FUNCTION c
ON AN ORDERED SYMMETRIC SPACE

PIOTR GRACZYK
Département de Mathématiques, Université d'Angers, 2 bd. Lavoisier, 49045 Angers, France

ABSTRACT
We prove by a direct computation an explicit formula for the c-function on the ordered symmetric spaces $SL(n,\mathbb{R})/SO(p,q)$. Our formula agrees with a formula conjectured in the general case by Faraut.

1. Ordered symmetric spaces and c-function.

In this section we present some basic facts concerning ordered symmetric spaces and their c-function. All these facts may be found in [FHO].

Let $M = G/H$ be a semisimple symmetric space where G is a connected semisimple Lie group with finite center and

$$(G^\sigma)_o \subset H \subset G^\sigma$$

for an involution σ of G.

Let the corresponding decomposition of the Lie algebra of G be

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}.$$

A *causal structure* on M is a field of regular cones $M \ni x \to C_x \subset T_xM$. The space M is called *causal* if there exists on M a G-invariant causal structure or, equivalently, if there exists a regular $\mathrm{Ad}(H)$-invariant cone C in $\mathfrak{q}$. The space M is said to be *ordered* if it is causal and the causal structure on M is *global*, i.e. there is no non trivial closed causal curve.

Let θ be a Cartan involution of G commuting with σ and K the corresponding maximal compact subgroup. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the related Cartan decomposition of $\mathfrak{g}$. The space G/H is ordered if and only if it is causal and $C \cap \mathfrak{k} = \{0\}$ or, equivalently, $(\mathfrak{p} \cap \mathfrak{q})^K \neq \{0\}$. In the case when the pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible (i.e. there is no non trivial ideal in $\mathfrak{g}$ which is invariant under σ) we have $\dim(\mathfrak{p} \cap \mathfrak{q})^{K\cap H} = 1$.

1991 Mathematics Subject Classification 43A85,43A90,43A05

Key words and phrases: ordered symmetric spaces,c-function, spherical functions

Then there exists $X_o \in (\mathfrak{p} \cap \mathfrak{q})^{K\cap H}$ such that $\mathrm{ad}X_o$ has eigenvalues $-1, 0, 1$ on $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$$

where $\mathfrak{g}_j = \{X \in \mathfrak{g} |\ \mathrm{ad}X_o(X) = jX\}$, $j = -1, 0, 1$. There exists a Cartan subspace $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ with $X_o \in \mathfrak{a}$. Let

$$\triangle_j = \{\alpha \in \triangle(\mathfrak{g}, \mathfrak{a}) |\ \mathfrak{g}_\alpha \subset \mathfrak{g}_j\} = \{\alpha \in \triangle(\mathfrak{g}, \mathfrak{a}) |\ \alpha(X_o) = j\}, \quad j = -1, 0, 1.$$

Denote by $\triangle_o^+$ a positive root system in $\triangle_o$ and $\triangle^+ = \triangle_o^+ \cup \triangle_1$. Then one puts

$$\mathfrak{n}_o = \sum_{\alpha \in \triangle_o^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_1 = \sum_{\alpha \in \triangle_1} \mathfrak{g}_\alpha = \mathfrak{g}_1 \quad \text{and} \quad \mathfrak{n} = \mathfrak{n}_o + \mathfrak{n}_1.$$

$\mathfrak{n}, \mathfrak{n}_o$ and $\mathfrak{n}_1$ are nilpotent subalgebras of $\mathfrak{g}$; moreover $\mathfrak{n}_1$ and $\bar{\mathfrak{n}}_1 = \mathfrak{g}_{-1} = \theta(\mathfrak{n}_1)$ are abelian. Denote by A, N, N_o, N_1 and $\bar{N}_1$ the analytic subgroups of G which have Lie algebras $\mathfrak{a}, \mathfrak{n}, \mathfrak{n}_o, \mathfrak{n}_1$ and $\bar{\mathfrak{n}}_1$ respectively. One knows that NAH is an open subset of G. If $g \in NAH$ one writes

