HARDY SPACES ON SYMMETRIC SPACES -

APPLICATIONS OF JORDAN ALGEBRAS AND TRIPLE SYSTEMS

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The theory of Hardy spaces on complex semigroups was generalized by Hilgert, 'Olafsson and Ørsted (1991) to a class of symmetric spaces called symmetric spaces of Hermitian type. In this lecture we present another, geometrical approach to Hardy spaces on such symmetric spaces, using also the technique of Jordan algebras and Jordan triple systems. This is a joint work with Jacques Faraut and a preliminary version of a part of an article in preparation.

In the first part of these notes we will introduce the symmetric spaces of Hermitian type and give some examples. Next we will show in which way the Jordan theory is used in order to study properties of symmetric spaces of Hermitian type, very useful in the study of Hardy spaces.

In the second part of the lecture we will define the Hardy spaces on some domains in the complexification of symmetric spaces of Hermitian type and explain how their theory is related to the relative holomorphic discrete series (RHDS). The results of the first part will be used to determine the spectrum of a Hardy space.

We will keep to the examples rather than to the general case.

I. Symmetric spaces of Hermitian type

Definition. — Let D be a complex bounded symmetric domain in a complex vector space V. The domain D identifies with a Hermitian symmetric space

$$D \cong G/K$$

where G is the connected component of identity in the group $\mathrm{Aut}(G)$ of holomorphic automorphisms of D and K is the isotropy subgroup of $0 \in D$. Let θ be the Cartan involution of the pair (G, K) and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding decomposition of the Lie algebra $\mathfrak{g}$ of G.

Every complex bounded symmetric domain is isomorphic to a bounded symmetric and circled domain. On the other side, each circled bounded symmetric domain is the open unit ball of the spectral norm of a positive Hermitian Jordan triple system (PHJTS) (see Loos(1977), Satake(1980)) and conversely.

Let now ν be a complex conjugation of V (that is an antilinear involution of V), preserving D

$$\nu(D) = D.$$

We define an involution σ of the group G by

$$\sigma(g) = \nu \circ g \circ \nu, \qquad g \in G = \mathrm{Aut}(D)_o.$$

Remark that σ commutes with θ because it preserves D. Defining

$$H = \{g \in G |\ \sigma(g) = g\}$$

one says that the symmetric space

$$X = G/H$$

is of **Hermitian type**. Denote by $\mathfrak{h}$ the Lie algebra of H and write

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$$

the decomposition of $\mathfrak{g}$ with respect to σ.

The notion of a symmetric space of Hermitian type was introduced by 'Olafsson(1991) ("compactly causal" in his terminology). His definition is different (but equivalent). Let us cite the following equivalent property:
If the pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible (that is there is no compact ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$) then G/H is of Hermitian type if there exists a regular H-invariant cone $C_{\mathfrak{q}} \subset \mathfrak{q}$ such that $\mathfrak{k} \cap C_{\mathfrak{q}} \neq \{0\}$, which in turn is equivalent to $\dim(\mathfrak{q} \cap \mathfrak{k})^{(H \cap K)} = 1$. In particular, symmetric spaces of Hermitian type are causal.

Classification. — The bounded symmetric domains are classified (equivalently see the classification of Hermitian symmetric spaces of non-compact type). In order to know all symmetric spaces of Hermitian type it suffices to know the classification of complex conjugations of bounded symmetric domains or, equivalently, the classification of real forms of Jordan triple systems V containing D. If ν is an involution preserving D then

$$V_o = \{z \in V |\ \nu(z) = z\}$$

is a sub-triple system of V and V_o is a real form of V:

$$V = V_o \oplus iV_o.$$

Inversely, knowing a real form V_o of V, the complex conjugation with respect to V_o preserves D . The domain

$$D_o = D \cap V_o = \{z \in D |\ \nu(z) = z\}$$

is, by definition, a real bounded symmetric domain. Loos(1977) and Jaffee(1975,1978) give a complete classification of real forms of PHJTS's and real bounded symmetric domains.

An important class of symmetric spaces of Hermitian type is to be considered when the domain D is of **tube type**, that is V is a simple complex Jordan algebra, W a Euclidean real form of V and

$$D \cong W + i\Omega$$

where Ω is the interior of the cone of squares in W. In this case the classification of symmetric spaces of Hermitian type corresponds to the classification of real forms of complex simple Jordan algebras (see [Hellwig], [Kayoya]).

If V is a simple complex Jordan algebra and if the real form V_o fixed by ν is itself an Euclidean Jordan simple algebra then the space $X = G/H$ is said to be of **Cayley type**. Harmonic analysis on these spaces using the Jordan algebra approach was intensively developed by Faraut and Chadli.

We should notice that the first classification of symmetric spaces of Hermitian type was done by 'Olaffson using root systems (see [Hilgert,'Olafsson]).

Instead of giving the exhaustive classification we will present in detail some representative examples. First let us recall the different notions of rank which are important in the sequel.

Rank and split rank. — The rank of the domaine D, denoted by r, is equal to the real rank of the Hermitian symmetric space G/K, i.e. is equal to the dimension of a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Note that the rank of D is equal to the rank of V treated as a Jordan triple system.

The **split rank** of the symmetric space $X = G/H$, denoted by r_o, is equal to the dimension of a maximal abelian subspace $\mathfrak{b} \subset \mathfrak{p} \cap \mathfrak{q}$.

If $r = r_o$ we say that the space $X = G/H$ is **split**; otherwise it is called **non-split**.

The real bounded symmetric domain $D_o = D \cap V_o$ identifies with the Riemannian symmetric space $H/H \cap K$ so the rank of D_o is equal to the real rank of the space $H/H \cap K$, i.e. to the dimension of a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{h}$.

