The plane Jacobian conjecture for rational curves

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Abstract[1]

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $f(x,y)$ be a nonzero polynomial of $\mathbb{K}[x,y]$. We prove that if the generic element of the family $(f-\lambda)_{\lambda\in\mathbb{K}}$ is rational polynomial, and if the Jacobian $J(f,g)$ is a nonzero constant for some polynomial $g\in\mathbb{K}[x,y]$, then $\mathbb{K}[f,g]=\mathbb{K}[x,y]$.

Introduction and notations

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $f(x,y)$ be an irreducible polynomial of $\mathbb{K}[x,y]$. Denote by f_x (resp. f_y) the x-derivative (resp. the y-derivative) of f. The plane Jacobian conjecture says the following:

(JC) If $J(f,g)=f_xg_y-f_yg_x\in\mathbb{K}^*$ then $\mathbb{K}[f,g]=\mathbb{K}[x,y]$

If $\mathbb{K}[f,g]=\mathbb{K}[x,y]$, then f has one place at infinity, i.e. f has one point at infinity and it is analytically irreducible at this point. On the other hand, by Abhyankar-Moh Lemma (see [1]), if f has one place at infinity and $J(f,g)=f_xg_y-f_yg_x\in\mathbb{K}^*$ then $\mathbb{K}[f,g]=\mathbb{K}[x,y]$. In particular, (JC) is equivalent to the following: If $J(f,g)=f_xg_y-f_yg_x\in\mathbb{K}^*$ then f has one place at infinity.

Assume that f has one place at infinity. It follows from [1] that $f_\lambda=f-\lambda$ has one place at infinity for all $\lambda\in\mathbb{K}$. Moreover, given a nonzero polynomial $g\in\mathbb{K}[x,y]$, if for all $\tau\in\mathbb{K}$, f does not divide $g_\tau=g-\tau$, then the intersection $\mathrm{int}(f,g_\tau)$ of f with g_τ, defined to be the rank of the $\mathbb{K}$-vector space $\dfrac{\mathbb{K}[x,y]}{(f,g_\tau)}$, does not depend on $\tau\in\mathbb{K}$. Let more generally f,g be two polynomials of $\mathbb{K}[x,y]$: we say that the family $(g_\tau)_{\tau\in\mathbb{K}}$ is regular with respect to f if $\mathrm{int}(f,g_\tau)$ does not depend on τ. If $(g_\tau)_{\tau\in\mathbb{K}}$ is not regular with respect to f, then there is a finite set of elements, denoted $I(f,g)$, such that $\mathrm{int}(f,g_\tau)=i$ does not depend on $\tau\in\mathbb{K}-I(f,g)$. Moreover, for all $\tau\in I(f,g)$, $\mathrm{int}(f,g_\tau)\neq i$. In [2] (see also [3]), it has been proved that if $J(f,g)\in\mathbb{K}^*$ and if $I(f,g)=\emptyset$, then $\mathbb{K}[f,g]=\mathbb{K}[x,y]$. In particular, (JC) is equivalent to the following:

If $J(f,g)=f_xg_y-f_yg_x\in\mathbb{K}^*$ then $I(f,g)=\emptyset$

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The aim of this paper is to prove (JC) when the the generic element of the family $(f_\lambda)_{\lambda\in\mathbb{K}}$ is an irreducible polynomial with a rational parametrization. More precisely we prove the following:

Theorem. Let $(f_\lambda)_{\lambda\in\mathbb{K}}$ be a family of polynomials of $\mathbb{K}[x,y]$ whose generic element is rational. If $J(f,g)\in\mathbb{K}^*$ for some $g\in\mathbb{K}[x,y]$, then $I(f,g)=\emptyset$, in particular $\mathbb{K}[f,g]=\mathbb{K}[x,y]$.

The paper is organized as follows: in Section 1 we recall some basic properties about local and global intersection multiplicities. In particular we prove that under some conditions on the curves defined by f and g, the intersection of f with the Jacobian $J(f,g)$ coincides with that of f with the exact differential form dg (see Proposition 1.1. and Corollary 1.2). We also characterize polynomial maps from $\mathbb{K}^2$ to $\mathbb{K}^2$ with infinitely many empty fibers (see Proposition 1.4). In Section 2 we consider an irreducible rational curve f, then we characterize polynomials g such that the intersection of f with dg is 0. Using this characterization and previous results from [2] and [3], we prove our main Theorem (see Theorem 2.2).