$$g = n \exp(\mathcal{A}(g))h.$$

The function c on the ordered symmetric space M is given by the formula

$$c(\lambda) = \int_{\bar{N} \cap NAH} e^{(\rho+\lambda)(\mathcal{A}(\bar{n}))} d\bar{n}, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

if the integral converges; $\rho = \sum_{\alpha \in \triangle^+} m_\alpha \alpha$, m_α being the multiplicity of α. It is equal to the product

$$c(\lambda) = c_o(\lambda) c_{\mathcal{D}}(\lambda)$$

where c_o is the c-function of the Riemannian symmetric space G_o/K_o with $K_o = K \cap H$ and $G_o = N_o A K_o$ and $c_{\mathcal{D}}$ is given by

(1) $$c_{\mathcal{D}}(\lambda) = \int_{\mathcal{D}} e^{(\rho+\lambda)(\mathcal{A}(\bar{n}))} d\bar{n}, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

with $\mathcal{D} = \bar{N}_1 \cap NAH$. The domains $\bar{N} \cap NAH$ and $\mathcal{D}$ are bounded.

The knowledge of the function $c(\lambda)$ and of its domain is very important in the harmonic analysis on M. Spherical functions on M, defined by

$$\varphi_\lambda(x) = \int_H e^{(\rho-\lambda)(\mathcal{A}(hx))} dh$$

for $x \in \mathrm{Int}(\exp(C)H) \subset NAH$, exist exactly for these $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which (1) converges. Function $c(\lambda)$ appears also in the asymptotic behaviour of φ_λ.

The c-function for an ordered symmetric space M has been determined for M of rank 1 and $M = G/H$ with G complex by Faraut, Hilgert and Olafsson ([FHO]) and for the symmetric spaces of Cayley type by Faraut([F2]). In the last paper a formula for the c-function on an arbitrary ordered symmetric space has been conjectured.

In this note we compute the c-function on an important class of ordered symmetric spaces, the spaces $SL(n,\mathbb{R})/SO(p,q)$. These spaces may be identified with the spaces of symmetric matrices of determinant one with signature (p,q). They appear naturally as the determinant one surfaces of connected components of the Euclidean Jordan algebra $V = Sym(n,\mathbb{R})$, different from the components of the identity matrix I and $-I$. They are also the determinant one surfaces of the non-Riemannian orbits of the structure group of V acting on V. Some results of harmonic analysis on such spaces, in the real and complex case, were obtained in [F1].

The formula that we obtain verifies the conjecture of Faraut ([F2]). Our method is based on an explicit computation of the function $\mathcal{A}(\bar{n})$ in the considered case. Computing of the integral (1) involves some algebraic identities of determinants.

As the function c on Riemannian symmetric spaces is well known([GK]), it is sufficient to determine the function $c_{\mathcal{D}}$ defined in (1).

2. Some properties of determinants.

In this section we prove some elementary algebraic results used in the calculus of the c-function in the next section.

LEMMA 1. — *Let* $X = (x_{ij})_{\substack{i=1,\dots,q\\ j=1,\dots,p}}$ *be a matrix of dimension* $q \times p$. *Set* $z = x_{1p}$. *Then* $\det(XX^t - I)$ *is a quadratic function of* z*:*

$$\det(XX^t - I) = az^2 + bz + c$$

such that $a = \det(YY^t - I)$ *and its discriminant*

$$b^2 - 4ac = -4\det(AA^t - I)\det(BB^t - I) \tag{2}$$

where Y, A *and* B *are the submatrices of the matrix* X *defined by*

$$Y = (x_{ij})_{\substack{i=2,\dots,q\\ j=1,\dots,p-1}}, \quad A = (x_{ij})_{\substack{i=1,\dots,q\\ j=1,\dots,p-1}} \text{ and } B = (x_{ij})_{\substack{i=2,\dots,q\\ j=1,\dots,p}}.$$

Proof. — Denote by L the line $[x_{11} \dots x_{1,p-1}]$ and by C the column $\begin{bmatrix} x_{2p} \\ \vdots \\ x_{qp} \end{bmatrix}$.

