SPHERICAL ANALYSIS AND CENTRAL LIMIT THEOREMS

ON SYMMETRIC SPACES

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ABSTRACT

We prove some results on the kernel of the Abel transform on an irreducible Riemannian symmetric space $X = G/K$ with G noncompact and complex, in particular an estimate of this kernel. We also study the behaviour of spherical functions near the walls of Weyl chambers.

We show how these harmonic spherical analysis results lead to a new proof of a central limit theorem of Guivarc'h and Raugi in the complex and K-invariant cases. We present briefly this and other central limit theorems on X.

0. Introduction

Let G be a semisimple noncompact Lie group with finite center and K a maximal compact subgroup of G.

Harmonic spherical analysis on Riemannian symmetric spaces of the form $X = G/K$ is a very well developed and powerful tool in studying probabilities on such spaces, especially in the central limit problem and the investigating of Gaussian measures.

The study of central limit theorems on Riemannian symmetric spaces was initiated by Karpelevich, Shur and Tutubalin[16] and then developed by many authors in several parallel directions. Such theorems find applications in multivariate statistics and in some engineering problems.[8,22] Interesting results concerning properties of Gaussian measures on symmetric spaces were also obtained (see Section 2).

1991 Mathematics Subject Classification 43A85,43A90,43A05,60B15

Key words and phrases: symmetric spaces, spherical functions and transform, Abel transform, K-invariant probability measures, central limit theorem.

First author partially supported by KBN grant.

A natural tool in investigating K-invariant probability measures on Riemannian symmetric spaces is the spherical Fourier transform, using spherical functions. Analytical methods used to prove central limit theorems on $X = G/K$ often give interesting independent results in harmonic analysis on X.

In this paper we prove some estimates of the Abel transform kernel on X in the case when G is complex and we apply them to a new, analytical proof of a central limit theorem of Guivarc'h and Raugi[13]. Some properties of spherical functions near the walls of Weyl chambers in the general case are also proved and used.

This work is divided in three parts. In Section 1 we recall some basic facts about spherical functions, the spherical Fourier transform and K-invariant probability measures on symmetric spaces. We give and compare the definitions of different parameters of such measures and we define their "Iwasawa transform".

In Section 2 we propose a short survey of known central limit theorems on Riemannian symmetric spaces and some related results.

In Section 3, which is the original part of this paper, we prove an estimate of the kernel of the Abel transform on X in the complex case. To do this, we express this kernel by a formula involving a fundamental solution of a natural differential operator $\partial(\pi)$ on G. This formula is different and simpler than the formula for the Abel transform on $SL(n,\mathbb{C})/SU(n)$ due to Aomoto[2], generalized by Beerends[4].

Next we prove that if a K-invariant probability measure μ on X is not concentrated in the origin then the whole mass of μ^m concentrates when $m \to \infty$ far away from the walls of the Weyl chambers. To justify this we use non-zero lower estimates for some bounded positive spherical functions near a wall of the positive Weyl chamber. These results are true for arbitrary X of noncompact type.

We establish a relation between the Iwasawa transform of a K-invariant measure and the Abel transform on X. This leads to a new proof of a central limit theorem for K-invariant measures obtained more generally and by other methods by Guivarc'h and Raugi[13] and Babillot[3]. Our methods, based entirely on spherical harmonic analysis, work in the complex case and in rank one case.

We conclude by some remarks on the normalization of measures in the case of the considered central limit theorem.

1. Basic definitions and facts

We start by reviewing some basic facts concerning K-invariant measures on symmetric spaces.

1.1. K-invariant probability measures

A probability measure μ on $X = G/K$ is called *K-invariant* if μ is invariant with respect to the action of K on X. Then we write $\mu \in M^\natural(X)$. One may identify such measures with K-biinvariant measures on G. We denote the corresponding measures on G in the same way as on X. This makes it possible to define the convolution on $M^\natural(X)$. This convolution is commutative. We write μ^m to denote the m-th convolution power of μ.

Let $\mu \in M^\natural(X)$. Let $\mathfrak{a}$ be the Lie algebra of the Abelian group A in the Cartan decomposition of G and $\mathfrak{a}^+$ the positive Weyl chamber. We denote by $\mu_\mathfrak{a}$ and μ_+ the images of μ on $\mathfrak{a}$ and $\overline{\mathfrak{a}^+}$ respectively, that is $\mu_\mathfrak{a}(B) = \mu(K \exp BK)$ for $B \in \mathcal{B}_\mathfrak{a}$ and $\mu_+(B) = \mu(K \exp BK)$ for $B \in \mathcal{B}_{\overline{\mathfrak{a}^+}}$. Let W be the Weyl group acting on $\mathfrak{a}$. If f is a W-invariant bounded measurable function on $\mathfrak{a}$, then denoting also by f its K-invariant extension on G we have

$$\int_\mathfrak{a} f d\mu_\mathfrak{a} = \int_{\overline{\mathfrak{a}^+}} f d\mu_+ = \int_G f d\mu.$$

We will write briefly μ_+^m and $\mu_\mathfrak{a}^m$ to denote $(\mu^m)_+$ and $(\mu^m)_\mathfrak{a}$.

1.2. Spherical Fourier transform

We recall now some basic facts concerning the spherical functions and transforms on symmetric spaces, the principal tools of spherical harmonic analysis (see e.g. Helgason[14]).

A K-invariant $\mathcal{C}^\infty$ function ϕ on X is said to be *spherical* if $\phi(eK) = 1$ and ϕ is an eigenfunction of all G-invariant differential operators on X. On the level of the group G all the spherical functions are given by the Harish-Chandra formula:

$$\phi_\lambda(g) = \int_K e^{<i\lambda - \rho, \mathcal{H}(gk)>} dk, \quad g \in G,\ \lambda \in \mathfrak{a}_\mathbb{C}^* \tag{1}$$

where $g = k \exp \mathcal{H}(g) n$ is the Iwasawa decomposition of $g \in G$, $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ where Σ^+ denotes the set of the positive roots of multiplicity m_α and $< \cdot, \cdot >$ denotes the scalar product on $\mathfrak{a}$ and $\mathfrak{a}^*$ induced from the Killing form of the Lie algebra of G. Throughout all this paper we use the same notation for a $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and the dual element of $\mathfrak{a}_\mathbb{C}$, i.e. we write $\lambda(a) = < \lambda, a >$.

The spherical function $\phi_\lambda(g)$ is holomorphic in λ for every $g \in G$ fixed; it is invariant with respect to the Weyl group W acting on $\mathfrak{a}$:

$$\phi_\lambda \equiv \phi_{w\lambda}, \quad w \in W.$$

By the Helgason-Johnson theorem, the spherical functions are bounded if and only if λ belongs to the tube $T_\rho = \mathfrak{a}^* + iC(\rho)$, where $C(\rho)$ is the convex hull of the set $\{w\rho,\ w \in W\}$.

The *spherical Fourier transform* of $\mu \in M^\natural(X)$ is defined by

$$\hat{\mu}(\lambda) = \int_X \phi_\lambda(x) d\mu(x)$$

for $\lambda \in T_\rho$. It is holomorphic in the interior of the tube, continuous on the whole closed tube and W-invariant.

By the Lévy - Gangolli continuity theorem[7], if $\hat{\mu}_n(\lambda) \to \hat{\mu}(\lambda)$ on $\mathfrak{a}^*$ then $\mu_n \Rightarrow \mu$.

The Gaussian measures on X are defined by means of generators. A measure $\gamma \in M^\natural(X)$ is called *Gaussian* if it belongs to a continuous semigroup of probability measures whose generator is a second order G-invariant elliptic differential operator on X annihilating constants. If X is irreducible then all such operators are given by positive multiples of the Laplace-Beltrami operator Δ on X (see[10]).

The Gaussian measures on X form then the *heat semigroup* $\{\gamma_t\}_{t>0}$ whose spherical transform is given by

$$\hat{\gamma}_t(\lambda) = e^{-t(<\lambda,\lambda>+<\rho,\rho>)}, \quad t > 0. \tag{2}$$

The number $-(<\lambda,\lambda> + <\rho,\rho>)$ is the eigenvalue of Δ acting on ϕ_λ.