1 Intersection multiplicity

Let f,g be two nonzero polynomials of $\mathbb{K}[x,y]$. If $H(x,y)\in\mathbb{K}[x,y]$ then we set $V(H)=\{P\in\mathbb{K}^2, H(P)=0\}$. Let $P=(a,b)\in\mathbb{K}^2$, $P\in V(f)\cap V(g)$. Set $X=x-a, Y=y-b$ and let $F(X,Y)=f(X+a,Y+b), G(X,Y)=g(X+a,Y+b)$. We define the local intersection multiplicity of f,g at (a,b), denoted $\mathrm{int}_P(f,g)$, to be the rank of the $\mathbb{K}$-vector space $\dfrac{\mathbb{K}[[X,Y]]}{(F,G)\mathbb{K}[[X,Y]]}$, where $\mathbb{K}[[X,Y]]$ denotes the ring of formal power series in X,Y over $\mathbb{K}$. We define the Milnor number of f at P, denoted $\mu_P(f)$, to be the rank of the $\mathbb{K}$-vector space $\dfrac{\mathbb{K}[[X,Y]]}{(F_X,F_Y)\mathbb{K}[[X,Y]]}$. If $\mu_P(f)=0$ (resp. $\mu_P(f)>0$) then we say that P is a smooth (resp. singular) point of $V(f)$. We define the intersection multiplicity of f and g, denoted $\mathrm{int}(f,g)$, to be the rank of the $\mathbb{K}$-vector space $\dfrac{\mathbb{K}[x,y]}{(f,g)}$. We define the Milnor number of f, denoted $\mu(f)$, to be the rank of the $\mathbb{K}$-vector space $\dfrac{\mathbb{K}[x,y]}{(f,f_x,f_y)}$. Note that $\mathrm{int}(f,g)=\sum_{P\in V(f)\cap V(g)}\mathrm{int}_P(f,g)$ and $\mu(f)=\sum_{P\in V(f)}\mu_P(f)$. We finally set $\bar{\mu}(f)=\sum_{\lambda\in\mathbb{K}}\mu(f_\lambda)$ and we recall that $\bar{\mu}(f)=\mathrm{rank}_{\mathbb{K}}\dfrac{\mathbb{K}[x,y]}{(f_x,f_y)}$.

Let $P\in V(f)$, and assume, after a possible change of variables, that $P=(0,0)$. Assume that $f(x,y)$ is irreducible in $\mathbb{K}[[x,y]]$, and let $x(t),y(t)\in\mathbb{K}[[t]]$ be a primitive parametrization of f at $(0,0)$. Given $0\neq g\in\mathbb{K}[x,y]$, we have $\mathrm{int}_P(f,g)=O_t g(x(t),y(t))$, where O_t denotes the t-order. Given a differential form $w=a(x,y)dx+b(x,y)dy$, we set $\mathrm{int}_P(f,w)=O_t(a(x(t),y(t))x'(t)+b(x(t),y(t))y'(t))$, and we call this order the intersection multiplicity of w with f at P. If $w=dg=g_xd_x+g_ydy$, then $\mathrm{int}_P(f,w)=\mathrm{int}_P(f,g)-1$.

More generally, let $f=f_1\dots f_r$ be the factorization of f into irreducible components of $\mathbb{K}[[x,y]]$. We set $\mathrm{int}_P(f,dg)=\sum_{i=1}^r\mathrm{int}_P(f_i,dg)=\sum_{i=1}^r(\mathrm{int}_P(f_i,g)-1)=\mathrm{int}_P(f,g)-r$.

We finally set $\mathrm{int}(f,dg)=\sum_{\tau\in\mathbb{K},P\in V(f)\cap V(g_\tau)}\mathrm{int}_P(f,dg)$. With these notations we have the following proposition:

Proposition 1.1 Assume that $f(0,0)=g(0,0)=0$ and let $J=J(f,g)$ be the Jacobian of f and g.

If $\mu_{(0,0)}(f) = 0$ (i.e. $(0,0)$ is a smooth point of $V(f)$), then $\text{int}_{(0,0)}(f, J) = \text{int}_{(0,0)}(f, dg)$.

Proof. Let $(u = f, v)$ be a system of local coordinates at $(0,0)$ (this is possible since $(0,0)$ is a smooth point of $V(f)$). Write $g(u,v) = \sum_{ij} c_{ij} u^i v^j \in \mathbb{K}[[u,v]]$, then locally at $(0,0)$, $J(f,g) = f_u g_v - f_v g_u$, and $dg = g_u du + g_v dv$. But $f_u = 1, f_v = 0$, hence, if $u = 0$, $J(f,g) = g_v(0,v)$ and $dg = g_v(0,v)dv$. This proves the result.