Then $X = \begin{pmatrix} L & z \\ Y & C \end{pmatrix}$ and one has $\det(XX^t - I) = az^2 + bz + c$ with

$$a = \begin{vmatrix} 1 & C^t \\ C & YY^t + CC^t - I \end{vmatrix} = \det(YY^t - I),$$

$$b = 2\begin{vmatrix} 0 & C^t \\ YL^t & YY^t - I \end{vmatrix} \quad \text{and} \quad c = \begin{vmatrix} LL^t - 1 & LY^t \\ YL^t & YY^t + CC^t - I \end{vmatrix}.$$

Observing that

$$\det(BB^t - I) = \det(YY^t - I + CC^t) = \det(YY^t - I) - \begin{vmatrix} 0 & C^t \\ C & YY^t - I \end{vmatrix}$$

and that

$$c = \det(AA^t - I) - (LL^t - 1)\begin{vmatrix} 0 & C^t \\ C & YY^t - I \end{vmatrix} - \gamma, \quad \gamma = \begin{vmatrix} 0 & 0 & C^t \\ 0 & 0 & LY^t \\ C & YL^t & YY^t - I \end{vmatrix}$$

we use $\det(AA^t - I) = (LL^t - 1)\det(YY^t - I) + \begin{vmatrix} 0 & LY^t \\ YL^t & YY^t - I \end{vmatrix}$ and we see that (2) is equivalent to

$$\begin{vmatrix} 0 & LY^t \\ C & YY^t - I \end{vmatrix}^2 - \begin{vmatrix} 0 & C^t \\ C & YY^t - I \end{vmatrix}\begin{vmatrix} 0 & LY^t \\ YL^t & YY^t - I \end{vmatrix} = -\det(YY^t - I)\gamma$$

which follows from the next lemma. ∎

Lemma 2. — *Let S be a symmetric matrix of dimension $n \times n$ and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$. Define*

$$H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) = \begin{vmatrix} 0 & \mathbf{y}_1 \\ \mathbf{x}_1^t & S \end{vmatrix}\begin{vmatrix} 0 & \mathbf{y}_2 \\ \mathbf{x}_2^t & S \end{vmatrix} - \begin{vmatrix} 0 & \mathbf{x}_2 \\ \mathbf{x}_1^t & S \end{vmatrix}\begin{vmatrix} 0 & \mathbf{y}_2 \\ \mathbf{y}_1^t & S \end{vmatrix}$$

$$\text{and} \quad K(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) = -\det S \begin{vmatrix} 0 & 0 & \mathbf{x}_2 \\ 0 & 0 & \mathbf{y}_1 \\ \mathbf{x}_1^t & \mathbf{y}_2^t & S \end{vmatrix}.$$

Then $H(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) = K(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$.

Proof. — As S is a symmetric matrix, there exists an orthonormal matrix U such that $D = USU^t$ is diagonal. It is then easy to see that the equality of H and K is equivalent to the equality of analogous forms defined with D in place of S. Thus we may suppose that S is diagonal.

Both forms H and K are bilinear forms in $\mathbf{x}_1, \mathbf{x}_2$ when $\mathbf{y}_1, \mathbf{y}_2$ are fixed and conversely. Thus in order to establish their equality it is enough to show that $H = K$ for $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ in the canonical basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ of $\mathbb{R}^n$. If S is diagonal with diagonal entries $s_1, \ldots, s_n$, an easy direct computation gives for $i \neq j$

$$H(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_j) = -\prod_{k\neq i} s_k \prod_{l\neq j} s_l = K(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_j)$$
$$H(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j) = \prod_{k\neq i} s_k \prod_{l\neq j} s_l = K(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_i, \mathbf{e}_j)$$

and $H = K = 0$ in other cases. ∎

3. c-function on $SL(n,\mathbb{R})/SO(p,q)$.