PROPOSITION 1. — *Let $X = G/H$ be a symmetric space of Hermitian type. Then the split rank r_o of X equals to the rank of the real bounded symmetric domain D_o.*

Proof. — From the construction of the Harish-Chandra embedding we know that

$$V = \mathfrak{p}_+^{\mathbb{C}} \cong \mathfrak{p},$$

the isomorphism of $\mathfrak{p}_+^{\mathbb{C}}$ ans $\mathfrak{p}$ being $\mathbb{R}$-linear. From the definition of σ we deduce that $V_o \cong \mathfrak{p} \cap \mathfrak{h}$ and $iV_o \cong \mathfrak{p} \cap \mathfrak{q}$. The multiplication by i in V sending V_o isomorphically onto iV_o corresponds to the isomorphism

$$\mathrm{Ad}(\exp(\frac{\pi}{2}Z_o)) : \mathfrak{p}\cap\mathfrak{h} \to \mathfrak{p}\cap\mathfrak{q}$$

where Z_o is an element of the center of $\mathfrak{k}$ such that $\mathrm{ad}(Z_o)$ has $i, 0, -i$ as eigenvalues on $\mathfrak{g}_{\mathbb{C}}$. In particular, $\dim(\mathfrak{p}\cap\mathfrak{h})=\dim(\mathfrak{p}\cap\mathfrak{q})$. ∎

Corollary 2. — *If $X = G/H$ is non-split then*

$$r = 2r_o.$$

Proof. — From the Proposition 1 we deduce that it suffices to compare the ranks of V_o and V as Jordan triple systems. It follows from the remark of Loos(1977), p.11.10, that either $r = r_o$ (V "reduced" in terms of Loos) or $r = 2r_o$ (V "non-reduced"). The proof of Loos is based on properties of the Jordan algebra and field V_1 in a Peirce decomposition $V = V_1 \oplus V_{1/2} \oplus V_0$ of V. ∎

Examples. — **1.** $\mathbf{X = SU(1,1)/SO(1,1)}$. Let D be the unit disc in $V = \mathbb{C}$. Then $D = G/K$ with

$$G = SU(1,1) = \Big\{ \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \big| |\alpha|^2 - |\beta|^2 = 1,\ \alpha,\beta\in\mathbb{C}\Big\},$$

and $K = SO(2)$, the Cartan involution being $\theta(g) = (g^*)^{-1}$.

Take $\nu(z) = \bar z$, the ordinary conjugation on $V = \mathbb{C}$. Then $\sigma(g) = \bar g$ and

$$H = SO(1,1) = \Big\{ \pm \begin{pmatrix} \mathrm{ch}\, t & \mathrm{sh}\, t \\ \mathrm{sh}\, t & \mathrm{ch}\, t \end{pmatrix} \big| t\in\mathbb{R}\Big\}.$$

On the level of Lie algebras we have

$$\mathfrak{g} = \Big\{ \begin{pmatrix} ix & z \\ \bar z & -ix \end{pmatrix} \big| z\in\mathbb{C}, x\in\mathbb{R}\Big\},$$

$$\mathfrak{k} = \Big\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \big| x\in\mathbb{R}\Big\},\ \mathfrak{p} = \Big\{ \begin{pmatrix} 0 & z \\ \bar z & 0 \end{pmatrix} \big| z\in\mathbb{C}\Big\}$$

$$\mathfrak{h} = \Big\{ \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \big| x\in\mathbb{R}\Big\},\ \mathfrak{q} = \Big\{ \begin{pmatrix} ix & iy \\ -iy & -ix \end{pmatrix} \big| x,y\in\mathbb{R}\Big\}.$$

We have then

$$\mathfrak{p}\cap\mathfrak{q} = \mathbb{R}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathfrak{b}$$

so $r_o = 1 = r$. The space $X = SU(1,1)/SO(1,1)$ is split. It is a Cayley type space and in fact all Cayley type spaces are split.

We will see later that the space $X = SU(1,1)/SO(1,1)$ plays a fundamental role in harmonic analysis on symmetric spaces of Hermitian type - there are techniques of reduction to the $SU(1,1)/SO(1,1)$ case.

2. Hyperboloids $\mathbf{X = SO(2,n)/SO(1,n)}$. Let D be the Lie ball in $V = \mathbb{C}^n$

$$D = \{z = x + iy \in \mathbb{C}^n \mid N(z) = (\|x\|^2 + \|y\|^2 + 2\sqrt{\|x\|^2|y\|^2 - (x|y)^2})^{1/2} < 1.$$

Recall that

$$D = SO(2,n)/SO(2) \times SO(n)$$

with the Cartan involution $\theta(g) = I_{2,n}gI_{2,n}\}$. We have

$$\mathfrak{g} = \left\{ \begin{pmatrix} \mathfrak{so}(2) & B \\ B^t & \mathfrak{so}(n) \end{pmatrix} \middle| B \in M_{\mathbb{R}}(2,n) \right\},\ \mathfrak{k} = \begin{pmatrix} \mathfrak{so}(2) & 0 \\ 0 & \mathfrak{so}(n) \end{pmatrix}, \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \right\}.$$

We see that D is of rank 2.

There are several complex conjugations of V preserving D. They are all of the form

$$\nu(z) = I_{p,q}\bar{z}, \quad p + q = n.$$

Consider the case $q = 0$, that is $\nu(z) = \bar{z}$. The corresponding involution $\sigma(g) = I_{1,n+1}gI_{1,n+1}$ and

$$\mathfrak{h} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathfrak{so}(1,n) \end{pmatrix},\ \mathfrak{q} = \left\{ \begin{pmatrix} 0 & a & \mathbf{d} \\ -a & 0 & \mathbf{0} \\ & & \\ \mathbf{d}^t & \mathbf{0} & \mathbf{0} \end{pmatrix} \middle| a \in \mathbb{R}, \mathbf{d} \in \mathbb{R}^n \right\}.$$

Then

$$\mathfrak{p} \cap \mathfrak{q} = \left\{ \begin{pmatrix} 0 & 0 & \mathbf{d} \\ 0 & 0 & \mathbf{0} \\ & & \\ \mathbf{d}^t & \mathbf{0} & \mathbf{0} \end{pmatrix} \right\}$$

and the split rank of $X = G/H$ is equal to 1. The spaces $SO(2,n)/SO(1,n)$, which may be identified to hyperboloids in $\mathbb{R}^{n+2}$, are non-split.

