X^*$ as an infinite week direct product of groups A_k, each of which is isomorphic to $\mathbb{Z}(p^\infty)$. It is convenient for us to write elements $y \in Y$ as two-sided sequences $y = \{h_k\}_{k=-\infty}^{\infty},\ h_k \in A_k$. Let an automorphism $a \in \mathrm{Aut}(Y)$ be of the form $a\{h_k\}_{k=-\infty}^{\infty} = \{h_{k+1}\}_{k=-\infty}^{\infty}$.

Let $y = \{h_k\}_{k=-\infty}^{\infty} \in Y$, and $y \neq 0$. If $h_k = 0$, when $k < m,\ h_m \neq 0,\ h_n \neq 0$ and $h_k = 0$, when $k > n$, define then $l(y) = n - m + 1$.

Let $e_1 \in Y$ be an element of order p in A_1. Consider on X the positive function
$$\rho(x) = 1 + \frac{1}{4}(x, e_1) + \frac{1}{4}(x, (p-1)\ e_1).$$
Let $\mu_1 = \mu_2$ be distributions on the group X with the density $\rho(x)$ with respect the Haar distribution m_X. It is obvious that $\hat{\mu}_j(y) = 0$, when $y \notin A_1$. To see that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (8) we reason as in the proof of previous lemma. It suffices to verify (8) assuming that $u \neq 0$ and $v \neq 0$. Since v and av do not belong to A_1 simultaneously, it follows from this that the right-hand side of (8) vanishes. If the left-right side of (8) does not vanish, this implies that $u + v \in A_1$, and $u + av \in A_1$. Hence $(I - a)v \in A_1$. In case when $y \in A_1,\ y \neq 0$, we have $l(y) = 1$. On the other hand if $y = (I - a)v,\ y \neq 0$, then $l(y) \geq 2$. Taking into account that $Ker(I - a) = \{0\}$, it follows from this that $v = 0$. The obtained contradiction shows that equation (8) is fulfilled for any $u, v \in Y$.

Let ξ_1, ξ_2 be independent random variables taking on values in the group X and with distributions μ_1, μ_2. Put $\alpha = \tilde{a},\ \alpha \in \mathrm{Aut}(X)$. The fulfillment of equation (8) for any $u, v \in Y$ is equivalent to the fact that $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + \alpha(\xi_2)$ are independent. Obviously $\mu_1, \mu_2 \notin I(X)$ as required. Lemma 2 is proved.

Remark 1. It is easily seen, Lemma 2 is valid for any compact Abelian group X of the form $X = K^\infty$, where K is a compact group.

Proof of Theorem 1. Observe that $\left(X/X^{(p)}\right)^* \approx A(Y, X^{(p)}) = \{y \in Y : (px, y) = 1 \text{ for all } x \in X\} = \{y \in Y : (x, py) = 1 \text{ for all } x \in X\} = \{y \in Y : py = 0\}$. Since X is a separable compact totally disconnected Abelian group, then Y is a countable discrete torsion Abelian group. Let us represent Y as a week direct product of its p-primary subgroups

$$Y = \mathrm{P}^*_{p\in\mathcal{P}}\ Y_p, \tag{9}$$

where $\mathcal{P}$ stands for the set of prime numbers. Let us assume that the group X possesses the property (i). The independence of linear statistics $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$ and $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2)$ is equivalent to the fact that the characteristic functions $\hat\mu_j(y)$ satisfy equation (1). Note that the characteristic functions of the distributions $\nu_j = \mu_j * \bar\mu_j$ satisfy equation (1) too. Since $\hat\nu_j(y) = |\hat\mu_j(y)|^2 \geq 0$ we can assume that $\hat\mu_j(y) \geq 0$ from the very beginning and verify that each of the function $\hat\mu_j(y)$ takes the values 0 and 1 only. Then the sufficiency of condition (i) will be proved.

Let $y_0 \in Y$. We can represent $y_0 = \sum_{j=1}^n\ y_j$, where $y_j \in Y_{p_j}$. It is obvious that ${p_j}^{k_j}\ y_j = 0,\ j = 1, 2, ...n$ for some k_j. Consider subgroups $B_j = \{y \in Y_{p_j} : {p_j}^{k_j}\ y = 0\}$. Then $y_0 \in B = B_1 \times ... \times B_n$. Condition (i) implies that each subgroup B_j is finite. Hence the subgroup B is finite too. It is obvious that the subgroup B is invariant with respect to each automorphism $a \in \mathrm{Aut}(Y)$. It is clear that for any $u, v \in Y$ there exists a subgroup B of the described form such that $u, v \in B$. Consider the restriction of equation (8) on B and make now use of the fact that theorem 1 is valid for finite groups ([6]). It follows from this that characteristic functions $\hat\mu_j(y)$ take values 0 and 1 only, which is what had to be proved. The sufficiency of (i) is proved.