In this section we deal with $M = G/H = SL(n,\mathbb{R})/SO(p,q)$ where $n = p+q$. The involution σ on G is given by $\sigma(g) = I_{pq}(g^t)^{-1}I_{pq}$ with $I_{pq} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Then $H = G^\sigma$, $\mathfrak{h} = \mathfrak{so}(p,q) = \left\{ \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \mid A \in \mathfrak{so}(p),\ D \in \mathfrak{so}(q),\ B \in M(p,q,\mathbb{R}) \right\}$ and $\mathfrak{q} = \left\{ \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix} \mid A, D \text{ symmetric},\ \mathrm{tr}A + \mathrm{tr}D = 0,\ B \in M(p,q,\mathbb{R}) \right\}$.

The Cartan involution commuting with σ is given by $\theta(g) = (g^t)^{-1}$, the maximal compact subgroup $K = SO(n)$, $\mathfrak{k} = \mathfrak{so}(n)$, $\mathfrak{p}$ is composed of symmetric matrices of trace 0.

We have $\mathfrak{p} \cap \mathfrak{q} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A, D \text{ symmetric},\ \mathrm{tr}A + \mathrm{tr}D = 0 \right\}$ and $(\mathfrak{p} \cap \mathfrak{q})^{K\cap H} = \left\{ \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_q \end{pmatrix} \mid \alpha p + \beta q = 0 \right\}$ is of dimension 1.

If $X_o = \frac{1}{n}\begin{pmatrix} qI_p & 0 \\ 0 & -pI_q \end{pmatrix}$ then $X_o \in (\mathfrak{p} \cap \mathfrak{q})^{K\cap H}$ and $\mathrm{ad}X_o$ has eigenvalues -1,0,1 on $\mathfrak{g}$. A Cartan space $\mathfrak{a}$ contained in $\mathfrak{p} \cap \mathfrak{q}$ and containing X_o is the space of diagonal matrices of trace zero.

In the root system $\triangle = \triangle(\mathfrak{g}, \mathfrak{a}) = \{\alpha_{ij} \mid i \neq j\}$ with $\alpha_{ij}(\mathbf{t}) = t_i - t_j$ for $\mathbf{t} \in \mathfrak{a}$ with diagonal entries $t_1, ..., t_n$ we have the subsets

$$\triangle_o = \{\alpha_{ij} \mid 1 \leq i \neq j \leq p \ \text{ or } \ p+1 \leq i \neq j \leq n\}$$
$$\triangle_1 = \{\alpha_{ij} \mid 1 \leq i \leq p \ \text{ and } \ p+1 \leq j \leq n\}$$
$$\triangle_{-1} = \{\alpha_{ij} \mid p+1 \leq i \leq n \ \text{ and } \ 1 \leq j \leq p\}.$$

N is the upper triangular nilpotent group and $\bar{N}_1$ is the abelian nilpotent group of matrices $\bar{n}$ of the form

$$\bar{n} = \begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix} \tag{3}$$

where $X \in M(q,p,\mathbb{R})$.

In order to evaluate the integral (1) defining the c-function of M we are going to determine explicitely the function $\mathcal{A}(\bar{n})$ for $\bar{n} \in \mathcal{D} = \bar{N}_1 \cap NAH$.

We will denote by X_k the matrix constructed from $X = (x_{ij})_{\substack{i=1,\dots,q \\ j=1,\dots,p}}$ by taking up its first $k-1$ lines and by X_{-l} the matrix which remains after taking up l columns on the right in $X (k=1,...,q;\ l=1,...,p-1;\ X_o = X_1 = X)$:

$$X_k = (x_{ij})_{\substack{i=k,\dots,q \\ j=1,\dots,p}} \qquad X_{-l} = (x_{ij})_{\substack{i=1,\dots,q \\ j=1,\dots,p-l}}.$$

Lemma 3. — *If $\bar{n} \in \mathcal{D}$ is of the form (3) then* $\mathcal{A}(\bar{n}) = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}$ *with*

$$\begin{aligned}
t_1 &= \frac{1}{2}\log\Big(\frac{1}{\det(I - X_{-(p-1)}X^t_{-(p-1)})}\Big) \\
t_{p-l} &= \frac{1}{2}\log\Big(\frac{\det(I - X_{-(l+1)}X^t_{-(l+1)})}{\det(I - X_{-l}X^t_{-l})}\Big), \qquad & l = 0,\dots,p \\
t_{p+k} &= \frac{1}{2}\log\Big(\frac{\det(I - X_k X^t_k)}{\det(I - X_{k+1}X^t_{k+1})}\Big), \qquad & k = 1,\dots,q-1 \\
t_n &= \frac{1}{2}\log(1 - X_q X^t_q).
\end{aligned}$$