Corollary 1.2 Let the notations be as above and assume that for all $\tau \in \mathbb{K}$ and for all $P \in V(f) \cap V(g_\tau)$, $\mu_P(f) = 0$ (i.e. $V(f)$ and $V(g_\tau)$ meet at smooth points of $V(f)$). Let $J = J(f,g)$ be the Jacobian of f and g. We have $\text{int}(f, J) = \text{int}(f, dg)$. In particular, if $J \in \mathbb{K}^*$ then $\text{int}(f, dg) = 0$.

Proof. This results from Proposition 1.1

Corollary 1.3 *Let the notations be as above and assume that $V(f)$ is smooth. We have int$(f, f_y) =$* $\text{int}(f, dx)$.

Proof. This is a particular case of Corollary 1.2 where $g = x$.

Given $(\lambda, \tau) \in \mathbb{K}^2$, we set $i_{(\lambda,\tau)} = \text{int}(f_\lambda, g_\tau)$. The next Proposition gives a characterization of $J(f,g)$ when $i_{(\lambda,\tau)} = 0$ for infinitely many $(\lambda, \tau) \in \mathbb{K}^2$.

Proposition 1.4 Suppose that f and g are monic in y, i.e. $f = y^n + a_1(x)y^{n-1} + \ldots + a_n(x)$ and $g = y^m + b_1(x)y^{m-1} + \ldots + b_m(x)$ with $\deg_x a_i(x) \leq i$ (resp. $\deg_x b_j(x) \leq j$) for all $i \in \{1, \ldots, n\}$ such that $a_i(x) \neq 0$ (resp. for all $j \in \{1, \ldots, m\}$ such that $b_j(x) \neq 0$). If $i_{(\lambda,\tau)} = 0$ for infinitely many $(\lambda, \tau) \in \mathbb{K}^2$, then either $i_{(\lambda_0,\tau_0)} = +\infty$ for some $(\lambda_0, \tau_0) \in \mathbb{K}^2$, or $H(f,g) \in \mathbb{K}^*$ for some $H(\lambda, \tau) \in \mathbb{K}[\lambda, \tau]$. In particular either $\text{int}(f - \lambda_0, J(f,g)) = +\infty$ (hence $f - \lambda_0$, $g - \tau_0$, and $J(f,g)$ have a common component in $\mathbb{K}[x,y]$) or $J(f,g) = 0$.

Proof. Let $R(x, \lambda, \tau) \in \mathbb{K}[x, \lambda, \tau]$ be the y-resultant of $f - \lambda, g - \tau$ and write:

$$R(x, \lambda, \tau) = \sum_{k=0}^{s} A_k(\lambda, \tau) x^k$$

Let $(\lambda_0, \tau_0) \in V(A_1, \ldots, A_s) - V(A_0)$, where $V(A_1, \ldots, A_s) = \cap_{i=1}^s V(A_i)$. Then $i_{(\lambda_0,\tau_0)} = 0$. By hypothesis, $V(A_1, \ldots, A_s)$ is infinite. In particular there is a polynomial $H(\lambda, \tau) \in \mathbb{K}[\lambda, \tau] \setminus \mathbb{K}^*$ such that $V(H) \subseteq V(A_1, \ldots, A_s)$.

i) If $V(A_0, A_1, \ldots, A_s) \neq \emptyset$, then for all $(\lambda_i, \tau_i) \in V(A_0, A_1, \ldots, A_s)$, $R(x, \lambda_i, \tau_i) = 0$, in particular $\text{int}(f - \lambda_i, g - \tau_i) = +\infty$. It follows that $f = Qf_1, g = Qg_1$ with $Q \in \mathbb{K}[x,y] \setminus \mathbb{K}$, hence $\text{int}(f, J(f,g)) = +\infty$.

ii) If $V(A_s, \ldots, A_1, A_0) = \emptyset$, then $V(H) \subseteq \mathbb{K}^2 - V(A_0)$, consequently for all $(c,d) \in V(H)$, $R(x,c,d) = A_0(c,d) \neq 0$. Let $(X,Y) \in \mathbb{K}^2$ and let $\lambda_0 = f(X,Y), \tau_0 = g(X,Y)$. Since $R(x, \lambda_0, \tau_0) = \text{res}_y(f - \lambda_0, g - \tau_0) = 0$, then $H(\lambda_0, \tau_0) \neq 0$ (otherwise, $R(x, \lambda_0, \tau_0) = A_0(\lambda_0, \tau_0) = 0$, then $(\lambda_0, \tau_0) \in V(H) \cap V(A_0)$ which is a contradiction). This proves that for all $(X,Y) \in \mathbb{K}^2, H(f(X,Y), g(X,Y)) \neq 0$, in particular $H(f,g) \in \mathbb{K}^*$, and $J(f,g) = 0$.