In the cases $0 < q < n$ one obtains split spaces $X = SO(2,n)/SO(1,p) \times SO(1,q)$. Here the domain D is of tube type. The space X is of Cayley type for $p = 1$ or $q = 1$.

In the case $q = 0$ one obtains $D = B_{\mathbb{R}^n}$, the ordinary unit ball and one has $r = r_o = 1$. **3. $\mathbf{V = M_{\mathbb{C}}(p,q)}$.** In this case $D = SU(p,q)/S(U(p) \times U(q))$ is the unit ball in V and if $p \neq q$ then V is not a Jordan algebra but only a Jordan triple system and D is not of tube type.

If we take $\nu(z) = \bar{z}$, we will obtain a symmetric space of Hermitian type $SU(p,q)/SO(p,q)$. It is split because the real form $V_o = M_{\mathbb{R}}(p,q)$ is of the same rank equal to $\min(p,q)$ as V.

If $p = q$ and one takes $\nu(z) = z^*$, one obtains V_o =Herm$(m, \mathbb{C})$ and a Cayley type space $X = SU(p,p)/SL(p,\mathbb{C}) \times \mathbb{R}^+$.

One more involution exists in the case when $p = 2p_o$ and $q = 2q_o$ are even. Set

$$\nu(z) = -J_{p_o} \bar{z} J_{q_o}$$

where

$$J_k = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Interpreting the quaternion field $\mathbb{H}$ as

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \right\}$$

we see that

$$V_o = M_{\mathbb{H}}(p_o, q_o).$$

In particular the rank r_o of V_o equal to $\min(p_o, q_o)$ is twice smaller than r =min(p,q) and the corresponding symmetric space of Hermitian type is non-split. The involution σ corresponding to ν acts on $\mathfrak{g}$ in the following way

$$\sigma(X) = \begin{pmatrix} J_{p_o} & 0 \\ 0 & -J_{q_o} \end{pmatrix} \bar{X} \begin{pmatrix} J_{p_o} & 0 \\ 0 & -J_{q_o} \end{pmatrix}$$

and $\mathfrak{h}$ identifies with $\mathfrak{sp}(p_o, q_o)$. Note that

$$\mathfrak{p} = \left\{ X = \begin{pmatrix} 0 & \mathbf{C} \\ \mathbf{C}^* & 0 \end{pmatrix} \middle| \mathbf{C} \in M_{\mathbb{C}}(p,q) \right\}$$

and $X \in \mathfrak{p} \cap \mathfrak{q}$ if and only if

$$\mathbf{C} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \bar{\mathbf{b}} & -\mathbf{a} \end{pmatrix}$$

where $\mathbf{a}, \mathbf{b} \in M_{\mathbb{C}}(p_o, q_o)$. One sees easily directly that a maximal Abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ is of dimension $\min(p_o, q_o)$.

Note that this example is the only non-split one except an exceptional case (when $V = M(1, 2, \mathbb{O}_{\mathbb{C}})$) for which V is not a Jordan algebra.

4. Group case. The simple Hermitian groups considered as symmetric spaces are of Hermitian type.

Let G_o be such a group, $D_o = G_o/K_o$ a symmetric bounded complex domain and $D = D_o \times \bar{D}_o$, where $\bar{D}_o$ is the domain D_o equipped with the opposite complex structure. Set

$$\nu(z, w) = (w, z)$$

which corresponds to $\sigma(g_1, g_2) = (g_2, g_1)$ on the group $G = G_o \times G_o$. One finds then $H = \mathrm{diag}(G)$ and the space $X = G/H$ identifies with G_o. We have $\mathfrak{h} = \{(X, X) |\ X \in \mathfrak{g}_o\}$, $\mathfrak{q} = \{(X, -X) |\ X \in \mathfrak{g}_o\}$, $\mathfrak{p} = \mathfrak{p}_o \times \mathfrak{p}_o$ and $\mathfrak{p} \cap \mathfrak{q} = \{(X, -X) |\ X \in \mathfrak{p}_o\}$. One can take $\mathfrak{b} = \{(X, -X) | X \in \mathfrak{a}_o\}$ where $\mathfrak{a}_o$ is a Cartan subspace of $\mathfrak{p}_o$. We see that $r = 2r_o$ and G_o treated as a symmetric space of Hermitian type is non-split.

Remark that in this example the domain $D = D_o \times \bar{D}_o$ is reducible whereas the pair (D, ν) is irreducible, i.e. it may not be decomposed into product of two pairs $(D_1, \nu_1) \times (D_2, \nu_2)$. The space $G_o = G/H$ is also irreducible as a symmetric space.

The homomorphism Ψ. — In the study of complex bounded symmetric domains one uses frequently(see e.g. [Satake]) a Harish-Chandra homomorphism

$$\Psi : SL(2, \mathbb{C})^r \to G^{\mathbb{C}}$$

such that $\Psi \circ \theta_o = \theta \circ \Psi$ (the subscript $\circ$ referring to the space $SL(2, \mathbb{C})^r$, e.g. θ_o is the Cartan involution of this space). In particular one takes as the Cartan space $\mathfrak{a} \subset \mathfrak{p}$ the image by $\psi = d\Psi$ of

$$\mathfrak{a}_o = \left\{ T = \left(\begin{pmatrix} 0 & it_1 \\ -it_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & it_r \\ -it_r & 0 \end{pmatrix} \right) \middle| t_1, \ldots, t_r \in \mathbb{R} \right\} \subset \mathfrak{p}_o.$$

In order to apply such a homomorphism in our set-up, we must control its properties with respect to the involution σ.

In the split case one may construct Ψ in such a way that

$$\Psi \circ \sigma_o = \sigma \circ \Psi$$

and $\mathfrak{a} = \mathfrak{b}$. From the Jordan theory point of view it is related to the fact that the real form V_o contains then a Jordan frame (in general of tripotents) of the complex space V.

In the non-split case this cannot be true because if Ψ preserved both involutions, we would have $\psi(\mathfrak{b}_o) \subset \mathfrak{b}$. As $\mathfrak{b}_o$ and $\psi(\mathfrak{b}_o)$ are of dimension r greater than $\mathfrak{b}$ this gives a contradiction.