In particular $\bar{n} \in \mathcal{D}$ if and only if $\det(I - X_{-l}X^t_{-l}) > 0,\ l = 0,1,\dots,p-1$ and $\det(I - X_k X^t_k) > 0,\ k = 1,\dots,q$.

Proof. — Suppose $\bar{n} = nah$ with $n \in N,\ a \in A$ and $h \in H$. Then

$$\bar{n} I_{pq} \bar{n}^t = n(I_{pq}a^2)n^t$$

and in order to write explicitely a and $\mathcal{A}(\bar{n}) = \log a$ we use the fact that the determinants of lower principal minors of $\bar{n} I_{pq} \bar{n}^t$ and $I_{pq}a^2$ are identical.

One has

$$\bar{n} I_{pq} \bar{n}^t = \begin{pmatrix} I_p & X^t \\ X & XX^t - I_q \end{pmatrix}.$$

Denote by D_j the determinant of the j-th lower principal minor of $\bar{n} I_{pq} \bar{n}^t$. One has

$$D_k = \det(X_{q-k+1} X^t_{q-k+1} - I), \qquad k = 1, \dots, q \tag{4}$$

$$D_{q+l} = \det(X_{-l} X^t_{-l} - I), \qquad l = 1, \dots, p-1 \tag{5}$$

and

$$D_n = \det(\bar{n} I_{pq} \bar{n}^t) = (-1)^q .$$

In fact, (4) is obvious and in order to prove (5) we write

$$\bar{n} = \begin{pmatrix} I_p & & 0 \\ & & \\ X_{-l} & 0 & I_q \end{pmatrix} + \begin{pmatrix} & 0 & 0 \\ & & \\ 0 & Z & 0 \end{pmatrix}$$

and we find $D_{q+l} = \begin{vmatrix} I & Z^t \\ & \\ Z & X_{-l} X^t_{-l} - I + Z Z^t \end{vmatrix} = \det(X_{-l} X^t_{-l} - I)$. Denoting by $a_1, \dots, a_n$ the diagonal entries of a we have

$$\begin{aligned} D_1 &= -a_n^2 \\ D_k &= (-1)^k a^2_{n-k+1} \dots a^2_{n-1} a^2_n, & k &= 1, \dots, q \\ D_{q+l} &= (-1)^q a^2_{p-l+1} \dots a^2_p \dots a^2_n, & l &= 1, \dots, p \end{aligned}$$

and the lemma follows. ∎

Remark. — One knows from the general theory that the domain $\mathcal{D}$ in our case is given by X belonging to the unit ball in $M(q, p, \mathbb{R})$, that is $\mathcal{D} = \{\bar{n} |\ I - X X^t >> 0\}$. This coincides with the description of $\mathcal{D}$ given in Lemma 3 used in the sequel.

Denote by

$$p_\lambda(\bar{n}) = e^{(\lambda+\rho)(\mathcal{A}(\bar{n}))} , \qquad \bar{n} \in \mathcal{D}$$

the function integrated in (1). Put

$$\begin{aligned} \Delta_j &= \det(I - X_{-(p+1-j)} X^t_{-(p+1-j)}) , & j &= 2, \dots, p \\ \Delta_{p+k} &= \det(I - X_k X_k^t) , & k &= 1, \dots, q. \end{aligned}$$

Lemma 3 shows that $\mathcal{D} = \{\Delta_j > 0,\ j = 2, ..., n\}$.

Corollary. — *If $\bar{n} \in \mathcal{D} = \bar{N}_1 \cap NAH$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\mathbf{s} = \frac{1}{2}(\lambda + \rho)$ then*

$$p_\lambda(\bar{n}) = \Delta_2^{s_2 - s_1} \Delta_3^{s_3 - s_2} \dots \Delta_n^{s_n - s_{n-1}} . \tag{6}$$

In the following computation of the c-function we will use extensively the following analytic lemma([F2]).