1.3. Parameters of measures and Iwasawa transform

In the study of limit properties of K-invariant measures one has to control, like in the real case, some parameters of considered measures.

Dispersions $D_j\mu$ of a K-invariant probability measure μ have been defined[6,9,10] in the following way.

Definition. —

$$D_j\mu = \int Q_j(x)d\mu(x) \quad \text{with} \quad Q_j(x) = \frac{1}{2i\|\alpha_j\|^2}\frac{\partial\phi_\lambda}{\partial\alpha_j}\Big|_{\lambda=-i\rho}$$

where α_j are simple positive roots and $j = 1, ..., n = \text{rank}X$. One proves there that all the applications $D : M^\natural(X) \to [0,\infty]$ with the property $D(\mu_1 * \mu_2) = D\mu_1 + D\mu_2$ and such that $D\mu = \int Q(x)d\mu(x)$ for an analytic K-invariant function Q are given by positive linear combinations of D_j. One may compute the dispersions of μ knowing its spherical Fourier transform

$$D_j\mu = \frac{1}{2i\|\alpha_j\|^2}\frac{\partial\hat{\mu}(\lambda)}{\partial\alpha_j}\Big|_{\lambda=-i\rho}.$$

Now consider the spherical Fourier transform $\hat{\mu}$ of $\mu \in M^\natural(X)$ restrained to the hyperplane $\{\lambda - i\rho | \lambda \in \mathfrak{a}^*_{\mathbb{R}}\}$. It is a positive definite function of $\lambda \in \mathfrak{a}^*_{\mathbb{R}}$ and as $\hat{\mu}(-i\rho) = 1$ there exists a probability measure on $\mathfrak{a}$ whose ordinary Fourier transform is equal to $\hat{\mu}$ on the hyperplane $\{\lambda - i\rho | \lambda \in \mathfrak{a}^*_{\mathbb{R}}\}$.

Definition. — Let $\mu \in M^\natural(X)$. We define the transform $\tilde{\mu}$ of μ as the probability measure on $\mathfrak{a}$ verifying

$$\mathcal{F}(\tilde{\mu})(\lambda) = \hat{\mu}(\lambda - i\rho), \qquad \lambda \in \mathfrak{a}^*_{\mathbb{R}}. \tag{3}$$

This transform of K-invariant probability measures on G/K plays an important role in studying of asymptotic behaviour of μ^m. It was introduced in some special cases by Tutubalin[23].

LEMMA 1. — *(i) If $\tilde{\mu} = \tilde{\eta}$ for $\mu, \eta \in M^\natural(X)$ then $\mu = \eta$.*
(ii) $\widetilde{\mu^m} = \tilde{\mu}^m$ where $\tilde{\mu}^m$ denotes the ordinary m-th convolution of $\tilde{\mu}$ on $\mathfrak{a}$, $m > 1$.
(iii) $\tilde{\mu}$ is the image on $\mathfrak{a}$ of μ by the Iwasawa projection $g \to \mathcal{H}(g)$, that is $\tilde{\mu}(B) = \mu(\mathcal{H}^{-1}(B))$ for $B \in \mathcal{B}_{\mathfrak{a}}$.
(iv) If γ is Gaussian on X then $\tilde{\gamma}$ is Gaussian on $\mathfrak{a}$.

Proof. — Property (i) follows from the holomorphy and W-invariance of the spherical Fourier transform and a version of the Phragmen-Lindelöf theorem. (ii) is implied by convolution properties of the ordinary and spherical Fourier transforms.
To prove (iii) we deduce from (1) and the K-invariance of μ that

$$\hat{\mu}(\lambda - i\rho) = \int e^{i<\lambda, \mathcal{H}(g)>} d\mu(g).$$

Finally (iv) follows from (2) writing

$$\mathcal{F}\tilde{\gamma}_t(\lambda) = e^{-t<\lambda,\lambda>} e^{2it<\lambda,\rho>}. \tag{4}$$

$\square$

We now define after Tutubalin[23] some parameters of $\mu \in M^\natural(X)$ by means of classical parameters of $\tilde{\mu}$ - the mean $\mathbf{E}\tilde{\mu}$ and the covariance matrix $\mathbf{D}\tilde{\mu}$.

Definition. — Let $\mu \in M^\natural(X)$ and suppose that $\tilde{\mu}$ has finite mean and covariance. We define the mean and the covariance of μ putting

$$\mathbf{E}\mu = \mathbf{E}\tilde{\mu} \qquad \text{and} \qquad \mathbf{D}\mu = \mathbf{D}\tilde{\mu}.$$

We now give some properties of the mean and the covariance of the K-invariant probability measures on symmetric spaces.
(i) Knowing the spherical Fourier transform of μ we may compute the parameters of μ directly by the formulas:

$$(\mathbf{E}\mu)_k = \frac{1}{i}\frac{\partial}{\partial\lambda_k}\Big|_{\lambda=0}\hat{\mu}(\lambda - i\rho)$$

$$(\mathbf{D}\mu)_{kl} = -\frac{\partial^2}{\partial\lambda_k\partial\lambda_l}\Big|_{\lambda=0}\hat{\mu}(\lambda - i\rho) - (\mathbf{E}\mu)_k(\mathbf{E}\mu)_l.$$

(ii) For $\mu \neq \delta_{x_o}$ the mean $\mathbf{E}\mu = \big((\mathbf{E}\mu)_k\big)_{k=1,\dots n}$ is a non-zero vector in the positive Weyl chamber $\mathfrak{a}^+$.
To prove this one may establish easily the relation between $\mathbf{E}\mu$ and the dispersions $D_j\mu,\ j = 1, ..., n$ of μ. Namely, one has

$$< \alpha_j, \mathbf{E}\mu >= 2\|\alpha_j\|^2 D_j\mu.$$

As $D_j\mu \geq 0$ we see that $\mathbf{E}\mu \in \overline{\mathfrak{a}^+}$ for all $\mu \in M^\natural(X)$; on the other hand [10]Thm.2 implies that if $\mathbf{E}\mu = 0$ then $\mu = \delta_{x_o}$. In fact, the convexity of the function $t \to \phi_{i(\rho+t\alpha_j)}(a)$ where $t \in \mathbb{R}$ and $a \neq 0$ and the fact that $|\phi_{i(\rho+t\alpha_j)}(a)| < 1$ for $t < 0$ sufficiently near 0 imply that $Q_j(a) > 0$ for $a \neq 0$ and $\mathbf{E}\mu \in \mathfrak{a}^+$ for $\mu \neq \delta_{x_o}$.

(iii) $\mathbf{D}\mu$ is a positive definite $n \times n$ matrix.

(iv) One has also for $\mu_1, \mu_2 \in M^\natural(X)$

$$\mathbf{E}(\mu_1 * \mu_2) = \mathbf{E}\mu_1 + \mathbf{E}\mu_2 \quad \text{and} \quad \mathbf{D}(\mu_1 * \mu_2) = \mathbf{D}\mu_1 + \mathbf{D}\mu_2.$$

Example. — If γ_t is a Gaussian measure on G/K we have by (4)

$$D_j\gamma_t = t, \quad \mathbf{E}\gamma_t = 2t\rho \quad \text{and} \quad \mathbf{D}\gamma_t = 2t\mathrm{Id}$$

where Id denotes the identity $n \times n$ matrix; in the above formula $\mathbf{D}\gamma_t$ is written in the canonical basis on $\mathfrak{a}$ orthonormal with respect to $< \cdot, \cdot >$.

2. Central limit theorems on Riemannian symmetric spaces

First central limit theorem on a symmetric space was established in the case of the hyperbolic plane and space by Karpelevich, Shur and Tutubalin[16]. The problem was then studied from many different points of view by several authors and led to basically three kinds of central limit theorems which we are going to present now briefly.

Limit theorems of Lindeberg type for products of normalized independent identically distributed random variables with values in a symmetric space were proved by Terras[21,22] and Richards[19] on the spaces $GL(n,\mathbb{R})/O(n)$ of symmetric positive definite matrices and next by Zhang[26] on symmetric spaces which are symmetric cones of Euclidean Jordan algebras.