Let us return now to representation (9) of the group Y. Each of p-primary subgroups Y_p can be represented as a direct product $Y_p = D \times N$, where D is the maximal divisible subgroup of Y_p and N is a reduced countable p-primary group. The group D can be represented as a week direct product of groups A_k each of which is isomorphic to $\mathbb{Z}(p^\infty)$. If N is a bounded group, i.e. the orders of all elements of N are bounded jointly, then N is a week direct product of cyclic groups ([7, §17]). If N is unbounded, then N has a direct factor which is a week direct product of cyclic groups ([8, §77]).

Let us assume now that condition (i) is not valid. Then for some prime p the subgroup $E = \{y \in Y : py = 0\}$ is infinite. Consider the corresponding direct factor in (9) and its representation $Y_p = D \times N$. If E is infinite, then at least one of the following two cases takes place.

1. The group D is an infinite week direct product of subgroups $A_k \approx \mathbb{Z}(p^\infty)$. Then the group X has a direct factor G, $G \approx {\Delta_p}^\infty = D^*$, and therefore each a topological automorphism of the subgroup G can be extended to a topological automorphism of the group X. In this case we should consider independent random variables ξ_1 and ξ_2 with values in G and apply Lemma 2.

2. The group N is infinite. Then either N itself is of the form (3) or N has a direct factor of the form (3), and therefore in both cases the group X has a direct factor of the form (2). In this case we should consider independent random variables ξ_1 and ξ_2 with values in G and apply Lemma 1.

Theorem 1 is proved completely.

Theorem 2. *Let X be a separable connected compact Abelian group. Then there exist independent random variables ξ_1, ξ_2 taking on values in X and with distributions μ_j and automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$, such that linear statistics $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$ and $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2)$ are independent, whereas $\mu_j \notin \Gamma(X) * I(X),\ j = 1, 2$.*

Proof. Observe that the character group $Y = X^*$ is a countable torsion free group. Consider two possible cases.

(i) $f_p \notin \mathrm{Aut}(X)$ for some prime p.

Suppose that p is minimal having this property. It follows from the connectedness of X that $X^{(n)} = X$ for each integer n, $n \neq 0$. Therefore if $f_p \notin \mathrm{Aut}(X)$, then $\mathrm{Ker} f_p \neq \{0\}$. Let $p = 2$. Then $\mathrm{Ker} f_2 = \{x \in X : 2x = 0\} \neq \{0\}$. As had been proved in [3, §9.21], in this case there exist independent random variables ξ_1, ξ_2 taking on values in X and with distributions $\mu_1 \neq \mu_2$, such that linear statistics $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 - \xi_2$ are independent, whereas $\mu_j \notin \Gamma(X) * I(X),\ j = 1, 2$.

Let us assume now that $p \geq 3$. Since $\mathrm{Ker} f_n = A(X, Y^{(n)})$, then $Y^{(p)} \neq Y$. Take $y_0 \notin Y^{(p)}$ and consider on X the positive function $\rho(x) = 1 + \frac{1}{2}\mathrm{Re}(x, y_0)$. Let $\mu_1 = \mu_2$ be distributions on the group X with the densities $\rho(x)$ with respect to the Haar distribution m_X. Then the characteristic functions $\mu_j(y) = 0$, when $y \notin \{0,\ \pm y_0\}$.

Let ξ_1, ξ_2 be independent random variables taking on values in the group X and with distributions μ_1, μ_2. It is obvious that $\mu_j \notin \Gamma(X) * I(X),\ j = 1, 2$. Put $a = 1 - p$. Then $f_a \in \mathrm{Aut}(X)$. Consider the linear statistics $L_1 = \xi_1 + \xi_2$ and $L_2 = \xi_1 + a\xi_2$. The independence of L_1 and L_2 is equivalent to the fact that the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (8). Observe that $Ker(I - \tilde{f}_a) = \{0\}$ and $Im(I - \tilde{f}_a) = Y^{(p)}$. Since $p \geq 3$ the numbers 2 and p are mutually prime. It implies that

$$Im(I - \tilde{f}_a) \cap \{k y_0\}_{k=-2}^{2} = \{0\}.$$

Reasoning as in the proof of Lemma 1, it is easy to make sure that equation (8) is valid for any $u, v \in Y$, and hence L_1 and L_2 are independent.

(ii) $f_p \in \mathrm{Aut}(X)$ for each prime p.

This means that X is a divisible group and hence can be represented as a direct product of $\mathbf{a}$-adic solenoids $\Sigma_{\mathbf{a}}$, where $\mathbf{a} = (2, 3, 4 \ldots)$ (see [10, §25]). Apparently, it suffices to prove the theorem for the group $X = \Sigma_{\mathbf{a}}$. Then $Y \approx Q$, where Q is the discrete group of rational numbers.