Let us see in which way Ψ may be chosen on some examples.

Example 1 . — Consider a hyperboloid $X = SO(2, n)/SO(1, n)$. One has then $r = 2$, $r_o = 1$. One seeks to associate to the $\mathfrak{su}(1, 1)$-triple

$$H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

two mutually commuting $\mathfrak{su}(1, 1)$-triples in $\mathfrak{so}(2, n)$. One may find in this way two following Lie algebra homomorphisms from $\mathfrak{su}(1, 1)$ to $\mathfrak{so}(2, n)$

$$\psi_1 : \begin{pmatrix} ix & z \\ \bar{z} & -ix \end{pmatrix} \to \begin{pmatrix} \begin{matrix} 0 & x \\ -x & 0 \end{matrix} & \mathbf{0} & \begin{matrix} \mathrm{Re} z & \mathrm{Im} z \\ -\mathrm{Im} z & \mathrm{Re} z \end{matrix} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{matrix} \mathrm{Re} z & -\mathrm{Im} z \\ \mathrm{Im} z & \mathrm{Re} z \end{matrix} & \mathbf{0} & \begin{matrix} 0 & -x \\ x & 0 \end{matrix} \end{pmatrix}$$

$$\psi_2 : \begin{pmatrix} ix & z \\ \bar{z} & -ix \end{pmatrix} \to \begin{pmatrix} \begin{matrix} 0 & x \\ -x & 0 \end{matrix} & \mathbf{0} & \begin{matrix} -\mathrm{Re} z & \mathrm{Im} z \\ \mathrm{Im} z & \mathrm{Re} z \end{matrix} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{matrix} -\mathrm{Re} z & \mathrm{Im} z \\ \mathrm{Im} z & \mathrm{Re} z \end{matrix} & \mathbf{0} & \begin{matrix} 0 & x \\ -x & 0 \end{matrix} \end{pmatrix}$$

and define $\psi(X,Y) = \psi_1(X) + \psi_2(Y)$ on $\mathfrak{su}(1,1)^2$. One has $\psi_j \circ \theta_o = \theta \circ \psi_j$, $j = 1,2$, but

$$\sigma \circ \psi_1 = \psi_2 \circ \sigma_o \text{ and } \sigma \circ \psi_2 = \psi_1 \circ \sigma_o.$$

Taking

$$\mathfrak{a} = \psi(\mathfrak{a}_o) = \left\{ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{matrix} 0 & t_1 + t_2 \\ -t_1 + t_2 & 0 \end{matrix} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{matrix} 0 & -t_1 + t_2 \\ t_1 + t_2 & 0 \end{matrix} & \mathbf{0} & \mathbf{0} \end{pmatrix} \Bigg| \; t_1, t_2 \in \mathbb{R} \right\}$$

we see that

$$\mathfrak{b} = \mathfrak{a} \cap \mathfrak{q} = \left\{ \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{matrix} 0 & s \\ 0 & 0 \end{matrix} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \begin{matrix} 0 & 0 \\ s & 0 \end{matrix} & \mathbf{0} & \mathbf{0} \end{pmatrix} \Bigg| s \in \mathbb{R} \right\}$$

and

$$\mathfrak{b} = \psi \left\{ \left(\begin{pmatrix} 0 & it_1 \\ -it_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & it_2 \\ -it_2 & 0 \end{pmatrix} \right) \Big| t_1 = t_2 \right\}.$$

In the general case one knows how to construct homomorphisms $\psi_a : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}^{\mathbb{C}}$ starting from an idempotent(tripotent) $a \in V$ (see Chadli(1996) in the Cayley case, that may be generalized to all cases when D is of tube type, and Loos(1977), Lemma 9.7 for the general JTS case). These constructions use the Kantor-Koecher-Tits representation of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. In order to obtain a homomorphism ψ with the properties observed in Example 1 one uses the following fact.

Proposition 3. — *In the non-split case there exists a Jordan frame $\{f_1, f_2, \ldots, f_r\}$ of V such that*

$$\nu(f_j) = f_{r_o + j}, \quad j = 1, \ldots, r_o$$

and denoting $e_j = f_j + f_{r_o + j}$ the system $\{e_1, e_2, \ldots, e_{r_o}\}$ is a Jordan frame of $V_o = V^\nu$.

Example 2. — Consider the space $SU(p,q)/Sp(p_o,q_o)$ with $p = 2p_o \leq q = 2q_o$ (see pages 5-6). If we take the following Jordan frame of $V = M_{\mathbb{C}}(p,q)$

$$f_1 = \begin{pmatrix} \begin{matrix} 10\ldots0 \\ 00\ldots0 \\ \ldots \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, f_2 = \begin{pmatrix} \begin{matrix} 00\ldots0 \\ 01\ldots0 \\ \ldots \end{matrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \ldots f_{r_o+1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 10\ldots0 \\ 00\ldots0 \\ \ldots \end{matrix} \end{pmatrix}, \ldots$$

one verifies easily that for these f_j's Proposition 3 holds.

Sketch of proof. — If e is a primitive idempotent(tripotent) in V_o then in the non-split case the Jordan algebra $V_1(e)$ from the Peirce decomposition is of rank 2 and is a field. Take an element $a \in V_1(e)$ verifying $a^2 = -e$. Then the elements $f = \frac{1}{2}(e + ia)$ and $f' = \nu(f) = \frac{1}{2}(e - ia)$ are primitive idempotents(tripotents) of V and $f + f' = e$. ∎

Finally one obtains

THEOREM 4. — *In the non-split case, there exists a homomorphism*

$$\Psi : SL(2,\mathbb{C})^r \to G^{\mathbb{C}}, \quad \Psi : SU(1,1)^r \to G$$

such that

$$\sigma \circ \Psi_j = \Psi_{r_o+j} \circ \sigma_o, \; j = 1, \ldots, r_o.$$

In particular

$$\mathfrak{b} = \psi\{T \in \mathfrak{a}_o | t_j = t_{r_o+j}, \; j = 1, \ldots, r_o\}.$$

Decomposition $\mathbf{G \subset P^+K^{\mathbb{C}}H^{\mathbb{C}}}$. — Let $Z_o \in \mathfrak{c}(\mathfrak{k})$ be such that $\mathrm{ad}Z_o$ has the eigenvalues $i, 0, -i$. The decomposition of $\mathfrak{g}^{\mathbb{C}}$ into eigenspaces of $\mathrm{ad}Z_o$ is

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+_{\mathbb{C}} \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-_{\mathbb{C}}.$$

One sets $P^+ = \exp \mathfrak{p}^+_{\mathbb{C}}$ et $P^- = \exp \mathfrak{p}^-_{\mathbb{C}}$.