In this kind of central limit theorem one considers a sequence Y_m of independent random variables with values in one of the symmetric spaces mentioned above. One supposes that Y_m have a common probability law $\mu \in M^\natural(X)$ which is absolutely continuous with respect to the invariant measure on X and such that $\mu_{\mathfrak{a}}$ has the mean zero and identity covariance matrix. In order to normalize the product of m variables Y_j one considers the corresponding elements of the group G and one normalizes $Y_j = k_j \exp a_j k_j'$ written in the Cartan decomposition by putting

$$Y_j^\sharp(m) = k_j \exp(\frac{1}{\sqrt{m}} a_j) k_j'$$

i.e. one divides the logarithms of the eigenvalues of Y_j by $\sqrt{m}$. Then one has the following theorem:

THEOREM A. — *The sequence of products of normalized random variables*

$$S_m^\sharp = Y_1^\sharp(m) \circ \cdots \circ Y_m^\sharp(m)$$

converges weakly to a Gaussian random variable on X.

The proof is based on writing the spherical Fourier transform of $S_m^\sharp$ and studying of its limit when $m \to \infty$. Using the Lévy - Gangolli continuity theorem one obtains in the limit a Gaussian measure on X whose parameters are given precisely.

Another type of central limit theorems, Lindeberg - Feller type theorems for infinitesimal arrays of measures was obtained by Faraut[6] in the hyperbolic plane case and by the first author in the $GL(n,\mathbb{R})/O(n)$ case[9] and next in general, for an irreducible Riemannian symmetric space of non-compact type[10].

Given a family $\{\mu_{mj}\}_{m\in\mathbb{N},1\le j\le k_m}$ of K-invariant probability measures on X, let $\mu_m = \mu_{m1} * \ldots \mu_{mk_m}$. Denote by $\sigma(x)$ the geodesic distance of x from $x_o = eK$ and as in Section 1 the dispersions of μ by $D_j\mu$. We have

THEOREM B. —
Suppose that $\lim_{m\to\infty} D_1\mu_m = t > 0$ *and* $\lim_{m\to\infty} \sum_{j=1}^{k_m} \int (\sigma^3/(1+\sigma^2))d\mu_{mj} = 0$. *Then the measures μ_m converge weekly to the Gaussian measure γ_t.*

The proof of this theorem involves a Taylor expansion of the spherical functions on X

$$\phi_\lambda(\exp H) = 1 - (<\lambda,\lambda> + <\rho,\rho>)Q_j(\exp H) + R_{j,\lambda}(H)$$

where Q_j are the functions appearing in the definition of D_j. Once this expansion established with some rest estimates, one uses the Lévy-Gangolli theorem and one obtains Theorem B. In this type of central limit theorem one avoids normalizing measures which does not seem very natural for symmetric spaces.

Yet another kind of central limit theorem for convolutions of a fixed K-invariant measure μ was introduced and proved by Tutubalin[23].

THEOREM C. — *Let $\mu \in M^\natural(X)$ be such that*

$$\mathbf{E}\mu = \mathbf{E}\gamma \quad \text{and} \quad \mathbf{D}\mu = \mathbf{D}\gamma$$

for a Gaussian measure $\gamma \in M^\natural(X)$. Let $\mathcal{R}$ be the family of rectangles with respect to a fixed basis of $\mathfrak{a}$. Then, writing $R^+ = R \cap \overline{\mathfrak{a}^+} for R \in \mathcal{R}$, we have

$$\sup_{R\in\mathcal{R}} |\mu^m(K\exp R^+K) - \gamma^m((K\exp R^+K)| \to 0 \quad \text{as } m\to\infty.$$

This theorem was first obtained[23,20] for the spaces $SL(2,\mathbb{R})/SO(2)$ and $SL(2,\mathbb{C})/SU(2)$. An incomplete proof for $SL(n,\mathbb{C})/SU(n)$ was proposed in[24]. Next, in the random walks formulation it was proved for μ absolutely continuous by Virtser[25] and in the general case by Guivarc'h and Raugi[13]. These results do not require the hypothesis of K-invariance of μ. A geometric proof of Theorem C in the K-invariant case was given recently by Babillot[3]. Some related results were also obtained by Bougerol[5]. In this paper we prove some new properties of the Abel transform and the spherical functions on X in the complex case and we show the Theorem C by harmonic analysis methods in the complex and rank one case.

In this kind of limit theorem one proves that

$$\sup_{S\in\mathcal{S}} |\mu^{*m}(S) - \gamma^{*m}(S)| \to 0, \quad m \to \infty \tag{5}$$

where γ is a Gaussian measure on X and $\mathcal{S}$ is a class of natural subsets of the Cartan space $\mathfrak{a}$ of X - for instance intervals in the case of rank 1 and multidimensional rectangles for a higher rank. Remark that on $\mathbb{R}^n$ one may prove easily the following fact.

Proposition 1. — *Let μ be a probability measure on $\mathbb{R}^n$ and γ a Gaussian measure on $\mathbb{R}^n$ such that the means and the covariance matrices of μ and γ are equal. Let $X_1, X_2, ...$ be independent random variables on R^n with law μ. Then*

$$\frac{X_1 + X_2 + ... + X_m - m\mathbf{E}X_1}{\sqrt{m}} \overset{\mathrm{m}}{\Rightarrow} \gamma \quad \Longleftrightarrow \quad \sup_{R\in\mathcal{R}} |\mu^{*m}(R) - \gamma^{*m}(R)| \overset{\mathrm{m}}{\to} 0$$

where $\mathcal{R}$ is the family of all infinite rectangles $R_{\mathbf{x}} = \{\mathbf{y} | y_j \le x_j, \ j = 1, ...n\}$, $\mathbf{x} \in \mathbb{R}^n$.

This proposition shows that the Lindeberg central limit theorem on $\mathbb{R}^n$ is equivalent to a theorem of type (5) so the Theorem C is the most natural to expect and prove in the symmetric case.

Comparing Theorem B and Theorem C we see that in order to describe the limit behaviour of convolutions of a fixed measure $\mu \in M^{\natural}(X)$ one must control more parameters of μ than in the case of an infinitesimal array of measures. This phenomenon appears in the symmetric case and is different from the Euclidean case.

Central limit theorem given by Theorem C leads[3] to results on asymptotic behaviour of convolutions of measures on X and in particular to the results on the heat kernel behaviour obtained analytically by Anker and Setti[1]. One characterizes this behaviour on solid cones Γ_σ with angle σ around the ρ-axis in $\mathfrak{a}^+$ and the intersections of these cones with centered balls. We recall briefly these results.

Let $R(t)$ and $\sigma(t)$ be two positive functions with

$$t^{-1/2}R(t) \to \infty \quad \text{and} \quad t^{1/2}\sigma(t) \to \infty, \quad t \to \infty.$$

Denote $r_t = 2|\rho|t - R(t)$ and $R_t = 2|\rho|t + R(t)$.
Let $\Omega(t)$ be the intersection of the centered annulus of radia r_t, R_t with the cone $\Gamma_{\sigma(t)}$.

THEOREM D. — *Let $\mu \in M^\natural(X)$ with $\mathbf{E}\mu = 2\rho$, $\mathbf{D}\mu = 2\mathrm{Id}$. Then*

$$\lim_{m\to\infty} \mu^m(K\Omega(m)K) = 1.$$

In particular, Theorem D is true for Gaussian measures γ_t for which one obtains $\lim_{t\to\infty} \gamma_t(K\Omega(t)K) = 1$.

Let us finally mention two related results obtained recently.

The Cramer theorem on symmetric spaces was obtained in[11]. It says that if γ is a Gaussian measure on X and $\gamma = \mu_1 * \mu_2$ with $\mu_1, \mu_2 \in M^\natural(X)$ then μ_1 and μ_2 are also Gaussian. The proof uses again harmonic spherical analysis on X.

The Fourier transforms of positive finite measures on $\mathbb{R}^n$ are characterized, by the Bochner theorem, as positive definite functions on $\mathbb{R}^n$. A characterization of spherical Fourier transforms of K-invariant probability measures was obtained[12] in the complex case.