Corollary 1.5 Let f, g be non zero polynomials of $\mathbb{K}[x, y]$. If $J = J(f, g) \in \mathbb{K}^*$, then the following conditions hold:

1. For all $(\lambda, \tau) \in \mathbb{K}^2$, $\text{int}(f_\lambda, g_\tau) < +\infty$.

2. There exists a finite set $E \subset \mathbb{K}^2$ (possibly empty) such that for all $(\lambda, \tau) \in (\mathbb{K}^2 \setminus E)$, $i_{(\lambda,\tau)} > 0$.

Proof. Given f, g, there exists an automorphism of $\sigma : \mathbb{K}[x, y] \longmapsto \mathbb{K}[w, z]$ such that $\sigma(f)$ and $\sigma(g)$ are monic in z. Since the intersection multiplicity remains invariant, then our assertion results from Proposition 1.4.

2 The main result

Let f be a nonzero polynomial of $\mathbb{K}[x, y]$ and assume that $J(f, g) \in \mathbb{K}^*$ for some $g \in \mathbb{K}[x, y]$. Using Corollary 1.2 and Corollary 1.5, we shall assume that the following conditions hold:

1) $\text{int}(f_\lambda, g_\tau) < +\infty$ for all $\lambda, \tau \in \mathbb{K}$.

2) $\text{int}(f, g_\tau) > 0$ for all $\tau \in \mathbb{K}$.

3) $\text{int}(f_\lambda, g) > 0$ for all $\lambda \in \mathbb{K}$.

4) $\text{int}(f_\lambda, dg) = 0$ for all $\lambda \in \mathbb{K}$.

Assume that the family $(f_\lambda)_{\lambda \in \mathbb{K}}$ is generically rational, i.e. for almost all $\lambda \in \mathbb{K}$, f_λ is irreducible and rational. Also, assume that f is a generic element of this family. Let $x(t) = \dfrac{x_1(t)}{x_2(t)}, y(t) = \dfrac{y_1(t)}{y_2(t)}$ be a parametrization of f and assume that $\gcd(x_1(t), x_2(t)) = \gcd(y_1(t), y_2(t)) = 1$ and also that, without loss of generality, the parametrization is normal, i.e. for all $P \in V(f)$, $P = f(x(t_P), y(t_P))$ for some $t_P \in \mathbb{K}$ (see [7], Theorem 3). Let g be as above and let $g(t) = g(x(t), y(t)) = \dfrac{u(t)}{v(t)}$. If $\deg_t u(t) = \deg_t v(t)$ then there exists $\tau_0 \in \mathbb{K}^*$ such that $g(t) = \dfrac{u(t)}{v(t)} = \tau_0 + \dfrac{u_1(t)}{v(t)}$, where $\tau_0 \in \mathbb{K}$ and $\deg_t(u_1(t)) \neq \deg_t(v(t))$. In particular we have $g'(t) = \dfrac{u'(t)v(t) - u(t)v'(t)}{v(t)^2} = \dfrac{u_1'(t)v(t) - u_1(t)v'(t)}{v(t)^2}$.

(*) If $v(t) \in \mathbb{K}^*$, then $g(t) = \tau_0 + a.u_1(t), a \in \mathbb{K}^*$. Since $\text{int}(f, dg) = 0$, then $\deg_t u_1(t) = 1$, in particular $\text{int}(f, g_\tau) = 1$ for all $\tau \in \mathbb{K}$.

(**) If $u_1(t) \in \mathbb{K}$, then $\text{int}(f, g_{\tau_0})$ is either 0 or $+\infty$. This contradicts conditions 1) and 2).