THEOREM 5. — *Let G/H be a symmetric space of Hermitian type. Then*

$$G \subset P^+K^{\mathbb{C}}H^{\mathbb{C}}$$

Proof. —

(1) The case $G_o = SU(1,1)$. One has $H_o = SO(1,1) = \left\{ \pm \begin{pmatrix} \mathrm{ch}\, t & \mathrm{sh}\, t \\ \mathrm{sh}\, t & \mathrm{ch}\, t \end{pmatrix} \middle| t \in \mathbb{R} \right\}$. and $P^+_o = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \middle| z \in \mathbb{C} \right\}$. Set $h_t = \begin{pmatrix} \mathrm{ch}\, t & \mathrm{sh}\, t \\ \mathrm{sh}\, t & \mathrm{ch}\, t \end{pmatrix}$. The group $G_o = SU(1,1)$ acts on the unit disc D by

$$g.z = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \qquad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1.$$

Then $h_t.0 = \mathrm{th}\; t = \dfrac{e^{2t} - 1}{e^{2t} + 1} = \dfrac{u - 1}{u + 1}$ where $u = e^{2t}$. If one takes $t \in \mathbb{C}$, one gets $u \in \mathbb{C}^*$ and $h_t \in H^{\mathbb{C}}_o$, so $H^{\mathbb{C}}_o.0$ is the Riemann sphere without the points 1 et -1.

The Riemann sphere identifies with the quotient $G_o^{\mathbb{C}}/P_o^+K_o^{\mathbb{C}}$ and $H_o^{\mathbb{C}}.0$ contains $D = G_o/K_o$. It follows that

$$G_o \subset P_o^+K_o^{\mathbb{C}}H_o^{\mathbb{C}}$$

and that $P_o^+K_o^{\mathbb{C}}H_o^{\mathbb{C}}$ is dense in $G_o^{\mathbb{C}}$.

(2) The general case. The crucial step is to establish that

$$B \subset P^+K^{\mathbb{C}}H^{\mathbb{C}}.$$

In the proof one uses the homomorphism Ψ constructed in Theorem 4. Let us consider the non-split case. Let $b \in B$. Then $b = \Psi(\exp T)$ where $T \in \mathfrak{a}_o$ is such that $t_j = t_{r_o+j},\ j = 1, \ldots, r_o$.For each j

$$\exp\begin{pmatrix} 0 & it_j \\ -it_j & 0 \end{pmatrix} = \begin{pmatrix} \operatorname{ch} t_j & i \operatorname{sh} t_j \\ -i \operatorname{sh} t_j & \operatorname{ch} t_j \end{pmatrix} \in SU(1,1) \subset P_o^+K_o^{\mathbb{C}}H_o^{\mathbb{C}}.$$

We get $b = \Psi(\mathbf{p})\Psi(\mathbf{k})\Psi(\mathbf{h})$ with $\mathbf{p} \in P_o^+$, $\mathbf{k} \in K_o^{\mathbb{C}}$ and $\mathbf{h} \in H_o^{\mathbb{C}}$ such that $h = \exp(z_1X, z_2X, \ldots, z_{r_o}X, z_1X, \ldots, z_{r_o}X)$. Then $\Psi(\mathbf{p}) \in P^+, \Psi(\mathbf{k}) \in K^{\mathbb{C}}$ and $\Psi(\mathbf{h}) = \exp[z_1(\psi_1(X) + \psi_{r_o+1}(X)) + \cdots + z_{r_o}(\psi_{r_o}(X) + \psi_r(X)) \in H^{\mathbb{C}}$ because $\psi_j(X) + \psi_{r_o+j}(X) \in \mathfrak{h}$ for $j = 1, \ldots, r_o$.

Using the decomposition $G = KBH$ one gets $G \subset KP^+K^{\mathbb{C}}H^{\mathbb{C}}H$ and as K normalizes P^+, $G \subset P^+K^{\mathbb{C}}H^{\mathbb{C}}$. □

Remark. — Note that similarly one obtains the decomposition

$$G \subset H^{\mathbb{C}}K^{\mathbb{C}}P^-.$$

II. Hardy spaces of symmetric spaces of Hermitian type

Stein variety associated to a symmetric space of Hermitian type . Let $X = G/H$ be a symmetric space of Hermitian type and $X^{\mathbb{C}} = G^{\mathbb{C}}/H^{\mathbb{C}}$ its complexification. Let C be a regular G-invariant cone in $\mathfrak{g}$ and

$$\Gamma(C) = G\exp(iC) \subset G^{\mathbb{C}}$$

the corresponding Olshanski semigroup . Write $\pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/H^{\mathbb{C}}$, the canonical projection and $x_o = eH^{\mathbb{C}}$ the base-point of $X^{\mathbb{C}}$. Set

$$\Xi(C) = \pi(\Gamma(-C)) = \Gamma(-C).x_o$$

$\Xi(C)$ is the orbit of the semigroup $\Gamma(-C)$ in $G^{\mathbb{C}}/H^{\mathbb{C}}$.

The semigroup $\Gamma(C)$ acts on the left on $\Xi(C)$. If $\gamma \in \Gamma(C^o)$ and f is a holomorphic function on the interior $\Xi^o(C) = \Xi(C^o)$ then we set

$$T(\gamma)f(z) = f(\gamma^{-1}z), \qquad z \in \Xi(C^o).$$

The space $X = G/H$ is contained in the boundary of $\Xi(C)$. Neeb(1996) showed that the domains Ξ^o are Stein.