For many other earlier results concerning probability measures on symmetric spaces and more generally on Lie groups see Heyer[15], Letac[18] and Terras[22].

3. Abel transform estimates and a central limit theorem

First we will explain the relation between the transform $\mu \to \tilde{\mu}$ defined in Section 1 and the Abel transform on G/K.

For $a \in \mathfrak{a}$ denote by ν_a the image on $\mathfrak{a}$ of the Haar measure on K by the mapping $k \to \mathcal{H}(\exp a \cdot k)$. By a Kostant convexity theorem supp $\nu_a = C(a)$ where $C(a)$ is the convex hull of $\{wa | w \in W\}$. It follows by (1) that for $\lambda \in \mathfrak{a}^*_{\mathbb{R}}$

$$\phi_\lambda(\exp a) = \mathcal{F}(e^{-<\rho,H>} d\nu_a(H))(\lambda). \tag{6}$$

Moreover, if X is irreducible and $a \neq 0$ the measure ν_a is absolutely continuous on $\mathfrak{a}$ ([14]p.478). We denote its density by $F_a(H)$. It is a measurable function of $(a, H) \in (\mathfrak{a} \setminus \{0\}) \times \mathfrak{a}$.

Note that if for $f \in L_1^\natural(G/K)$ we write its Abel transform in the kernel form:

$$\mathcal{A}f(H) = \int_{\mathfrak{a}} K(a, H) f(a) j(a) da$$

where $j(a)$ is the jacobian of changing to polar coordinates, then $K(a, H) = e^{-<\rho,H>} F_a(H)$ for $a \neq 0$.

LEMMA 2. — *(i) Let $\mu \in M^\natural(X)$ and $\tilde{\mu}$ its transform on $\mathfrak{a}$ defined by (3). Then $\tilde{\mu}$ may be decomposed*

$$\tilde{\mu} = \tilde{\mu}_{ac} \quad + \quad \mu(\{x_o\})\ \delta_o$$

where $\tilde{\mu}_{ac}$ is absolutely continuous on $\mathfrak{a}$ with the density

$$H \to \int_{\mathfrak{a}\setminus\{0\}} F_a(H)d\mu_{\mathfrak{a}}(a)$$

and $x_o = eK$, e being the unity of G.

(ii) For any $B \in \mathcal{B}_{\mathfrak{a}}$

$$\tilde{\mu}(B) = \int_{\mathfrak{a}} \nu_a(B)d\mu_{\mathfrak{a}}(a) = \int_{\overline{\mathfrak{a}^+}} \nu_a(B)d\mu_+(a).$$

Proof. — By (1) we have for $a \neq 0$

$$\phi_{\lambda-i\rho}(\exp a) = \int_{\mathfrak{a}} e^{i<\lambda,H>}d\nu_a(H) = \int_{\mathfrak{a}\setminus\{0\}} e^{i<\lambda,H>}F_a(H)dH$$

and $\phi_{\lambda-i\rho}(e) = 1$. Hence using Fubini theorem, for $\lambda \in \mathfrak{a}^*_{\mathbb{R}}$

$$\mathcal{F}\mu(\lambda) = \int e^{i<\lambda,H>}\Big[\int_{\mathfrak{a}\setminus\{0\}} F_a(H)d\mu_{\mathfrak{a}}(a)\Big]dH + \quad \mu_{\mathfrak{a}}(0)$$

and (i) follows. Then (ii) is obvious using $\nu_a = \nu_{wa}$ following from (6) and the W-invariance of ϕ_λ . ∎

Lemma 2 shows that the knowledge of the properties of the function $F_a(H)$, so equivalently of the Abel transform kernel $K(a,H)$ is useful to study the relations between μ and $\tilde{\mu}$. We study some of these properties now.

3.1. Growth of Abel transform kernel in the complex case

In this section we assume that $X = G/K$ with G complex and we study some properties of the Abel transform kernel $K(a,H)$ and of the densities $F_a(H)$.

The spherical functions on X may be expressed in a more explicit form than (1) by the formula[14]

$$\phi_\lambda(\exp a) = \frac{\pi(\rho)}{\pi(i\lambda)} \frac{\sum_{w\in W} \epsilon(w)e^{i<w\lambda,a>}}{\prod_{\alpha\in\Sigma^+} 2\mathrm{sh} <\alpha,a>}, \quad a \in \mathfrak{a}, \tag{7}$$

where $\pi(\lambda) = \prod_{\alpha\in\Sigma^+} <\alpha,\lambda>$ and ϵ denotes the determinant of $w : \mathfrak{a} \mapsto \mathfrak{a}$.

By (6), for $a \in \mathfrak{a}\setminus\{0\}$ fixed we have $\mathcal{F}(K(a,H))(\lambda) = \phi_\lambda(\exp a)$. Putting

$$U_a(H) = \frac{1}{\pi(\rho)} \prod_{\alpha\in\Sigma^+} 2\mathrm{sh} <\alpha,a> K(a,H)$$

we have $U_a(H) \geq 0$ for $a \in \overline{\mathfrak{a}^+}$, supp$U_a = C(a)$ and

$$\mathcal{F}(U_a)(\lambda) = \frac{\sum_{w\in W} \epsilon(w) e^{i<w\lambda,a>}}{\pi(i\lambda)}, \qquad \lambda \in \mathfrak{a}^*_{\mathbb{R}}. \tag{8}$$

In order to study $K(a,H)$ or $F_a(H)$ in the complex case it suffices to deal with the function $U_a(H)$ for $a \in \overline{\mathfrak{a}^+}$.

Our first aim will be to give a more explicit expression for $U_a(H)$; next we will deduce from it some growth properties of $U_a(H)$ and $K(a,H)$.

To do this we use a fundamental solution T of the operator $\partial(\pi)$ on $\mathfrak{a}$ defined by the property

$$\partial(\pi) e^{<\lambda,y>} = \pi(y) e^{<\lambda,y>}. \tag{9}$$

We denote by Σ_o^+ the set of simple positive roots $\{\alpha_1,...,\alpha_n\}$ on $\mathfrak{a}$ and we set ${}^+\mathfrak{a} = \{a \in \mathfrak{a} | \quad a = \sum_{j=1}^n c_j\alpha_j, \quad c_j > 0\}$. For $x,y \in {}^+\mathfrak{a}$ we write $x \leq y$ if $x = \sum_{j=1}^n c_j\alpha_j,\ y = \sum_{j=1}^n d_j\alpha_j$ and $c_j \leq d_j$, $j=1,...,n$.

Proposition 2. — *Let T be a distribution on $\mathfrak{a}$ defined for $f \in \mathcal{C}_c^\infty(\mathfrak{a})$ by*

$$(T,f) = \int_{\mathbb{R}^q_+} f(\sum_{\alpha_k\in\Sigma^+} x_k\alpha_k) dx_1...dx_q$$

where $q = |\Sigma^+|$. Then

(i) $\partial(\pi)T = \delta_o$.

(ii) T is a continuous nonnegative function on $\overline{{}^+\mathfrak{a}}$.

(iii) There exists $c > 0$ such that $T(x) \leq c\|x\|^{q-n}$ for all $x \in \overline{{}^+\mathfrak{a}}$.

(iv) If $x,y \in \overline{{}^+\mathfrak{a}}$ and $x \leq y$ then $T(x) \leq T(y)$.

Proof. — (i) follows easily using $\frac{d}{dt}\mathbf{1}_{[0,\infty)}(t) = \delta_o$ and $\partial(\pi) = \frac{\partial^q}{\partial\alpha_1...\partial\alpha_q}$.