LEMMA 4. — *Let $f(z) = az^2 + bz + c$ be a quadratic function with $a < 0$ and a positive discriminant Δ. Let α, β be the roots of f. Then for* $\mathrm{Re}(s) > -1$

$$\int_\alpha^\beta f(z)^s dz = 2^{-2s-1} \Delta^{s+\frac{1}{2}} |a|^{-s-1} B(s+1, \frac{1}{2}).$$

THEOREM 1. — *The function $c_{\mathcal{D}}$ on the ordered symmetric space $M = SL(n, \mathbb{R})/SO(p,q)$ is given by*

$$c_{\mathcal{D}}(\lambda) = \prod_{\substack{p+1 \le i \le n \\ 1 \le j \le p}} B(\frac{\lambda_i - \lambda_j + 1}{2}, \frac{1}{2}) = \prod_{\alpha \in \Delta_{-1}} B(\frac{< \lambda, \alpha >}{2} + \frac{1}{2}, \frac{1}{2})$$

for λ such that $\mathrm{Re}(\lambda_i - \lambda_j) > -1$, $p+1 \le i \le n$; $1 \le j \le p$.

Proof. — We first remark that the unique entry of the matrix X defining $\bar{n}$ by (3) which appears in only one factor of (6) is x_{1p}. It is natural to start computing of $\int_{\mathcal{D}} p_\lambda(\bar{n}) d\bar{n}$ by integrating with respect to x_{1p}.

By Lemma 1

$$\Delta_{p+1} = (-1)^q \det(XX^t - I) = a x_{1p}^2 + b x_{1p} + c$$

with $a = -\det(I - YY^t)$, where Y is the submatrix of X given by

$$Y = (x_{ij})_{\substack{i=2,\dots,q \\ j=1,\dots,p-1}}$$

and the discriminant equal to $4\det(I - X_{-1}X_{-1}^t)\det(I - X_2X_2^t) = 4\Delta_p\Delta_{p+2} > 0$. As $\mathcal{D}$ is bounded and $\Delta_{p+1} > 0$ on $\mathcal{D}$ it follows that $a < 0$. Let $\alpha < \beta$ be the roots of $\Delta_{p+1}(x_{1p})$. Then $\bar{n} \in \mathcal{D}$ if and only if $\Delta_j > 0$, $j \ne p+1$, $\det(I - YY^t) > 0$ and $x_{1p} \in (\alpha, \beta)$. Using Lemma 4 we see that

$$\int_\alpha^\beta \Delta_{p+1}^{s_{p+1}-s_p} dx_{1p} = (\Delta_p\Delta_{p+2})^{s_{p+1}-s_p+\frac{1}{2}} \det(I - YY^t)^{-s_{p+1}+s_p-1} B(s_{p+1} - s_p + 1, \frac{1}{2})$$

for $\mathrm{Re}(s_{p+1} - s_p) > -1$ and the integral diverges otherwise.

Next we integrate in the same way, using Lemma 1 and Lemma 4, succesively with respect to $dx_{1,p-1}, \dots, dx_{12}$. In order to integrate with respect to dx_{11} we remark that

$$\Delta_2 = 1 - (x_{11}^2 + x_{21}^2 + \dots + x_{q1}^2)$$

and we use a change of variables

$$x_{11} = x \sqrt{1 - (x_{21}^2 + \dots + x_{q1}^2)}. \tag{7}$$

Finally, after the integration with respect to all the variables in the first line of X we obtain

$$\int_{\mathcal{D}} p_\lambda(\bar{n})d\bar{n} = \prod_{j=1}^{p} B\big(s_{p+1} - s_j + \frac{p-j+2}{2}, \frac{1}{2}\big)$$