Example. — Let $X = SU(1,1)/SO(1,1)$. The space X is isomorphic to the product of two circles without the diagonal

$$X \cong S^1 \times S^1 \setminus \triangle.$$

Then

$$\Xi^o \cong D^1 \times D^1 \setminus \triangle$$

where D^1 is the unit disc.

A geometric realization of the domains Ξ in the case of symmetric spaces of Cayley type was obtained by Chadli(1995).

Hardy spaces. — One defines the Hardy spaces on the domains Ξ in the following manner:

$$H^2(\Xi) = \{f \text{ holomorphic on } \Xi^o | \ \|f\|^2 = \sup_{\gamma \in \Gamma(C)^o} \int_X |f(\gamma^{-1}x)|^2 dx < \infty\}$$

THEOREM 3. — *(i) $H^2(\Xi)$ is a Hilbert space.*

(ii) The boundary value operator

$$bf(x) = \lim_{\gamma \to e, \gamma \in \Gamma(C)^o} f(\gamma^{-1}x)$$

where the limit is taken in $L^2(X)$ is an isometric embedding of $H^2(\Xi)$ in $L^2(X)$.

(iii) Let λ be the left regular representation of G in $L^2(X)$. Then

$$b(T(g)f) = \lambda(g)(bf), \qquad f \in H^2(\Xi),\ g \in G,$$

that is b intertwines T and λ.

(iv) The image $b(H^2(\Xi))$ is the biggest C- dissipative subspace of $L^2(X)$.

The proof of this theorem (Hilgert,'Olafsson,Ørsted(1991)) follows the basic ideas of the proof of the theorem of Olshanski on the analogous properties of Hardy spaces on complex semigroups.

At the moment we do not even know whether the Hardy spaces defined above are non-zero. This is where a study of holomorphic representations of G must begin. It will in fact give a decomposition of $H^2(\Xi)$ into unitary irreducible representations.

Relative holomorphic discrete series. — The **relative discrete series**(RDS) contains by definition the unitary irreducible representations $(\pi, \mathcal{H})$ of G, square integrable with respect to G/H:

$$\mathcal{H} \hookrightarrow L^2(G/H)$$

If moreover, $(\pi, \mathcal{H})$ is a highest weight representation one says that it is in the **relative holomorphic discrete series** (RHDS). It extends then to a holomorphic representation of $G^{\mathbb{C}}$.
As $L^2(G/H) \hookrightarrow \mathcal{D}'(G/H)$, each representation of RDS is realizable in $\mathcal{D}'(G/H)$. It follows that there exists $f_o \neq 0$ in $(\mathcal{H}_\pi^{-\infty})^H$. The embedding $\mathcal{H} \hookrightarrow L^2(G/H)$ is defined for $f \in \mathcal{H}^\infty$ by

$$f \to \ [g \mapsto (f|\pi(g)f_o)].$$

A highest weight representation $(\pi, \mathcal{H})$ belongs to the RHDS if and only if for all $f \in \mathcal{H}^\infty$

$$\int_{G/H} |(f|\pi(g)f_o)|^2 d\dot{g} < \infty.$$

The holomorphic representations of $\Gamma(C)$ are in 1-1 correspondence with C-dissipative representations of G. Such representations are of highest weight. They are obtained by holomorphic induction from a representation (τ, V_τ) of K, which is unitary, irreducible and $H \cap K$-spherical in the RHDS case. Moreover

$$\dim(\mathcal{H}_\pi^{-\infty})^H = \dim(V_\tau^{H\cap K}) = 1.$$

Note λ the highest weight of τ and π, v_λ a highest weight vector of τ such that $\|v_\lambda\| = 1$ and $v_o \in V_\tau$, $\|v_o\| = 1$ an $H \cap K$ -invariant vector. The function

$$\phi_\lambda(k) = (v_o|\tau(k)v_o)_{V_\tau}$$

is a spherical function of the pair $(K, K \cap H)$.

We have in our case

$$\mathcal{H}^{-\infty} \hookrightarrow \mathrm{Hol}(D, V_\tau).$$

Let $f_\lambda \equiv v_\lambda$ be a highest weight vector of π and $f_o \in (\mathcal{H}_\pi^{-\infty})^H$ an H-invariant distribution vector such that $f_o(0) = v_o$. The representation $(\pi, \mathcal{H})$ is square-integrable if and only if

$$F_\lambda(g) = (f_\lambda|\pi(g)f_o)_{\mathcal{H}} \in L^2(G/H).$$

We define the spectrum of the Hardy space $H^2(\Xi)$

$$\mathrm{Spec}(H^2(\Xi)) = \{\lambda | F_\lambda \in L^2(G/H)\}.$$

This definition is justified by the following theorem.

THEOREM 6. —

$$H^2(\Xi) = \bigoplus_{\pi \in \mathrm{SDHR}, \lambda \in iC^*} \mathcal{H}_\pi$$

where λ is the highest weight of π.

For the proof of this theorem as well as more details on the RHDS we send the reader to the lecture notes of D.Achab and P.Graczyk in [Paris(1996)]. One may also find there a detailed proof of the following theorem. One uses in it the decomposition $G \subset P^+ K^{\mathbb{C}} H^{\mathbb{C}}$, with the notation $g = p_+ \mu(g) h$.

THEOREM 7. —

$$\int_{G/H} |F_\lambda(x)|^2 dx = \text{const} \int_{B^+} \phi_\lambda(\mu(b)^2) J(b) db$$

where $J(b) = J(X)$ if $b = \exp X$ and J is the Jacobian of the coordinates $G = KB^+H$

$$J(X) = \prod_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{b})} (\mathrm{sh} < \alpha, X >)^{m_+^\alpha} (\mathrm{ch} < \alpha, X >)^{m_-^\alpha}, \qquad X \in \mathfrak{b}^+$$

with m_+^α and m_-^α the dimensions of the intersections of the root space $\mathfrak{g}^\alpha$ with the spaces $\mathfrak{g}_+$ and $\mathfrak{g}_-$ from the decomposition $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ with respect to the involution $\sigma \circ \theta$.