To prove (ii) we express $\alpha_{n+1},...,\alpha_q$ in the basis of simple positive roots by

$$\alpha_k = \sum_{j=1}^n a_{kj}\alpha_j, \quad a_{kj} \in \mathbb{N}\cup\{0\},\ k = n+1,...,q$$

and we write $\quad \sum_{k=1}^q x_k\alpha_k = \sum_{j=1}^n y_j\alpha_j \quad$ where

$$y_j = x_j + \sum_{k=n+1}^q x_k a_{kj}, \qquad j = 1,...,n. \tag{10}$$

The change of variables given by (10) and $y_k = x_k$ for $k = n+1,...,q$ gives

$$(T,f) = \int_{\mathbb{R}^n_+} f(\sum_{j=1}^n y_j\alpha_j)\Psi(y_1,...,y_n) dy_1...dy_n$$

where $\Psi(y_1, ..., y_n) = \int_{\Delta(y_1,...,y_n)} dy_{n+1}...dy_q$ with the set $\Delta(y_1, ..., y_n)$ defined by

$$(y_{n+1}, ..., y_q) \in \Delta(y_1, ..., y_n) \quad \Leftrightarrow \quad y_{n+1}, ..., y_q \geq 0 \ \text{ and}$$

(11) $$\sum_{k=n+1}^{q} y_k a_{kj} \leq y_j, \quad j = 1, ..., n.$$

Then Ψ is continuous, supported by $\overline{\mathbb{R}^n_+}$ and $T(\sum_{j=1}^n y_j \alpha_j) = \Psi(y_1, ..., y_n)$. Now (ii) and (iv) follow from (11) observing that

$$\Delta(y_1, ..., y_n) \subset \{(y_{n+1}, ..., y_q) \mid \ 0 \leq y_k \leq \max_{1\leq j\leq n} |y_j|, k = n+1, ..., q\}$$

and that if $x \leq y$ then $\Delta(x) \subset \Delta(y)$. ∎

Now we may prove our main result on the Abel transform kernel.

Theorem 1. — *(i) $U_a(H) = \sum_{w\in W} \epsilon(w) T(wa - H)$ for $a, H \in \mathfrak{a}$.*
(ii) There exists $c > 0$ such that if $a \in \overline{\mathfrak{a}^+}$ and $H \in C(a)$ we have

$$U_a(H) \leq c\|a - H\|^{q-n}.$$

Proof. — Observe that directly from the definition of T, for $\lambda \in \mathfrak{a}$

$$(T, e^{i<\lambda+i\rho,H>}) = \prod_{k=1}^{q} \int_0^\infty e^{i<\lambda,\alpha_k>x_k} e^{-<\rho,\alpha_k>x_k} dx_k = \frac{1}{\pi(\rho - i\lambda)}.$$

Denote $V(H) = \sum_{w\in W} \epsilon(w) T(wa - H)$. Then $V^\vee(H) = V(-H)$ integrates $e^{-<\rho,H>}$ and $V^\vee = T * (\sum_{w\in W} \epsilon(w)\delta_{-wa})$. It follows from (8) that

$$(V^\vee, e^{i<\lambda+i\rho,H>}) = \frac{\sum_{w\in W} \epsilon(w) e^{-i<w(\lambda+i\rho),a>}}{\pi(-i(\lambda + i\rho))} = (U_a^\vee, e^{i<\lambda+i\rho,H>})$$

for each $\lambda \in \mathfrak{a}$. Hence $V = U_a$.

To prove (ii) recall that if $a \in \mathfrak{a}^+$ and $w \in W$ then $a - wa \in {}^+\mathfrak{a}$ ([11]). It is easy to see that if $a \in \overline{\mathfrak{a}^+}$ and $H \in C(a)$ then $a - H \in \overline{{}^+\mathfrak{a}}$. If $wa - H \in \overline{{}^+\mathfrak{a}}$ for $w \neq id$ then $wa - H \leq a - H$ and by Proposition 2 (ii) and (iv) we have $0 \leq T(wa - H) \leq T(a - H)$. (ii) then follows from (i) and Proposition 2(iii). ∎

Corollary 1. — *Let $K(a, H)$ be the kernel of the Abel transform on $X = G/K$ with G complex. Then there exists $c > 0$ such that for $a \in \mathfrak{a}^+$ and $H \in C(a)$*

$$K(a, H) \leq c \frac{\|a - H\|^{q-n}}{\prod_{\alpha\in\Sigma^+} \mathrm{sh} < \alpha, a >}$$

where $q = |\Sigma^+|$ *and* $n =$*rank*X.

Remark. — The formula for $U_a(H)$ in Theorem 1(i) may serve to compute directly $U_a(H)$ so also the kernel $K(a,H)$ of the Abel transform on X.

For example, for $X = SL(3,\mathbb{C})/SU(3)$ we find easily $T(u\alpha_1 + v\alpha_2) = \min^+(u,v)$. If $a \in \mathfrak{a}^+$ and $H \in C(a)\cap\mathfrak{a}^+$ with $a = A\alpha_1 + B\alpha_2$ and $H = u\alpha_1 + v\alpha_2$ then one computes

$$U_a(H) = \min{}^+(2A - B, A - u, B - v, 2B - A).$$

This method of computation of $K(a,H)$ is simpler and more straightforward than that based on an Aomoto formula for the Abel transform[2], proposed by Beerends[4].

As an application of Theorem 1 we will now prove a concentration property of the measures ν_a. Intuitively, ν_a being images of the same measure - the Haar measure on K - by the Iwasawa projection $\mathcal{H}$ translated by a, they should concentrate on some neighbourhoods of a. We show that this is true uniformly for $a \in \mathfrak{a}^+$ in a positive distance from the walls of the chamber. We show that the same is also true in rank one case.

In the proof of this and other theorems which follow we use the same notation c for changing positive constants.

Theorem 2. — *Let* $\delta > 0$ *and* $\mathfrak{a}^+_\delta = \{a \in \mathfrak{a}^+ \mid \min_{\alpha\in\Sigma^+_o} < \alpha, a > \geq \delta\}$. *(i) If* G *is complex, then there exist* $c, d > 0$ *such that for all* $a \in \mathfrak{a}^+_\delta$ *and* $M > 0$

$$\nu_a\{H \in \mathfrak{a} \mid \ \|H - a\| > M\} \leq ce^{-dM}.$$

(ii) If X *is of rank one then there exist* $c > 0$ *such that for all* $a > \delta$ *and* M *sufficiently big*

$$\nu_a\{H \in \mathfrak{a} \mid \ |t - a| > M\} \leq ce^{-\rho M}.$$

Proof. — (i) Denote $B(a,M) = \{H \in \mathfrak{a} \mid \ \|H - a\| \leq M\}$. There exists $c > 0$ such that $\mathrm{sh}x > ce^x$ for $x \geq \delta$. It follows that for $a \in \mathfrak{a}^+_\delta$

$$\begin{aligned} \nu_a(B(a,M)^c) &= \frac{1}{\pi(\rho)}\int_{B(a,M)^c} \frac{e^{<\rho,H>}U_a(H)}{\prod_{\alpha\in\Sigma^+} 2\mathrm{sh} < \alpha, a >} dH \\ &\leq \ c\int_{B(a,M)^c\cap C(a)} e^{<\rho, H-a>}U_a(H)dH. \end{aligned} \tag{12}$$

Let $\beta,\ 0 < \beta < \pi/2$ be the maximal angle between ρ and α when $\alpha \in \Sigma^+_o$. Then if H runs over the convex hull $C(a)$ the angle between ρ and $H - a$ belongs to $[\pi - \beta, \pi]$. Writing $k = \cos\beta$ we have $k > 0$ and $< \rho, H - a > \leq \ -k\|\rho\| \ \|H - a\|$. Using this estimate and Theorem 1(ii) we deduce from (12) that for $a \in \mathfrak{a}^+_\delta$

$$\nu_a(B(a,M)^c) \leq c\int_{B(a,M)^c} \|H - a\|^{q-n} e^{-k\|\rho\| \ \|H-a\|} dH.$$

Passing in the last integral to the polar coordinates on $\mathfrak{a}$ translated by a ends the proof.