Let H be a subgroup in Y of the form $H = \left\{\frac{m}{5^n}\right\}_{m,n\in\mathbb{Z}}$. Denote by $G = H^*$. Consider on the group H the function

$$f(h) = \begin{cases} \exp\{-h^2\}, & if \quad h \in H^{(2)}, \\ \\ c\exp\{-h^2\}, & if \quad h \notin H^{(2)}, \end{cases} \tag{10}$$

where $-1 < c < 1$. It is easily seen that $f(h)$ is the characteristic function of a distribution $\pi \in M^1(G)$, $\pi = \gamma * \lambda$, where $\gamma \in \Gamma(G)$ and $\lambda \in M^1(K)$, $K = A(G, H^{(2)}) \approx \mathbb{Z}(2)$. Consider on the group Y the function

$$g(y) = \begin{cases} \hat{\pi}(y), & if \quad y \in H, \\ \\ 0, & if \quad y \notin H. \end{cases}$$

The function $g(y)$ is positive definite on Y (see [11, §32]), and by the Bochner's theorem $g(y)$ defines a distribution $\mu \in M^1(X)$ such that $\hat{\mu}(y) = g(y)$. It is obvious that $\mu \notin \Gamma(X) * I(X)$.

Let ξ_1, ξ_2 be independent random variables taking on values in the group X and with distributions $\mu_1 = \mu_2 = \mu$. Let us verify that the linear statistics $L_1 = \xi_1 + 2\xi_2$ and $L_2 = 2\xi_1 - \xi_2$ are independent. This is equivalent to check that the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\hat{\mu}_1(u + 2v)\hat{\mu}_2(2u - v) = \hat{\mu}_1(u)\hat{\mu}_2(2u)\hat{\mu}_1(2v)\hat{\mu}_2(-v), \quad u, v \in Y. \tag{11}$$

It follows from (10) that (11) is fulfilled when $u, v \in H$. One sees that (11) holds true both when $u \in H$, $v \notin H$ and $v \in H$, $u \notin H$. Let us assume that $u, v \notin H$. Then the right-hand side of (11) vanishes. If the left-hand side of (11) does not vanish, then $u + 2v \in H$ and $2u - v \in H$, but this implies that $5u \in H$, and hence $u \in H$. The obtained contradiction shows us that equation (11) is fulfilled for any $u, v \in Y$. This means that the linear statistics L_1 and L_2 are independent. Theorem 2 is proved completely.

Remark 2. Let X be the same as in Theorem 2. Let us assume that automorphisms $\alpha_j, \beta_j \in \mathrm{Aut}(X)$ satisfy the condition

$$\alpha_1\beta_1 + \alpha_2\beta_2 = 0. \tag{12}$$

Then there exist independent identically distributed with a nondegenerated Gaussian distribution random variables ξ_1, ξ_2 taking on values in X such that linear statistics $L_1 = \alpha_1(\xi_1) + \alpha_2(\xi_2)$ and $L_2 = \beta_1(\xi_1) + \beta_2(\xi_2)$ are independent. It is therefore reasonable to ask either one can construct automorphisms α_j, β_j in Theorem 2 satisfying condition (12), and independent random variables ξ_1, ξ_2 with distributions $\mu_1 = \mu_2 \notin \Gamma(X) * I(X)$.

In the case (ii) this has been already done in the proof of Theorem 2, so that let (i) is fulfilled. Let us assume that $f_p \notin \mathrm{Aut}(X)$, where $p \geq 3$, and $L_1 = \xi_1 + a\xi_2$, $L_2 = a\xi_1 - \xi_2$, where $a = 1 - p$. Put $q = a^2 + 1$. The independence of L_1 and L_2 is equivalent to the fact that the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\hat{\mu}_1(u+av)\hat{\mu}_2(au-v)=\hat{\mu}_1(u)\hat{\mu}_2(au)\hat{\mu}_1(av)\hat{\mu}_2(-v), \quad u,v\in Y.$$

If $f_q \notin \mathrm{Aut}(X)$, then we reason as in the proof of Theorem 2 in case (i), taking $y_0 \notin Y^{(q)}$.

If $f_q \in \mathrm{Aut}(X)$, then we use the scheme of the proof of Theorem 2 in case (ii). Namely, let y_0 be an arbitrary element of Y, $H=\left\{\frac{m}{q^n}y_0\right\}_{m,n\in \mathbb{Z}}$ and $G=H^*$. Consider $\pi \in M^1(G)$ of the form $\pi=\gamma * \lambda$, where $\gamma \in \Gamma(G)$, $\lambda \in M^1(K)$, $K=A(G,H^{(a)})\approx \mathbb{Z}(a)$ and construct μ as had been done in (ii).

As for the groups X for which $f_2 \notin \mathrm{Aut}(X)$, the answer above mention question is, generally, negative. Really, let $X=T$ be the circle group. Then $\mathrm{Aut}(T)\approx \mathbb{Z}(2)$, so that the consideration of arbitrary linear statistics L_1 and L_2 reduced to $L_1=\xi_1+\xi_2$ and $L_2=\xi_1\pm\xi_2$. But it is well known (see [3, §9]) that if ξ_1 and ξ_2 are independent identically distributed with distribution μ random variables with values in T such that $\xi_1+\xi_2$ and $\xi_1-\xi_2$ are independent, then $\mu \in \Gamma(X) * I(X)$.

On the other hand let $X=T^2$. Then there exist independent identically distributed random variables $\xi_1,\ \xi_2$ taking on values in X with a distribution $\mu \notin \Gamma(X) * I(X)$ such that linear statistics $L_1=\xi_1+\xi_2$ and $L_2=\xi_1-\xi_2$ are independent (see [3, §9]).

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