In the proof of this theorem one remarks that

$$F_\lambda(g) = (v_\lambda | \tau(\mu(g)) v_o)_{V_\tau}.$$

One applies the Schur orthogonality relations. A reduction to the $SU(1,1)/SO(1,1)$ case is done and computations are made on $SU(1,1)/SO(1,1)$.

Determining of the spectrum. — Our task now is to study for which highest weights λ the integral obtained in Theorem 7

$$I(\lambda) = \int_{B^+} \phi_\lambda(\mu(b)^2) J(b) db$$

converges. Let us do the computations in the more interesting non-split case.

We will first explicit Weyl chambers and multiplicities of the roots in $\Delta(\mathfrak{g}, \mathfrak{a})^+$ and $\Delta(\mathfrak{g}, \mathfrak{b})^+$. We will write $\mathbf{t} = \psi(T)$ where

$$T = \left(\begin{pmatrix} 0 & i\frac{t_1}{2} \\ -i\frac{t_1}{2} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & i\frac{t_r}{2} \\ -i\frac{t_r}{2} & 0 \end{pmatrix} \right)$$

We take $\mathfrak{a} = \psi(\mathfrak{a}_o)$ and $\mathfrak{b}$ obtained as in Theorem 4, with the notation $s_j = t_j = t_{r_o+j}$, $j = 1, \dots, r_o$. The root system $\Delta(\mathfrak{g}, \mathfrak{a})$ is of BC_r type. Let the positive Weyl chamber $\mathfrak{a}^+$ be defined as the image by the homomorphism ψ of the elements $T \in \mathfrak{a}_o$ verifying

$$0 < t_1 < \dots < t_r.$$

Then $\Delta(\mathfrak{g}, \mathfrak{a})^+$ contains the following roots α with multiplicities m_α:
$\alpha(\mathbf{t}) = t_j,\ 1 \le j \le r,\ m_\alpha = 1$
$\alpha(\mathbf{t}) = \frac{t_j + t_k}{2},\ 1 \le j < k \le r,\ m_\alpha = a$
$\alpha(\mathbf{t}) = \frac{t_k - t_j}{2},\ 1 \le j < k \le r,\ m_\alpha = a$
$\alpha(\mathbf{t}) = \frac{t_j}{2},\ 1 \le j \le r,\ m_\alpha = 2b$
for some parameters $a, b \in \mathbb{N}$. In the tube case, $b = 0$ and $a = d$, the degree of the simple Euclidean Jordan algebra V.

The positive Weyl chamber $\mathfrak{b}^+$ is then given by

$$\mathfrak{b}^+ = \psi\{T |\ 0 < s_1 < \dots < s_{r_o}\}$$

and the root system $\Delta(\mathfrak{g}, \mathfrak{b})^+$ consists of the following roots
$\beta(\mathbf{s}) = s_j,\ 1 \le j \le r_o,\ m_\beta = a + 2$
$\beta(\mathbf{s}) = \frac{s_j + s_k}{2},\ 1 \le j < k \le r_o,\ m_\beta = 4a$ (if $r_o \ge 2$)
$\beta(\mathbf{s}) = \frac{s_k - s_j}{2},\ 1 \le j < k \le r_o,\ m_\beta = 4a$ (if $r_o \ge 2$)
$\beta(\mathbf{s}) = \frac{s_j}{2},\ 1 \le j \le r_o,\ m_\beta = 4b.$

Let $X \in \mathfrak{b}^+$ be the image by ψ of

$$\left(\begin{pmatrix} 0 & i\frac{s_1}{2} \\ i\frac{-s_1}{2} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & i\frac{s_{r_o}}{2} \\ i\frac{-s_{r_o}}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{is_1}{2} \\ \frac{-is_1}{2} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \frac{is_{r_o}}{2} \\ \frac{-is_{r_o}}{2} & 0 \end{pmatrix}\right).$$

A direct computation on $SU(1,1)/SO(1,1)$ shows that

$$\mu\left(\exp\begin{pmatrix} 0 & \frac{is_1}{2} \\ \frac{-is_1}{2} & 0 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{\sqrt{\mathrm{ch} s_1}} & 0 \\ 0 & \sqrt{\mathrm{ch} s_1} \end{pmatrix}.$$

Denoting by $\mathfrak{t}$ a Cartan subspace of $\mathfrak{k}$ and by $H \in \mathfrak{t}^{\mathbb{C}}$ the image by ψ of an element

$$\left(\begin{pmatrix} \frac{u_1}{2} & 0 \\ 0 & -\frac{u_1}{2} \end{pmatrix}, \dots, \begin{pmatrix} \frac{u_r}{2} & 0 \\ 0 & -\frac{u_r}{2} \end{pmatrix}\right) \in i\mathfrak{t}_o$$

we get

$$\mu(\exp X) = \exp H,\ H \in i\mathfrak{t},\ \text{with}$$
$$u_j = u_{r_o+j} = -\ln \mathrm{ch} s_j,\ j = 1, \dots, r_o.$$

As $0 < s_1 < \ldots < s_{r_o}$, we have $u_1 = u_{r_o+1} \geq \ldots \geq u_{r_o} = u_r$. We define $(i\mathfrak{t})^+$ by the condition

$$u_1 \geq u_{r_o+1} \geq u_2 \geq \ldots \geq u_{r_o} \geq u_r$$

so $H \in (i\mathfrak{t})^+$.

Let the highest weight $\lambda \in (i\mathfrak{t})^+$ be given by a vector $(m_1, \ldots, m_r)$, that is

$$\lambda(H) = m_1 u_1 + \cdots + m_r u_r.$$

The corresponding representation π is in the RDS if and only if $m_j = m_{r_o+j},\ j = 1, \ldots, r_o$ and we have

$$m_1 = m_{r_o+j} \geq \ldots \geq m_{r_o} = m_r.$$

For the spherical functions on $(K, K \cap H)$ one may prove the following estimate (in fact it holds for any symmetric pair (K, K_o) with K compact reductive).