(ii) In the rank one case we have the following formula for the kernel of the Abel transform[17]

$$K(a,t) = c\frac{(\mathrm{ch}a - \mathrm{ch}t)^{\kappa-\frac{1}{2}}}{\mathrm{ch}^{\beta+\frac{1}{2}}a\mathrm{sh}^{2\kappa}a}\ {}_2F_1(\frac{1}{2}+\beta, \frac{1}{2}-\beta, \frac{1}{2}+\kappa, \frac{\mathrm{ch}a-\mathrm{ch}t}{2\mathrm{ch}a}),\ 0 \le t \le a$$

where $\kappa = \frac{1}{2}(m_\alpha + m_{2\alpha} - 1)$ and $\beta = \frac{1}{2}(m_{2\alpha} - 1)$, α being the simple positive root. As the hypergeometric function ${}_2F_1(a,b,c,z)$ is holomorphic for $|z| < 1$ we see that the hypergeometric factor of $K(a,t)$ is bounded.

It is easy to see that for $\kappa \ge \frac{1}{2}$ and $|a| > \delta$ one has $K(a,t) \le ce^{-\rho a}$ which gives $F_a(t) \le ce^{\rho(t-a)}$ and the estimate for $\nu_a(B(a,M)^c)$ follows.

The only case when $\kappa = \frac{1}{2}$ is the hyperbolic plane $SL(2,\mathbb{R}/SO(2)$. One has then $F_a(t) = e^{t/2}(\mathrm{ch}a - \mathrm{ch}t)^{-1/2}$, $0 \le t \le a$. For M sufficiently big and $a - t > M$ we have $\mathrm{ch}a - \mathrm{ch}t > \mathrm{ch}a - \mathrm{ch}(a-M) > \frac{1}{2}\mathrm{ch}a$ and $F_a(t) \le ce^{\frac{1}{2}(t-a)}$ which implies (ii).
∎

3.2. Asymptotic behaviour of μ^m near the walls of Weyl chambers

In this section we consider $X = G/K$ arbitrary and we show that if $\mu \ne \delta_{x_o}$ then the convolution powers μ^m concentrate in a positive distance from the boundary of the Weyl chambers.

Our first objective is to show that for each wall of $\mathfrak{a}^+$ one may find a non-constant bounded spherical function which is greater than a positive constant on this wall and near it.

Next we use these functions to study the behaviour of μ^m, $\mu \in M^\natural(X)$ on certain neighbourhoods of the boundary of $\mathfrak{a}^+$.

Let I be a nonempty subset of the set of simple positive roots Σ_o^+. We denote by C_I the wall of $\mathfrak{a}^+$ determined by I:

$$C_I = \{a \in \mathfrak{a} \mid <\alpha, a>= 0 \ \text{ for } \alpha \in I;\ <\beta, a>> 0 \ \text{ for } \beta \in \Sigma_o^+ \setminus I\}.$$

We set for $\delta > 0$

$$B_I^\delta = \{a \in \overline{\mathfrak{a}^+} \mid <\alpha, a>< \delta \text{ if } \alpha \in I\ ; <\beta, a> \ge \ \text{ if } \beta \in \Sigma_o^+ \setminus I\}.$$

Proposition 3. — *(i) Let $\lambda \in \overline{\mathfrak{a}^+}$. Then*

(13) $$\inf_{a \in \overline{\mathfrak{a}^+}} e^{<\rho-\lambda, a>}\phi_{-i\lambda}(\exp a) > 0.$$

(ii) Let $I \subset \Sigma_o^+$, $I \ne \emptyset$ and $I \ne \Sigma_o^+$. Set $\xi = \rho - \epsilon\alpha$ for an $\epsilon > 0$ and $\alpha \in I$ such that $\xi \in C(\rho) \cap \mathfrak{a}^+$. Then

(14) $$\inf_{a \in B_I^\delta} \phi_{-i\xi}(\exp a) > 0.$$

Proof. — By [14]Prop.IV.6.3 we may write

$$\phi_{-i\lambda}(\exp a) = e^{<\lambda-\rho,a>} \int_{\bar{N}} e^{<\lambda-\rho,\mathcal{H}(\exp a\bar{n}(\exp a)^{-1})>-<\lambda+\rho,\mathcal{H}(\bar{n})>} d\bar{n}$$

where $\bar{N}$ is the image by the Cartan involution of the nilpotent group N in the Iwasawa decomposition of G and $d\bar{n}$ is an appropriately normalized Haar measure on $\bar{N}$.

As $\lambda \in \overline{\mathfrak{a}^+}$ and $\rho \in \mathfrak{a}^+$ it follows, using [14]Prop.IV.6.6, that for all $a \in \overline{\mathfrak{a}^+}$

$$e^{<\rho-\lambda,a>}\phi_{-i\lambda}(\exp a) \geq \int_{\bar{N}} e^{-<2\rho+\lambda,\mathcal{H}(\bar{n})>} d\bar{n} > 0$$

which proves (13). The second part of the Proposition follows from (13).

∎

Remark. — In the statement of Proposition 3(ii) one may take any $0 < \epsilon < 1$; indeed $\rho - \epsilon\alpha \in C(\rho) \cap \mathfrak{a}^+$ if and only if $\epsilon \in (0,1)$. Moreover, (14) remains true for $\xi \in C(\rho) \cap \mathfrak{a}^+$ of the form

$$\xi = \rho - \sum_{\alpha\in I} \epsilon_\alpha \alpha \quad \text{with} \quad \epsilon_\alpha \geq 0.$$

However, if we take $\xi \in C(\rho) \cap \mathfrak{a}^+$ such that $\rho - \xi \notin \mathrm{Vect} I$ then (14) fails; in particular this is the case of any $\xi \in \mathrm{Int} C(\rho)$.

We now use Proposition 3 to prove that for non-trivial $\mu \in M^\natural(X)$ the mass of μ^m does not concentrate near to the walls of Weyl chambers.

THEOREM 3. — *Let $\mu \in M^\natural(X)$ and $\mu \neq \delta_{x_o}$. For any $\delta > 0$ let*

$$B^\delta = \{a \in \overline{\mathfrak{a}^+} \mid \min_{\alpha\in\Sigma_o^+} < \alpha, a > < \delta\}.$$

Then for each $\delta > 0$ $\qquad \lim_{m\to\infty} \mu_+^m(B^\delta) = 0.$

Proof. — Observe that

$$B^\delta = \bigcup_{\emptyset\neq I\subset\Sigma_o^+} B_I^\delta$$

so to prove the theorem it is sufficient to show that

$$\mu_+^m(B_I^\delta) \to 0 \tag{15}$$

for each nonempty subset of simple positive roots I.

If $I \neq \Sigma_o^+$ we consider a spherical function $\phi_{-i\xi}$ as in Proposition 3(ii). As we saw in the beginning of this section, for $a \neq 0$

$$\phi_{-i\xi}(\exp a) = \int e^{<\xi, H>} e^{-<\rho, H>} F_a(H) dH.$$

By strict convexity of the exponential function and the fact that $\phi_{-i\rho} = \phi_{-iw\rho} \equiv 1$ for $w \in W$ it follows that $\phi_{-i\xi}(\exp a) < 1$. As $\mu \neq \delta_{x_o}$, it implies that $0 \leq \hat{\mu}(-i\xi) < 1$. Using Proposition 3 we have

$$\hat{\mu}(-i\xi)^m = \int \phi_{-i\xi} d\mu_+^m \ \geq \ c \int_{B_I^\delta} d\mu_+^m = c\mu_+^m(B_I^\delta)$$

for a positive constant c and all $m \in \mathbb{N}$. Letting $m \to \infty$ we get (15).

If $I = \Sigma_o^+$ one may take in the above estimate any positive bounded non-constant spherical function on X. $\square$

Remark. — The case $I = \Sigma_o^+$ in the proof of Theorem 3 corresponds to showing that

$$\mu_+^m(\{a \in \overline{\mathfrak{a}^+} \mid \max_{\alpha \in \Sigma_o^+} < \alpha, a > < \delta\}) \to 0$$

for any $\delta > 0$. In fact, one may prove in a similar way, using the spherical function ϕ_o that for any $X = G/K$ one has (cf. [23] Lemma 1 in the case of $SL(2, \mathbb{R})/SO(2)$ and $SL(2, \mathbb{C})/SU(2)$)

PROPOSITION 4. — *There exists $c > 0$ such that for $m \to \infty$*

$$\mu_{\mathfrak{a}}^m(\{a \in \mathfrak{a} \mid \ \|a\| < cm\}) \to 0.$$

3.3. Central limit theorem

In this paragraph we prove by harmonic analysis methods developed in paragraphs 3.1 and 3.2 the central limit theorem concerning the asymptotic behaviour of convolution powers μ^m of a K-invariant probability measure on $X = G/K$ with X irreducible and G complex or G of rank one. (Theorem C of Section 2).