(***) If $v(t) \notin \mathbb{K}$ and $u_1(t) \notin \mathbb{K}$, since $\deg_t u_1(t) \neq \deg_t v(t)$, then $u_1'(t)v(t) - u_1(t)v'(t) \notin \mathbb{K}^*$. Now $\text{int}(f, dg) = 0$ implies that $r(u_1'(t)v(t) - u_1(t)v'(t))$ divides $x_2(t)y_2(t)$ (where r stands for the radical). Suppose, without loss of generality, that $g(0, y) \neq g$, i.e. g is not a polynomial of $\mathbb{K}[y]$. Let $x = X + cY, y = Y$ where c is a generic element of $\mathbb{K}^*$. Then $\tilde{f}(X, Y) = f(X + cY, Y)$ is a rational polynomial of $\mathbb{K}[X, Y]$ whose parametrization is $X(t) = x(t) - cy(t) = \dfrac{x_1(t)y_2(t) - cy_1(t)x_2(t)}{x_2(t)y_2(t)}, Y(t) = y(t)$. But $g(x - cy, y) = g(x, y) + c \cdot h(x, y, c)$ and $h \notin \mathbb{K}^*$ (otherwise we get $g(x - cy, y) = g(x, y) + ac, a \in \mathbb{K}$ and the substitution $c = \dfrac{x}{y}$ leads to a contradiction), hence

$$g(X(t), Y(t)) = g(x(t) - cy(t), y(t) = \frac{\bar{u}(t,c)}{\bar{v}(t)}$$

and $\bar{u}'(t,c)\bar{v}(t) - \bar{u}(t,c)\bar{v}'(t) \notin \mathbb{K}[t]$. Consequently there exists $c_0 \in \mathbb{K}^*$ such that $r(\bar{u}'(t,c_0)\bar{v}(t) - \bar{u}(t,c_0)\bar{v}'(t))$ does not divide $x_2(t)y_2(t)$. In particular, we may assume that the case (***) does not hold. We finally get the following:

Proposition 2.1 Let the notations be as above. If conditions 1) to 4) are satisfied, then $\text{int}(f, g_\tau) = 1$ for all $\tau \in \mathbb{K}$.

We can now state and prove our main theorem:

Theorem 2.2 Let $f \in \mathbb{K}[x, y]$, and assume that the generic element of the family $(f_\lambda)_{\lambda \in \mathbb{K}}$ is irreducible and rational. If $J = J(f, g) \in \mathbb{K}^*$ for some $g \in \mathbb{K}[x, y]$, then $\mathbb{K}[f, g] = \mathbb{K}[x, y]$.

Proof. We shall assume that f is a generic element of the family $(f_\lambda)_{\lambda \in \mathbb{K}}$. By Corollary 1.5, we can also assume that conditions 1) to 4) are satisfied. In particular, if $(x(t), y(t))$ is a proper parametrization of f, then we may assume that $g(x(t), y(t)) = \tau_0 + a.u_1(t)$, where $a \in \mathbb{K}^*$. Finally for all $\tau \in \mathbb{K}$, $g_\tau(x(t), y(t)) = \tau_0 - \tau + a.u_1(t)$ with $\deg_t u_1(t) = 1$. Hence $\text{int}(f, g_\tau) = 1$ for all $\tau \in \mathbb{K}$. Our assertion results from the following proposition:

Proposition 2.3 Let f, g be two nonzero polynomials of $\mathbb{K}[x, y]$ and let $i = \text{int}(f, g)$. Let $I(f, g) = \{\tau \in \mathbb{K}^* | \text{int}(f, g - \tau) \neq i\}$. If $J(f, g) \in \mathbb{K}^*$ and $I(f, g) = \emptyset$, then $\mathbb{K}[f, g] = \mathbb{K}[x, y]$.

Proof. See [2], (8.5.) and (8.6) or [3], Lemma 1. and Proposition 3.

Remark 2.4 i) Let $(f_\lambda)_{\lambda \in \mathbb{K}}$ be a pencil of polynomials of $\mathbb{K}[x, y]$. If for all $\lambda \in \mathbb{K}$, f_λ is irreducible and rational then, by [6], f has one place at infinity. Moreover, there exists an automorphism of $\mathbb{K}^2$ which transforms f into a coordinate of $\mathbb{K}^2$. This conclusion is false if we assume that the pencil $(f_\lambda)_{\lambda \in \mathbb{K}}$ is generically rational. For example, the pencil $(f_\lambda)_{\lambda \in \mathbb{K}}$, where $f = y - xy^2$, is generically rational, but f has two points at infinity.

ii) In [4] (Theorem 2.5., see also the corrigendum in [5]), R. Heitmann proved that the plane Jacobian conjecture is equivalent to the following: if $f - \lambda$ is irreducible for all $\lambda \in \mathbb{K}$ and if $\frac{dx}{f_y} = dg$ for some $g \in \mathbb{K}[x, y]$ then $V(f)$ has genus zero. Our result implies that the condition on the irreducibility is unecessary.

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