PROPOSITION 8. — *There exists a constant $0 < c < 1$ such that for all $Y \in (i\mathfrak{t})^+$*

$$ce^{<\lambda,X>} \leq \phi_\lambda(\exp X) \leq e^{<\lambda,X>}.$$

In our case we have

$$\begin{aligned}\phi_\lambda(\mu(\exp X)^2) =& \phi_\lambda(\exp 2H) \approx e^{\lambda(2H)} \\ =& \exp[4m_1(-\ln \mathrm{ch} s_1) + \cdots + 4m_{r_o}(-\ln \mathrm{ch} s_{r_o})] \\ =& \prod_{j=1}^{r_o} \left(\frac{1}{\mathrm{ch} s_j}\right)^{4m_j} \approx \prod_{j=1}^{r_o} e^{-4m_j s_j}.\end{aligned}$$

The Jacobian $J(X)$ may be estimated

$$J(X) \approx \exp(\langle \sum_{\beta \in \Delta(\mathfrak{g},\mathfrak{b})^+} m_\beta \beta, X\rangle) = \exp(2\rho_\mathfrak{b}(X)).$$

If $\rho_1 < \ldots < \rho_{r_o}$ are coordinates of $\rho_\mathfrak{b}$ we have $\rho_\mathfrak{b}(X) = \rho_1 s_1 + \cdots + \rho_{r_o} s_{r_o}$. The functional $\rho_\mathfrak{b}$ extends naturally to $\mathfrak{a}$ by setting $\rho_{r_o+j} = \rho_j$.
Finally

$$I(\lambda) \approx \int_{\{0 < s_1 < \ldots < s_{r_o}\}} \prod_{j=1}^{r_o} e^{2(\rho_j - 2m_j)s_j} ds_1 \ldots ds_{r_o}.$$

As $\rho_1 - 2m_1 < \ldots < \rho_{r_o} - 2m_{r_o} = \rho_r - 2m_r$ and the integral converges if and only if all the factors $\rho_j - 2m_j$ in the exponent are negative, we obtain $I(\lambda) < \infty \Leftrightarrow \rho_r - 2m_r < 0$. The same reasoning in the split case gives a condition $\rho_r - m_r < 0$.

THEOREM 9. — $\lambda \in \mathrm{Spec}(H^2(\Xi))$ *if and only if*

$$m_r > \rho_r \ \textit{if} \ X = G/H \ \textit{is split}$$
$$m_r > \frac{\rho_r}{2} \ \textit{if} \ X = G/H \ \textit{is non} - \textit{split}.$$

Remark 1. — The caracterization of the RHDS obtained by Hilgert,'Olafsson and Ørsted (1991) contained another condition on λ, analogous to the Harish-Chandra condition for the holomorphic discrete series(HDS). Our condition is more explicit and easier to verify.

Remark 2. — If one applies our approach in order to find a condition for the holomorphic discrete series of G, one gets a condition $m_r > \rho_r^{(\mathfrak{a})}$ where $\rho_j^{(\mathfrak{a})}$ are the coordinates of $\rho_{\mathfrak{a}}$, the half-sum of positive roots of the system $\Delta(\mathfrak{g}, \mathfrak{a})$.

In the split case, this is the same condition as for the spectrum of $H^2(\Xi)$, so the RHDS is contained in the HDS .

In the non-split case (if $r_o \geq 2$) we have

$$\rho_r^{(\mathfrak{a})} = \frac{1}{2}(1 + a(2r_o - 1) + b)$$
$$\rho_r^{(\mathfrak{b})} = \rho_r = \frac{1}{2}(2 + a + 4a(r_o - 1) + 2b).$$

One sees that

$$\rho_r^{(\mathfrak{a})} > \frac{\rho_r^{(\mathfrak{b})}}{2}$$

so the condition on the RHDS is weaker than the condition on the HDS. The relative holomorphic discrete series contains representations which do not belong to the holomorphic discrete series. This is also the case when $r_o = 1$.

Example. — On the hyperboloids $SO(2,n)/SO(1,n)$ we have $\rho_2^{(\mathfrak{a})} = (n-1)/2$ and $\rho_1^{(\mathfrak{b})} = \rho_2^{(\mathfrak{b})} = n/2$. For $n > 2$ the relative holomorphic discrete series contains representations which are not square-integrable on $SO(2,n)$. It was observed by Molchanov(1994).

Remark 3. — In the non-split case, when D is of tube type, using the relation $n = r + \frac{d}{2}r(r-1)$ the RHDS condition may be written down

$$m_r > \frac{n}{r} - \frac{d+2}{4}$$

whereas the HDS condition is

$$m_r > \frac{n}{r} - \frac{1}{2}$$

which becomes $m_r \geq \frac{n}{r}$, m_r being integer.

REFERENCES

Chadli M.(1995), Domaine complexe associé à un espace symétrique de type Cayley, *CRAS*,**321**, 1157-1162.

Chadli M.(1996), Espace de Hardy d'un espace symétrique de type Cayley, Thèse de doctorat, Paris VI.

Hilgert J.,'Olafsson G.(1997), Causal symmetric spaces, geometry and harmonic analysis, Academic Press.

Hilgert J.,'Olafsson G.,Ørsted B.(1991), Hardy spaces on affine symmetric spaces,*J.Reine und Angew.Math.* **415**, 189-218.

Jaffee A.(1975), Real forms of Hermitian symmetric spaces, *Bull.A.M.S.* **81**,456-458.

Jaffee A.(1978), Antiholomorphic automorphisms of the exceptional symmetric domains, *J.Diff.Geom.* **13**, 79-86.

Kayoya J.B. (1994) Thèse de doctorat, Paris VI.

Loos O.(1977), Bounded symmetric domains and Jordan pairs, The University of California, Irvine.

Molchanov V.F. (1994), Holomorphic discrete series for hyperboloids of Hermitian type, preprint.

Neeb K.H.(1996), Biinvariant domains of holomorphy in Olshanskii semigroups, preprint.

'Olafsson G., Symmetric spaces of Hermitian type, *Diff.Geometry and Appl.* **1**, 195-233.

Paris(1996), Journées "Programme de Gelfand-Gindikin", Université Paris VI.

Satake I.(1980), Algebraic Structures of Symmetric Domains, Princeton University Press.