The transform $\mu \to \tilde{\mu}$ defined in Section 1 may be useful in the study of μ^m on the condition that we are able to inverse it to some extent. In[23] Tutubalin proposes an inversion formula in the case $SL(2, \mathbb{R})/SO(2)$ and $SL(2, \mathbb{C})/SU(2)$ which may be generalized to any symmetric space of rank 1. In the case of higher rank this kind of formula, based on integration by parts argument, becomes too complicated due to the complicated form of the kernel of the Abel transform appearing in the transform $\mu \to \tilde{\mu}$(Lemma 2).

Here we use some natural mutual estimates of μ and $\tilde{\mu}$ the idea of which is based on the fact that far away from the walls of Weyl chambers we control the behaviour of ν_a appearing in the absolutely continuous part of $\tilde{\mu}$(Theorem 2) and near the walls μ^m tends to 0 (Theorem 3)(cf.[24]).

Let C be a Borel subset of $\mathfrak{a}$. To compare μ and $\tilde{\mu}$ on C we use the M-hull $C^M = \{a \in \mathfrak{a} \mid \inf_{x\in C} \|a - x\| \leq M\}$ of C and the "interior" M-hull $C^{-M} = C \setminus (C^c)^M$.

As the atoms of μ_+ and $\tilde{\mu}$ in the origin are equal we may suppose that $\mu(\{x_o\}) = 0$. Take $\delta > 0$. By Lemma 2(ii) we may write:

$$\tilde{\mu}(C) \ \leq \ \int_{\mathfrak{a}_\delta^+} \nu_a(C) d\mu_+(a) + \mu_+(B^\delta) \tag{16}$$

where $\mathfrak{a}_\delta^+$ and $B^\delta = \overline{\mathfrak{a}^+} \setminus \mathfrak{a}_\delta^+$ were defined in Theorem 2 and Theorem 3. Now, wanting to make use of the known behaviour of $\nu_a(B(a,M)^c)$ for $a \in \mathfrak{a}_\delta^+$ we write the integral in (16) as a sum of integrals on $\mathfrak{a}_\delta^+ \cap C^M$ and on $\mathfrak{a}_\delta^+ \setminus C^M$ for an $M > 0$ and we have

$$\int_{\mathfrak{a}_\delta^+} \nu_a(C) d\mu_+(a) \leq \ \mu_+(C^M) + \int_{\mathfrak{a}_\delta^+ \setminus C^M} \nu_a(C) d\mu_+(a). \tag{17}$$

Finally, noting that if $a \in \mathfrak{a}_\delta^+ \setminus C^M$ then $C \subset B(a,M)^c$ it follows from (16) and (17) that

$$\tilde{\mu}(C) \ \leq \ \mu_+(C^M) + \sup_{a\in\mathfrak{a}_\delta^+} \nu_a(B(a,M)^c) + \mu_+(B^\delta) \tag{18}$$

for any $\mu \in M^\natural(X)$, $C \in \mathcal{B}_\mathfrak{a}$, $\delta > 0$ and $M > 0$.

Similarly, we get

$$\tilde{\mu}(C^M) \geq \mu_+(C) - \sup_{a\in\mathfrak{a}_\delta^+} \nu_a(B(a,M)^c) - \mu_+(B^\delta). \tag{19}$$

Combining (19) with (18) applied to C^{-M} we obtain

Lemma 3. — *For any $\mu \in M^\natural(X)$, $C \in \mathcal{B}_\mathfrak{a}$, $\delta > 0$ and $M > 0$*

$$|\mu_+(C) - \tilde{\mu}(C)| \ \leq \tilde{\mu}(C^M \setminus C^{-M}) + \sup_{a\in\mathfrak{a}_\delta^+} \nu_a(B(a,M)^c) + \mu_+(B^\delta).$$

A natural class of subsets of the Euclidean space $\mathfrak{a}$ on which one would like to compare μ and $\tilde{\mu}$ is formed by the multidimensional rectangles in different basis of $\mathfrak{a}$.

Fix a basis of $\mathfrak{a}$. Let $R_{\mathbf{x}}$ denote the rectangle $R_{\mathbf{x}} = \{\mathbf{y} \mid y_j \leq x_j, j = 1, ..., n\}$ and $R_{\mathbf{x},\mathbf{y}}$ the bounded rectangle $R_{\mathbf{y}} \setminus R_{\mathbf{x}},\ \mathbf{x} \leq \mathbf{y}$. We denote all the family of such rectangles by $\mathcal{R}$. We also write $\mathbf{x} + M = (x_1 + M, ..., x_n + M)$ for $M \in \mathbb{R}$.

On considering families of rectangles, it is more natural to use the maximum norm with respect to the concerned basis. We denote balls in this norm by $B_{\max}$.

In exactly the same way as Lemma 3 one proves

LEMMA 3'. — *For any $\mu \in M^\natural(X)$, $R_{\mathbf{x}} \in \mathcal{R}$, $\delta > 0$ and $M > 0$*

$$|\mu_+(R_{\mathbf{x}}) - \tilde{\mu}(R_{\mathbf{x}})| \ \leq \tilde{\mu}(R_{\mathbf{x}-M,\mathbf{x}+M}) + \sup_{a \in \mathfrak{a}_\delta^+} \nu_a(B_{\max}(a, M)^c) + \mu_+(B^\delta).$$

Remark. — The most natural choice of a basis for the family of rectangles on which we compare μ_+ and $\tilde{\mu}$ is that of the dual basis $\{\alpha_1^*, ..., \alpha_n^*\}$ for the basis of simple positive roots. The walls of rectangles are then parallel to the walls of the chamber $\mathfrak{a}^+$.

Proof of Theorem C. — Suppose that $X = G/K$ with G complex or rank$X$=1. In our notation the limit property of Theorem C is equivalent to

$$\sup_{\mathbf{x} \in \mathfrak{a}} |\mu_+^m(R_{\mathbf{x}}) - \gamma_+^m(R_{\mathbf{x}})| \to 0 \quad \text{as } m \to \infty.$$

By Lemma 3', for any δ and M positive

$$\begin{aligned} |\mu_+^m(R_{\mathbf{x}}) - \gamma_+^m(R_{\mathbf{x}})| \ &\leq |\tilde{\mu}^m(R_{\mathbf{x}}) - \tilde{\gamma}^m(R_{\mathbf{x}})| + \tilde{\mu}^m(R_{\mathbf{x}-M,\mathbf{x}+M}) + \tilde{\gamma}^m(R_{\mathbf{x}-M,\mathbf{x}+M}) \\ &\quad + \mu_+^m(B^\delta) + \gamma_+^m(B^\delta) + 2 \sup_{a \in \mathfrak{a}_\delta^+} \nu_a(B_{\max}(a, M)^c). \end{aligned} \tag{20}$$

Let $\epsilon > 0$. Fixing $\delta > 0$ choose by Theorem 2 an $M > 0$ such that

$$2 \sup_{a \in \mathfrak{a}_\delta^+} \nu_a(B_{\max}(a, M)^c) \ < \ \epsilon.$$

Next, for m sufficiently big, by Theorem 3

$$\mu_+^m(B^\delta) + \gamma_+^m(B^\delta) \ < \ \epsilon.$$

By the central limit theorem on $\mathfrak{a}$ and Proposition 1

$$\lim_{m \to \infty} \sup_{\mathbf{x} \in \mathfrak{a}} |\tilde{\mu}^m(R_{\mathbf{x}}) - \tilde{\gamma}^m(R_{\mathbf{x}})| = 0$$

so to prove the theorem it suffices to show that the second and third term on the right-hand side of (20) converge to 0 for any $M > 0$, uniformly in $\mathbf{x} \in \mathfrak{a}$. This property of ordinary convolutions of non-atomic measures on an Euclidean space may be proved by a standard Fourier analysis argument(cf.[23](3) in the rank one case and [24]Lemma 5).