$$\times \int_{\tilde{\mathcal{D}}} \tilde{\Delta}_2^{s_2-s_1} \ldots \tilde{\Delta}_p^{s_p-s_{p-1}} \Delta_{p+2}^{s_{p+2}-s_p+\frac{1}{2}} \Delta_{p+3}^{s_{p+3}-s_{p+2}} \ldots \Delta_n^{s_n-s_{n-1}} dx_{21} \ldots dx_{qp} \tag{8}$$

where $\tilde{\Delta}_j = \det(I - Y_{-(p-j)}Y^t_{-(p-j)})$, $j = 2, \ldots, p$ (one sets $Y_o = Y$) and $\tilde{\mathcal{D}} = \{\tilde{\Delta}_j > 0,\ j = 2, \ldots, p,\ \Delta_k > 0,\ k = p+2, \ldots, n\}$ is isomorphic to the unit ball in $M(q-1, p, \mathbb{R})$.
Let $\tilde{c}$ be the c-function of the space $SL(n-1, \mathbb{R})/SO(p, q-1)$. Then (8) may be written

$$c_{\mathcal{D}}(\lambda) = \tilde{c}_{\tilde{\mathcal{D}}}(\tilde{\lambda}) \prod_{j=1}^{p} B\big(\frac{\lambda_{p+1} - \lambda_j}{2} + \frac{1}{2}, \frac{1}{2}\big)$$

where $\tilde{\lambda} = (\lambda_1 + \frac{1}{2}, \ldots, \lambda_p + \frac{1}{2}, \lambda_{p+2} + \frac{1}{2}, \ldots \lambda_n + \frac{1}{2})$.
The theorem follows by induction if we prove it for the space $SL(p+1, \mathbb{R})/SO(p, 1)$. This is easy and follows by a series of changes of variables analogous to (7) and by Lemma 4. $\square$

Remark. — In the case of the ordered symmetric spaces $SL(n, \mathbb{C})/SU(p, q)$ of Hermitian matrices of signature (p, q) and determinant 1, using a similar argument as in the proof of Theorem 1, one derives for λ such that $\mathrm{Re}(\lambda_i - \lambda_j) > 0,\ p+1 \leq i \leq n;\ 1 \leq j \leq p$ the formula

$$c_{\mathcal{D}}(\lambda) = \frac{(2\pi)^{pq}}{\prod_{\alpha \in \Delta_{-1}} < \lambda, \alpha >}$$

which was obtained by a different method for all ordered symmetric spaces of Olshanskii type in [FHO].

One can apply our method of computation of the c-function to other ordered symmetric spaces which appear as the determinant one surfaces of the non-Riemannian orbits of the structure group on a Euclidean Jordan algebra: the spaces $SL(n, \mathbb{H})/SO(p, q, \mathbb{H})$(quaternion case) and $SL(3, \mathbb{O})/SO(1, 2, \mathbb{O})$(the exceptional octonion case).

In [F1] an integral formula for spherical functions φ_λ on $M = SL(n, \mathbb{R})/SO(p, q)$ was obtained. Theorem 1 allows to determine the set of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ for which these spherical functions exist.

Corollary. — *The spherical functions* φ_λ *on* $M = SL(n, \mathbb{R})/SO(p, q)$ *are defined for* λ *such that* $\mathrm{Re}(\lambda_i - \lambda_j) > -1,\ p+1 \leq i \leq n;\ 1 \leq j \leq p$.

Acknowledgements. We thank Jacques Faraut and Jean-Jacques Loeb for helpful discussions and comments on the subject of the paper.

REFERENCES

[F1] Faraut J., Algèbres de Volterra et transformation de Laplace sphérique sur certains espaces symétriques ordonnés, *Symp.Math.* **29**(1987), 183-196.

[F2] Faraut J., Fonctions sphériques sur un espace symétrique de type Cayley, 1994, to appear in *Contemporary Mathematics.*

[FHO] Faraut J., Hilgert J. and Olafsson G., Spherical functions on ordered symmetric spaces, *Ann. Inst. Fourier* **44**(1994), 927-966..

[GK] Gindikin S.G. and Karpelevitch F.I., Plancherel measure of Riemannian symmetric spaces of non-positive curvature, *Dokl.Akad.Nauk SSSR* **145**(1962), 252-255.