This ends the proof of the theorem. ∎

3.4. Normalization on the space $SL(2,\mathbb{C})/SU(2)$

In[23,5] one points out that in Theorem C, for a given measure $\mu \in M^\natural(X)$ with finite $\mathbf{E}\mu$ and $\mathbf{D}\mu$ one may not find a Gaussian measure γ on X such that μ^m and γ^m approach each other in the sense of Theorem C. Here we propose a normalization procedure in the case $SL(2,\mathbb{C})/SU(2)$ which allows to overcome this phenomenon.

Let $\delta_r^\natural$ be the K-invariant version of the Dirac measure δ_r concentrated on the diagonal matrix with non-zero entries e^r and e^{-r}.

One computes easily

$$\mathbf{E}\delta_r^\natural = \frac{r\mathrm{ch}r - \mathrm{sh}r}{\mathrm{sh}r} \quad \text{and} \quad \mathbf{D}\delta_r^\natural = \frac{\mathrm{sh}^2 r - r^2}{\mathrm{sh}^2 r}$$

and one verifies that $\mathbf{E}\delta_r^\natural \geq \mathbf{D}\delta_r^\natural$ for all $r > 0$. Moreover, the function $r \to \mathbf{E}\delta_r^\natural - \mathbf{D}\delta_r^\natural$ is an increasing bijection of $[0, \infty[$ on itself.

If γ_t is a Gaussian measure on $SL(2,\mathbb{C})/SU(2)$ then $\mathbf{E}\gamma_t = \mathbf{D}\gamma_t = 2t$.

Let now $\mu \in M^\natural(SL(2,\mathbb{C})/SU(2))$ with finite $\mathbf{E}\mu > 0$ and $\mathbf{D}\mu > 0$.

1) If $\mathbf{E}\mu = \mathbf{D}\mu$ then there exists a Gaussian measure γ such that μ^m and γ^m approach each other in the sense of Theorem C (we write briefly $\mu^m \sim \gamma^m$).

2) If $\mathbf{E}\mu < \mathbf{D}\mu$ there exists a unique $r > 0$ such that $\mathbf{D}\mu - \mathbf{E}\mu = \mathbf{E}\delta_r^\natural - \mathbf{D}\delta_r^\natural$ and $\mathbf{E}(\mu * \delta_r^\natural) = \mathbf{D}(\mu * \delta_r^\natural)$. There exists a Gaussian measure γ such that $(\mu * \delta_r^\natural)^m \sim \gamma^m$.

3) If $\mathbf{E}\mu > \mathbf{D}\mu$ we take the only $r > 0$ such that $\mathbf{E}\mu - \mathbf{D}\mu = \mathbf{E}\delta_r^\natural - \mathbf{D}\delta_r^\natural$. If $\mathbf{E}\mu = \mathbf{E}\delta_r^\natural$ then $\mu^m \sim \delta_r^{\natural m}$. If $\mathbf{E}\mu > \mathbf{E}\delta_r^\natural$ taking γ Gaussian such that $\mathbf{E}\gamma = \mathbf{D}\gamma = \mathbf{E}\mu - \mathbf{E}\delta_r^\natural$ we have $\mu^m \sim (\gamma * \delta_r^\natural)^m$. Finally, if $\mathbf{E}\mu < \mathbf{E}\delta_r^\natural$ we choose γ Gaussian such that $\mathbf{E}\gamma = \mathbf{D}\gamma = \mathbf{E}\delta_r^\natural - \mathbf{E}\mu$. Then $(\gamma * \mu)^m \sim \delta_r^{\natural m}$.

4. Acknowledgements

We would like to thank Jacques Faraut for suggesting the problem and for many valuable discussions. We thank Philippe Bougerol for his comments and calling our attention to several references. We also thank Jean-Philippe Anker for helpful comments and a short proof of Proposition 3. We are grateful to Herbert Heyer for the possibility of presenting of these results during the Oberwolfach Probability on Groups Conference 1994.

5. References

1. J.Ph.Anker and A.Setti, Asymptotic finite propagation speed for heat diffusion on certain Riemannian manifolds, *J.Funct.Anal.* **103**(1992), 50-61.

2. K. Aomoto, Sur les transformations d'horisphère et les équations intégrales qui s'y rattachent, *J.Fac.Sci.Univ.Tokyo* **14**(1967), 1-23.

3. M. Babillot, A probabilistic approach to heat diffusion on symmetric spaces, *J.Theor.Prob.* **7**(1994), 599-607.

4. R. Beerends, On the Abel transform and its inversion, *Ph.D. thesis* University of Leiden, 1987.

5. P. Bougerol, Comportement asymptotique des puissances de convolution d'une probabilité sur un espace symétrique, *Asterisque* **74**(1980), 29-45.

6. J. Faraut, Dispersion d'une mesure de probabilité sur $SL(2,\mathbb{R})$ biinvariante par $SO(2,\mathbb{R})$ et théorème de la limite centrale, Oberwolfach, 1975.

7. R. Gangolli, Isotropic infinitely divisible measures on symmetric spaces, *Acta Math.* **111**(1964), 213-246.

8. M.E. Gertsenshtein and V.B. Vasil'ev, Waveguides with random inhomogeneities and Brownian motion in the Lobachevsky plane, *Theory Prob.Appl.* **4** (1959), 391-398.

9. P. Graczyk, A central limit theorem on the space of positive definite symmetric matrices, *Ann.Inst.Fourier* **42**(1992), 857-874.

10. P. Graczyk, Dispersions and a central limit theorem on symmetric spaces, *Bull.Sciences Math.* **118**(1994), 105-116.

11. P. Graczyk, Cramer Theorem on Symmetric Spaces of Noncompact Type, *J.Theor.Prob.* **7**(1994),609-613.

12. P. Graczyk and J.J. Loeb, Bochner and Schoenberg theorems on symmetric spaces in the complex case, *Bull.Soc.Math.France* **122**(1994), 571-590.

13. Y. Guivarc'h and A. Raugi, Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence, *Zeit.Wahr.* **69**, 187-242.

14. S. Helgason, Groups and Geometric Analysis, Academic Press, New York, 1984.

15. H. Heyer, Probability Measures on Locally Compact Groups, Springer Verlag, Berlin 1977.

16. F.I.Karpelevich, M.G.Shur,V.N.Tutubalin, Limit theorems for the compositions of distributions in the Lobachevsky plane and space, *Theory Prob.Appl.* **4**(1959), 399-402.

17. T. Koornwinder, A new proof of a Paley-Wiener type theorem for the Jacobi transform, *Ark.Math.* **13**(1975), 145-159.

18. G. Letac, Problèmes classiques de probabilités sur un couple de Gelfand, Analytical methods in Probability theory, *Proceedings Oberwolfach 1980*, LNM 861, 93-120.

19. D.St.P. Richards, The central limit theorem on spaces of positive definite matrices, *J.Multivariate Anal.* **29**(1989), 326-332.

20. V.V. Sazonov, V.N. Tutubalin, Probability distributions on topological groups, *Theory Prob. Appl.* **11**(1966), 1-45.

21. A. Terras, Noneuclidean harmonic analysis, the central limit theorem and long transmission lines with random inhomogeneities, *J.Multivariate Anal.* **15**(1984), 261-276.

22. A. Terras, Harmonic Analysis on Symmetric Spaces and Applications I,II, Springer Verlag, New York,1985,1988.

23. V.N. Tutubalin, On the limit behaviour of compositions of measures in the plane and space of Lobachevsky, *Th. Prob.Appl.* **7**(1962), 189-196.

24. V.N. Tutubalin, The limiting behaviour of compositions of measures in certain homogeneous spaces, *Izv. Akad. Nauk SSSR Ser.Mat.***27** (1963), 1301-1342 (in Russian).

25. A.D. Virtser, Central limit theorem for semisimple Lie groups, *Th. Prob.Appl.* **15**(1970),667-687.

26. G. Zhang, The Asymptotics of Spherical Functions and the Central Limit Theorem on Symmetric Cones, preprint, University of Odense 1993.